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A General Family of the Srivastava-Gupta Operators Preserving Linear Functions

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Abstract. The general sequence of positive linear operators containing some well-known operators as special cases were introduced in the earlier work by Srivastava and Gupta [9], which reproduce only the constant functions. In the present sequel, we provide a general sequence of operators which preserve not only the constant functions, but also linear functions.

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1. Introduction and Preliminaries

In the year 2003, Srivastava and Gupta [9] introduced a general sequence of positive linear operators defined by

$$V_{n,c}(f,x) = n\sum_{k=1}^{\infty} p_{n,k}(x,c)\int_0^{\infty} p_{n+c,k-1}(t,c)f(t)dt + p_{n,0}(x,c)f(0) \qquad (n\in\mathbb{N}:=\{1,2,3,\cdots\}), \tag{1}$$

where

$$p_{n,k}(x,c) = \frac{(-x)^k}{k!}\,\phi_{n,c}^{(k)}(x).$$

The following special cases of the operator defined by (1) are worthy of mention here:

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- If $c = 0$ and $\phi_{n,c}(x) = e^{-nx}$, then we get

$$p_{n,k}(x,0) = e^{-nx}\,\frac{(nx)^k}{k!};$$

- If $c \in \mathbb{N}$ and $\phi_{n,c}(x) = (1+cx)^{-\frac{n}{c}}$, then we obtain

$$p_{n,k}(x,c) = \frac{\left(\frac{n}{c}\right)_k}{k!}\,\frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}},$$

where, and in what follows, $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined, for $\lambda \in \mathbb{C}$, by

$$(\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) \quad (n \in \mathbb{N});$$

- If $c = -1$ and $\phi_{n,c}(x) = (1-x)^n$, then

$$p_{n,k}(x,-1) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Here, for the last case when $c = -1$, we have $x \in [0,1]$ whereas, for $c \in \mathbb{N} \cup \{0\}$, we have $x \in [0,\infty)$. These operators were further discussed by Ispir and Yüksel [6], Atakuta and Büyükyazici [2], Deo [3], Gupta and Tachev [5], Kumar [7] and Yadav [13], and other authors (see also the closely-related works by Acar *et al.* [1], Maheshwari [8], Srivastava and Gupta [10], Srivastava and Zeng [11], and Verma and Agrawal [12]).

The moments of the above-defined operator $V_{n,c}(f,x)$ of order r $(r \in \mathbb{N})$ are given, in terms of the Gauss hypergeometric function ${}_2F_1$ and Kummer's confluent hypergeometric function ${}_1F_1$, as follows:

$$V_{n,c}(e_r,x) = \begin{cases} \dfrac{r!\cdot(nx)}{(n-c)(n-2c)\cdots(n-rc)} \\ \qquad \cdot\,{}_2F_1\left(\dfrac{n}{c}+1, 1-r; 2; -cx\right) & (c \in \mathbb{N} \cup \{-1\}) \\ \\ \dfrac{r!\cdot(nx)}{n^r}\;{}_1F_1(1-r;2;-nx) & (c = 0), \end{cases} \tag{2}$$

where ${}_\ell F_m(\alpha_1,\cdots,\alpha_\ell;\beta_1,\cdots,\beta_m;x)$ denotes the generalized hypergeometric function with ℓ numerators parameters $\alpha_1,\cdots,\alpha_\ell$ and m denominator parameters $\beta_1,\cdots,\beta_m$ defined by

$${}_\ell F_m(\alpha_1,\cdots,\alpha_\ell;\beta_1,\cdots,\beta_m;x) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\ell}(\alpha_j)_n}{\prod_{j=1}^{m}(\beta_j)_n}\,\frac{x^n}{n!} \tag{3}$$

$$\Big(\ell, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \ell \leqq m+1;\ \ell \leqq m \ \text{ and } \ |x| < \infty;$$
$$\ell = m+1 \ \text{ and } \ |x| < 1;\ \ell = m+1,\ |x| = 1 \ \text{ and } \ \Re(\omega) > 0\Big),$$

where

$$\omega := \sum_{j=1}^{m} \beta_j - \sum_{j=1}^{\ell} \alpha_j$$
$$\big(\alpha_j \in \mathbb{C} \ \ (j = 1, \cdots, \ell);\ \ \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \ \ (j = 1, \cdots, m)\big).$$

The operators $V_{n,c}(f, x)$ are known to preserve the constant functions only except for the case $c = 0$. We observe that in (1) the suffix n in the basis function $p_{n,k}(x, c)$ has a difference of c under summation and integration. But, if we have a difference of $2c$ in place of c under summation and integration, we may get the modified operators which preserve the constant functions as well as linear functions.

2. Modified Operators Preserving Linear Functions

Here, in this section, we introduce a modification of the operators $V_{n,c}(f, x)$ which do preserve linear functions as well. For m an integer and $c \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define

$$G_{n,c}(f, x) = [n + (m+1)c] \sum_{k=1}^{\infty} p_{n+mc,k}(x, c) \cdot \int_0^{\infty} p_{n+(m+2)c,k-1}(t, c) f(t)dt + p_{n+mc,0}(x, c) f(0), \tag{4}$$

where $p_{n,k}(x, c)$ is defined by (1). The above-mentioned two cases lead to the familiar Phillips operators and the genuine Baskakov-Durrmeyer type operators for $c = 0$ and $c \in \mathbb{N}$, respectively. In the special case when $m = 0$ and $c = 1$, the operators $G_{n,c}(f, x)$ were considered by Finta [4]. who also estimated some converse results. Moreover, for $c = -1$, the operators $G_{n,c}(f, x)$ take the following form:

$$G_{n,-1}(f, x) = (n-m-1) \sum_{k=1}^{n-m-1} p_{n-m,k}(x, -1) \cdot \int_0^1 p_{n-m-2,k-1}(t, -1) f(t)dt + p_{n-m,0}(x, -1) f(0) + p_{n-m,n-m}(x, -1) f(1), \tag{5}$$

The moments of the operators $G_{n,c}(f, x)$ of order r $(r \in \mathbb{N})$ are given, in terms of the Gauss hypergeometric function ${}_2F_1$, as follows:

$$G_{n,c}(e_r,x) = \begin{cases} \dfrac{r!\cdot x\Gamma\left(\dfrac{n}{c}-r+m+1\right)}{c^{r-1}\,\Gamma\left(\dfrac{n}{c}+m\right)} \\ \quad \cdot {}_2F_1\left(\dfrac{n}{c}+m+1, 1-r; 2; -cx\right) & (c\in\mathbb{N}\cup\{-1\}) \\ \\ \dfrac{r!\cdot(nx)}{n^r}\;{}_1F_1(1-r;2;-nx) & (c=0). \end{cases} \tag{6}$$

Finally, by applying this last result (6), it can easily be verified that the operators $G_{n,c}(f,x)$ preserve not only the constant functions, but also linear functions.

3. Concluding Remarks and Observations

The present investigation was motivated essentially by the fact that the widely-studied Srivastava-Gupta operator $V_{n,c}(f,x)$ preserves only the constant functions, but not linear functions. Here, in this paper, we have successfully provided a general sequence $G_{n,c}(f,x)$ of positive linear operators which preserve not only the constant functions, but also linear functions. We have also considered several recent developments on the subject of positive linear operators.

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