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Pre-irresolute functions in closure spaces

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Abstract. The preopen sets are used to define pre-open functions, pre-closed functions, pre-continuous functions, contra-pre-continuous functions and pre-irresolute functions which are investigated. They are also used to introduce a new type of connectedness and compactness in closure spaces, they called p-connectedness and p-compactness respectively

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1. Introduction

Kazimierz Kuratowski was a Polish mathematician and logician, he defined [14] closure operator by the following: Let X be a set and $P(X)$ its power set. A Kuratowski Closure Operator is a function $cl : P(X) \to P(X)$ with the following properties:

(i) $cl\,(\phi) = \phi$ (Preservation of Nullary Union)

(ii) $A \subseteq cl\,(A)$ for every subset $A \subseteq X$ (Extensivity)

(iii) $cl\,(A \cup B) = cl\,(A) \cup cl\,(B)$ for any subsets $A, B \subseteq X$ (Preservation of Binary Union).

(iv) $cl\,(cl\,(A)) = cl\,(A)$ for every subset $A \subseteq X$ (Idempotence)

If the last axiom(iv), idempotence, is omitted, then the axioms define a preclosure operator. A consequence of the third axiom(iii) is: $A \subseteq B$ then $cl\,(A) \subseteq cl\,(B)$ (Preservation of Inclusion). Then cl, together with the underlying set X, is called closure space and is

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denoted by (X, cl).
In 1966, Eduard Cech defined closure operator by the following:

Let X be a set and $P(X)$ its power set. A function $c : P(X) \to P(X)$ with the following properties:

(i) $c(\phi) = \phi$.

(ii) $A \subseteq c(A)$ for every subset $A \subseteq X$.

(iii) $c(A \cup B) = c(A) \cup c(B)$ for any subsets $A, B \subseteq X$.

Then c, together with the underlying set X, is called a Cech closure space and is denoted by (X, c). If c also satisfies: $c(c(A)) = c(A)$ for every subset $A \subseteq X$, then (X, c) is a topological space.
In 2009, Jeeranunt Khampakdee [15] defined closure operator by the following: A function $c : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a closure operator on X and the pair (X, c) is called a closure space if the following axioms are satisfied:

(i) $c(\phi) = \phi$.

(ii) $A \subseteq c(A)$ for every $A \subseteq X$.

(iii) $A \subseteq B \Rightarrow c(A) \subseteq c(B)$, for all $A, B \subseteq X$.

The concept of closure operator and closure spaces are very usefull material in several branches of Science, such as Topology [2],[3],[4],[5]Computer Science Theory[18], Biochemistry[6].
The purpose of this paper is to study the concept of preopen sets in closure spaces. Closure spaces were introduced by E.Cech [2] in 1966 and then studied by many mathematicians, see e.g. [2],[3],[4],[5],[7],[10],[8],[9] and [13]. Closure spaces are sets endowed with a grounded, extensive and monotone closure operator. Mashhure et al[17] introduced the concept of preopen sets and pre-continuous functions. The preopen sets and local dense sets are same in topological space, also the pre-continuity and almost-continuity (in the sense Hussain)[12] are same in topological spaces. Halgwrd M.Darwesh [11] used the teqnique of mashhoury[12] to introduce and study the concept of preopen sets in closure spaces, and then he showed that its differ to local dense sets. However he defined the concept of pre-continuous functions in closure spaces and then he showed that the concepts of pre-continuity and almost-continuity (in the sense Hussain) [11] are independent concepts.
In this paper, in Section 3, we introduce the notion of pre-open(pre-closed) functions, contra-pre-continuous and study some of their properties.
In Section 4, we introduce and discuss pre-irresolute functions in closure spaces. We establish some basic properties of pre-irresolute functions
In Section 5, we introduce the notion of p-connectedness and study some of their properties.
In Section 6, we introduce the notion of p-compactness and study some of their properties.

2. Preliminaries

A function $c : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a closure operator on X and the pair (X, c) is called a closure space[15] if the following axioms are satisfied:

(i) $c(\phi) = \phi$.

(ii) $A \subseteq c(A)$ for every $A \subseteq X$.

(iii) $A \subseteq B \Rightarrow c(A) \subseteq c(B)$, for all $A,\ B \subseteq X$.

Definition 1. *[15] A closure operator c on a set X is called additive if* $c(A \cup B) = c(A) \cup c(B)$, *for all* $A, B \subseteq X$.

Definition 2. *[15] A closure operator c on a set X is called idempotent if* $cc(A) = c(A)$, for all $A \subseteq X$.

Definition 3. *[15] A subset* $A \subseteq X$ *is closed in the closure space* (X, c) if $c(A) = A$. *It is called open, if its complement in X is closed. The empty set and the whole space are both open and closed.*

Definition 4. *[11] A subset A of a space* (X, c) *is said to be a preopen set, if there exists an open set G such that* $A \subseteq G \subseteq c(A)$. *The complement of a preopen set is called preclosed.*

Theorem 1. *[11] A subset A of a space* (X, c) *is preclosed if and only if there exists a closed set F such that* $X \backslash c(X \backslash A) \subseteq F \subseteq A$.

Proposition 1. *[15] Let* (X, c) *be a closure space and* $\{G_\alpha\}_{\alpha \epsilon J}$ *be a collection of subsets of X. Then* $\bigcup_{\alpha \epsilon J} c(G_\alpha) \subseteq c(\bigcup_{\alpha \epsilon J} G_\alpha)$.

Proposition 2. *[15] The union(intersection)of any family of open(closed)sets in a closure space* (X, c) *is open(closed).*

Proposition 3. *[11] The union(intersection)of any family of preopen(preclosed)sets in a closure space* (X, c) *is preopen(preclosed).*

Definition 5. *[11] The interior operator* $i : P(X) \to P(X)$ *corresponding to the closure operator c on X is given by;* $i(A) = X \backslash c(X \backslash A)$.

Theorem 2. *[11] Let A be a subset of a closure* (X, c). *If* $x \in c(A)$, *then* $G \cap A \neq \phi$, *for each open subset G of X containing x.*

Proposition 4. *[11] Let A be a subset of a closure* (X, c) *and c is idempotent on X, then* $x \in c(A)$ *if and only if* $G \cap A \neq \phi$, *for each open subset G of X containing x.*

Proposition 5. *[11] Let c be an idempotent closure operator on a set X. If A is preopen in X and $B \subseteq A \subseteq c(B)$, then B is preopen.*

Definition 6. *[15] A closure space (Y, v), is said to be a subspace of (X, c), if $Y \subseteq X$ and $v(A) = c(A) \cap Y$, for each subset $A \subseteq Y$.*

Theorem 3. *[11] Let $A \subseteq Y \subseteq X$, where (Y, v) is a subspace of (X, c). If A is preopen in X, then A is preopen in Y.*

Proposition 6. *[1] The product of a family $\{(X_\alpha, c_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod_{\alpha \in I}(X_\alpha, c_\alpha)$, is the closure space $(\prod_{\alpha \in I} X_\alpha, c)$, where $\prod_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$ and c is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha \in I}(X_\alpha, c) \to (X_\alpha, c)$, $\alpha \in I$, i.e., is defined by $c(A) = \prod_{\alpha \in I} c_\alpha \pi_\alpha(A)$, $A \subseteq \prod_{\alpha \in I} X_\alpha$. Then the projection function π_α is continuous.*

Proposition 7. *[16] Let $\{(X_\alpha, c_\alpha) : a \in J\}$ be a family of closure spaces. Then $F_a$is closed in (X_α, c_α), for all $a \in J$ if and only if $\prod_{a \in J} F_\alpha$is closed in $\prod_{a \in J}(X_\alpha, c_a)$.*

Proposition 8. *[16] Let $\{(X_\alpha, c_\alpha) : a \in J\}$ be a collection of closure spaces, $G \subseteq \prod_{\alpha \epsilon J} X_\alpha$. If G is a open in $\prod_{\alpha \epsilon J}(X_\alpha, c_\alpha)$ and π_α is a project function, then $\pi_\alpha(G)$ is a open in (X_α, c_α).*

Definition 7. *[16] Let $(X, c_1)$and (Y, c_2) be closure spaces. A function $f : (X, c_1) \to (Y, c_2)$ is called open (respectively, closed) if the image of every open (respectively, closed) set in (X, c_1) is open (respectively, closed) in (Y, c_2).*

Proposition 9. *[16] A function $f : (X, c_1) \to (Y, c_2)$ is said to be continuous if $f(c_1(A)) \subseteq c_2 f(A)$ for every subset A of X.*

Proposition 10. *[16] Let (X, c_1) and (Y, c_2) be closure spaces. If $f : (X, c_1) \to (Y, c_2)$ is a continuous function, then the inverse image under f of each open set in $(Y, c_2)$is open in (X, c_1).*

Proposition 11. *[16] Let (X, c_1), (Y, c_2) and (Z, c_3)be closure spaces, let $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ be functions. Then:*

(i) If f and g are open, then so is gof.

(ii) If gof is open and f is a continuous surjection, then g is open.

(iii) If gof is open and g is a continuous injection, then f is open.

Proposition 12. *[16] Let (X, c_1) and (Y, c_2) be closure spaces and let $f : (X, c_1) \to (Y, c_2)$ be a function. If f is open, then for every $y \in Y$ and every closed subset F of (X, c_1) such that $f^{-1}(\{y\}) \subseteq F$, there exists a closed subset K of (Y, c_2) such that $y \in K$ and $f^{-1}(K) \subseteq F$.*

Proposition 13. *[11] Let (X, c_1) and (Y, c_2) be closure spaces. A function $f : (X, c_1) \to (Y, c_2)$ is called pre-continuous if the inverse image of every open set in (Y, c_2) is preopen in (X, c_1).*

Proposition 14. *[11] Let (X, c_1) and (Y, c_2) be closure spaces. A function $f : (X, c_1) \to (Y, c_2)$ is called pre-continuous if and only if the inverse image of every closed set in (Y, c_2) is preclosed in (X, c_1).*

Proposition 15. *[11] Let (X, c_1), (Y, c_2) and (Z, c_3)be closure spaces. If $f : (X, c_1) \to (Y, c_2)$ and $g : (Y, c_2) \to (Z, c_3)$ are pre-continuous and continuous respectively.Then $gof : (X, c_1) \to (Z, c_3)$ is pre-continuous.*

Proposition 16. *[11] Let (X, c_1), (Y, c_2) and (Z, c_3) be closure spaces. Let $f : (X, c_1) \to (Y, c_2)$ be a surjective open continuous function and $g : (Y, c_2) \to (Z, c_3)$ is a function such that $gof : (X, c_1) \to (Z, c_3)$ is pre-continuous. Then $g : (Y, c_2) \to (Z, c_3)$ is pre-continuous.*

Definition 8. *[16] A closure space (X, c) is said to be connected if ϕ and X are the only subsets of X which are both closed and open.*

Definition 9. *[16] A collection $\{G_a\}_{a\in J}$ of sets in a closure space(X, c) is called a cover of a subset $B$of X if $B \subseteq \bigcup_{\alpha\in J} G_a$ if holds, and an open cover if G_a is open for each $\alpha \in J$. Furthermore, a cover $\{G_a\}_{a\in J}$ of a subset B contains a finite subcover, if there exists a finite subset J_0 of J such that $B \subseteq \bigcup_{\alpha\in J_0} G_a$.*

Definition 10. *[16] A subset A of a closure space (X, c) is compact if every open cover of A contains a finite subcover.*

3. Pre-Open (Pre-Closed) Functions and Contra-Pre-Continuous

In the present section, we define and study some properties of pre-open (preclosed)functions and contra-pre-continuous.

Definition 11. *Let (X, c_1) and (Y, c_2) be two closure spaces. Let $f : (X, c_1) \to (X, c_2)$ be a function. Then f is pre-open (or preopen) if $f(G)$ is preopen in Y, for every open subset G of X.*

Definition 12. *Let (X, c_1) and (Y, c_2) be two closure spaces. Let $f : (X, c_1) \to (X, c_2)$ be a function. Then f is pre-closed (or preclosed) if $f(K)$ is preclosed in Y, for every closed subset K of X.*

Proposition 17. *Let $(X\ , c_1)$, (Y, c_2) and (Z, c_3) be closure spaces, let f: $(X, c_1) \to (Y, c_2)$ and g: $(Y, c_2) \to (Z, c_3)$ be functions. Then:*

(i) If f is open and g is preopen, then gof is preopen.

(ii) If gof is preopen and f is a continuous surjection, then g is preopen.

Proof.

(i) *Let G be an open subset of (X, c_1). Since f is open, $f(G)$ is open in (Y, c_2). Hence $g(f(G))$ is preopen in (Z, c_3). Thus, gof is preopen.*

(ii) *Let G be an open subset of (Y, c_2). Since f is a continuous function, $f^{-1}(G)$ is open in $(X\ , c_1)$. Since gof is preopen, $gof(f^{-1}(G)) =\ g(f(f^{-1}(G)))$ is preopen in (Z, c_3). But f is surjection, so that $gof(f^{-1}(G)) = g(G)$. Hence, $g(G)$ is preopen in (Z, c_3). Therefore, g is preopen.*

Proposition 18. *Let (X, c_1), (Y, c_2) and (Z, c_3) be closure spaces, let $f : (X, c_1) \rightarrow (Y, c_2)$ and $g : (Y, c_2) \rightarrow (Z, c_3)$ be functions. If gof is open and g is a pre-continuous injection, then f is preopen.*

Proof. Let G be an open subset of (X, c_1). Since gof is open, $g(f(G))$ is open in (Z, c_3). As g is pre-continuous, $g^{-1}(g(f\ (G)))$ is preopen in (Y, c_2). But g is injective, so that $g^{-1}(g(\ f\ (G))) = f\ (G)$ is preopen in (Y, c_2). Therefore, f is preopen.

Proposition 19. *Let (X, c_1) and (Y, c_2) be closure spaces. If $f : (X, c_1) \rightarrow (Y, c_2)$ is a bijection, then the following statements are equivalent:*

(i) *The inverse function $f^{-1} :\ (Y, c_2) \rightarrow (X, c_1)$ is pre-continuous.*

(ii) *f is a preopen function.*

(iii) *f is a preclosed function.*

Proof. Obvious

Definition 13. *A closure space (X, c) is said to be a T_p-space if every preopen set in (X, c) is open.*

The closure space in the following example is a T_p-space.

Example 1. *Let $X = \{1, 2, 3\}$ and defined a closure operator $c : P(X) \rightarrow P(X)$ by:*

$$c\,(A) = \begin{cases} A & if\ A \in \{\phi, \{1\}, \{2\}, \{3\}\} \\ X & Otherwise \end{cases}$$

Clearly (X, c) is a T_p-space, since every preopen set is open set

Proposition 20. *Let (X, c_1) and (Y, c_2) be closure spaces and (Y, c_2)be a T_p-space. If $f : (X, c_1) \rightarrow (Y, c_2)$ and $g : (Y, c_2) \rightarrow (Z, c_3)$ are pre-continuous, then gof is pre-continuous.*

Proof. Let H be open in (Z, c_3). Since g is pre-continuous, $g^{-1}\,(H)$ is preopen in (Y, c_2). But (Y, c_2) is a T_p-space, hence $g^{-1}(H)$is open in (Y, c_2). Thus $f^{-1}\left(g^{-1}\,(H)\right) = (gof)^{-1}(H)$ is preopen in (X, c_1). Therefore, gof is pre-continuous

Theorem 4. *Let (X,c) be a closure space, $\{(Y_\alpha,c_\alpha):\alpha\in J\}$ be a family of closure spaces and $f:(X,c)\to\prod_{\alpha\in J}(Y_\alpha,c_\alpha)$ be a function. If f is pre-continuous and π_α is a projection function, then $\pi_\alpha of$ is pre-continuous for each $\alpha\in J$.*

Proof. Assume that $f:(X,c)\to\prod_{\alpha\in J}(Y_\alpha,c_\alpha)$ is pre-continuous for all $\alpha\in J$. Since π_α is continuous, $\pi_\alpha of$ is pre-continuous for each $\alpha\in J$ by Proposition 15.

Definition 14. *Let (X,c_1) and (Y,c_2) be closure spaces and let $f:(X,c_1)\to(Y,c_2)$ be a function. Then f is contra-pre-continuous if the inverse image under f of every open subset of (Y,c_2) is preclosed in (X,c_1).*

Proposition 21. *Let (X,c_1) and (Y,c_2) be closure spaces and let $f:(X,c_1)\to(Y,c_2)$ be a function. Then f is contra-pre-continuous if and only if the inverse image under f of every closed subset of (Y,c_2) is preopen in (X,c_1).*

Proof. Let F be a closed subset in (Y,c_2). Then Y/F is open in(Y,c_2). Since f is contra-pre-continuous, $f^{-1}(Y/F)$is preclosed. But $f^{-1}(Y/F)=X/f^{-1}(F)$, thus $f^{-1}(F)$ is preopen in (X,c_1).
Conversely, let G be an open subset in (Y,c_2). Then Y/G is closed in (Y,c_2). Since the inverse image of each closed subset in $(Y,c_2)$is preopen in (X,c_1), $f^{-1}(Y/G)$ is preopen in (X,c_1). But $f^{-1}(Y/G)=X/f^{-1}(G)$, thus
$f^{-1}(G)$ is preclosed. Therefore, f is contra-pre-continuous.

Proposition 22. *Let (X,c_1), (Y,c_2) and (Z,c_3) be closure spaces, let $f:(X,c_1)\to(Y,c_2)$ and $g:(Y,c_2)\to(Z,c_3)$ be functions. If gof is contra-pre-continuous and g is a closed injection, then f is contra-pre-continuous.*

Proof. Let H be a closed subset of (Y,c_2). Since g is closed, $g(H)$ is closed in (Z,c_3). As gof is contra-pre-continuous, $(gof)^{-1}(g(H))=f^{-1}(g^{-1}(g(H)))$ is preopen in (X,c_1) by Proposition 21. But g is injective, hence
$f^{-1}\left(g^{-1}(g(H))\right)=f^{-1}(H)$. Therefore, f is contra-pre-continuous.

Proposition 23. *Let (X,c_1) and (Z,c_3)be closure spaces and (Y,c_2) be a T_p-space. If $f:(X,c_1)\to(Y,c_2)$ and $g:(Y,c_2)\to(Z,c_3)$ are contra-pre-continuous functions, then gof is pre-continuous.*

Proof. Let H be closed in (Z,c_3). Since g is contra-pre-continuous, $g^{-1}(H)$is preopen in (Y,c_2). But $(Y,c_2)$is a T_p-space, hence $g^{-1}(H)$is open in (Y,c_2). As f is contra-pre-continuous by Proposition 21, $f^{-1}\left(g^{-1}(H)\right)=(gof)^{-1}(H)$ is preclosed in (X,c_1). Therefore, gof is pre-continuous by Proposition 15.

The following statement is evident:

Proposition 24. *Let (X,c_1), (Y,c_2) and (Z,c_3)be closure spaces and let $f:(X,c_1)\to(Y,c_2)$ and $g:(Y,c_2)\to(Z,c_3)$ be functions. If f is contra-pre-continuous and g is continuous, then gof is contra-pre-continuous.*

As a direct consequence of Proposition 24, we have:

Proposition 25. *Let* (X, c)*be a closure space,* $\{(Y_\alpha, c_\alpha) : \alpha \in J\}$*be a family of closure spaces and* $f : (X, c) \to \prod_{\alpha\in J} (Y_\alpha, c_\alpha)$ *be a function. If* f *is contra-pre-continuous and* π_α *is a projection function, then* $\pi_\alpha of$ *is contra-pre-continuous for each* $\alpha \in J$.

4. Pre-Irresolute Functions

In view of the definition of Pre-irresolute Functions, we define Pre-irresolute Functions as:

Definition 15. *Let* (X, c_1) *and* (Y, c_2) *be closure spaces. A function* $f : (X, c_1) \to (Y, c_2)$ *is called pre-irresolute if* $f^{-1}(G)$ *is preopen in* (X, c_1) *for every preopen set* G *in* (Y, c_2).

Proposition 26. *Let* (X, c_1) *and* (Y, c_2)*be closure spaces and* $f : (X, c_1) \to (Y, c_2)$ *be a function. Then* f *is pre-irresolute if and only if* $f^{-1}(B)$ *is preclosed in* (X, c_1)*, whenever* B *is preclosed in* (Y, c_2).

Proof. Let B be a preclosed subset of (Y, c_2). Then Y/B is preopen in (Y, c_2). Since $f : (X, c_1) \to (Y, c_2)$ is pre-irresolute, $f^{-1}(Y/B)$ is preopen in (X, c_1). But $f^{-1}(Y/B) = X\ /f^{-1}(B)$, so that $f^{-1}(B)$is preclosed in (X, c_1).
Conversely, let A be a preopen subset in (Y, c_2). Then Y/A is preclosed in (Y, c_2). By the assumption, $f^{-1}(Y/A)$ is preclosed in (X, c_1). But $f^{-1}(Y/A) = X/f^{-1}(A)$. Thus $f^{-1}(A)$ is preopen in (X, c_1). Therefore, f is pre-irresolute.

Clearly, every pre-irresolute function is pre-continuous. The converse need not be true as can be seen from the following example.

Example 2. *Let* $X = \{1, 2, 3\} = Y$ *and define a closure operator* c_1 *on* X *by:*

$$c_1(A) = \begin{cases} A & if\ A \in \{\phi, \{3\}\} \\ \{1,2\} & if\ A = \{1\} \\ \{2,3\} & if\ A = \{2\} \\ X & \text{Otherwise} \end{cases}$$

And also define a closure operator c_2 *on* Y *by:*

$$c_2(A) = \begin{cases} A & if\ A = \phi \\ \{1,3\} & if\ A = \{1\} \\ \{2,3\} & if\ A = \{2\} \\ Y & \text{Otherwise} \end{cases}$$

Let $f : (X, c_1) \to (Y, c_2)$ *be the function defined by:*

$$f(x) = \begin{cases} 1 & if\ x = 1 \\ 2 & if\ x = 2 \\ 3 & if\ x = 3 \end{cases}$$

The family of all open sets with respect to c_1=$\{\phi, \{1, 2\}, X\}$

$$PO(X,\ c_1) = \{\phi, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

The family of all open sets with respect to c_2=$\{\phi, X\}$

$$PO(X,\ c_2) = \{\phi, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}.$$

Then f *is pre-continuous but not pre-irresolute because* $\{3\}$ *is preopen in* (Y, c_2) *but* $f^{-1}(\{3\}) = \ \{3\}$ *is not preopen in* (X, c_1).

Proposition 27. *Let* (X, c_1), (Y, c_2) *and* (Z, c_3) *be closure spaces. If* $f : (X, c_1) \to (Y, c_2)$ *is a pre-irresolute function and* $g : (Y, c_2) \to (Z, c_3)$ *is a pre-continuous function, then the composition* $gof : (X, c_1) \to (Z, c_3)$ *is pre-continuous.*

Proof. Let G be an open subset of (Z, c_3). Then $g^{-1}(G)$ is a preopen subset of (Y, c_2) as g is pre-continuous. Hence, $f^{-1}(g^{-1}(G))$ is preopen in (X, c_1) because f is pre-irresolute. Thus, gof is pre-continuous.

The following statements are evident:

Proposition 28. *Let* (X, c_1), (Y, c_2) *and* (Z, c_3) *be closure spaces. If* $f : (X, c_1) \to (Y, c_2)$ *and* $g : (Y, c_2) \to (Z, c_3)$ *are pre-irresolute, then* $gof : (X, c_1) \to (Z, c_3)$*is pre-irresolute.*

Proof. Obvious.

Proposition 29. *Let* (X, c_1) *and* (Z, c_3) *be closure spaces and* (Y, c_2) *be a* T_P*-space. If* f: $(X, c_1) \to (Y, c_2)$ *is a pre-continuous function and* g: $(Y, c_2) \to (Z, c_3)$ *is a pre-irresolute function, then the composition* gof: $(X, c_1) \to (Z, c_3)$*is pre-irresolute.*

Proof. Obvious.

Proposition 30. *Let* (X, c_1) *and* (Y, c_2) *be closure spaces and* $f : (X, c_1) \to (Y, c_2)$ *be a bijective function. If* f *and* f^{-1}*are continuous, then* f *and* f^{-1} *are pre-irresolute.*

Proof. Let B be a preopen subset of (Y, c_2). Then there exists an open set H in (Y, c_2) such that $B \subseteq H \subseteq c_2(B)$, hence $f^{-1}(B) \subseteq f^{-1}(H) \subseteq f^{-1}(c_2(B))$. Since f^{-1} is continuous, $f^{-1}(c_2(B)) \subseteq c_1 f^{-1}(H)$. But f is continuous. Thus $f^{-1}(H))$ is open in (X, c_1). Hence, $f^{-1}(B)$ is preopen in (X, c_1). Therefore, f is pre-irresolute. Let A be a preopen subset of (X, c_1). Then there exists an open set G in (X, c_1) such that $A \subseteq G \subseteq c_1(A)$. Hence, $f(A) \subseteq f(G) \subseteq f(c_1(A))$. As f is continuous, $f(c_1 A) \subseteq c_2 f(A)$. Since $f^{-1}$is continuous and $f(G)$ is the inverse image of G under f^{-1}, $f(G)$ is open in (Y, c_2). Thus, $f(A)$ is preopen in (Y, c_2). But $f(A)$ is the inverse image of A under f^{-1}, therefore $f^{-1}$is pre-irresolute.

5. P-Connectedness

As another application of preopen sets, a new kind of Connectedness, namely *p*-Connectedness, is introduced.

Definition 16. *A closure space $(X, c)$is said to be p-connected if ϕ and X are the only subsets of X which are both preopen and preclosed. Clearly, if (X, c) is p-connected, then (X, c) is connected. The converse is not true as can be seen from the following example.*

Example 3. *Let X=$\{1, 2, 3\}$ and define a closure operator c on X by:*

$$c(A) = \begin{cases} A & if\ A = \phi \\ \{1,3\} & if\ A = \{1\} \\ \{2,3\} & if\ A = \{2\} \\ X & \text{Otherwise} \end{cases}$$

The family of all open sets =$\{\phi, X\}$

$$PO(X,\ c) = \{\phi, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}.$$

We have (X, c) is connected, but it is not p-connected, because $\{1, 2\}$ is preopen set and preclosed set.

Proposition 31. *Let (X, c) be a closure space. Then the following statements are equivalent:*

(i) X is p-connected.

(ii) X cannot be expressed as the union of two disjoint, non-empty, preclosed subsets.

(iii) X cannot be expressed as the union of two disjoint, non-empty, preopen subsets.

Proof. Statement(i) implies statement(ii): Suppose that $X = U \cup V$, where U and V are non-empty, disjoint, preclosed subsets of (X, c). Then $U = X/V$ and U is preopen. Thus, U is a subset of X which is both preopen and preclosed but U is neither X nor ϕ. Hence, (X, c) is not *p*-connected.
Statement (ii) implies statement (iii): Suppose that $X = A \cup B$ where A and B are disjoint non-empty preopen subsets of (X, c). Then $X/A = B$ and $X/B = A$ are both complements of preopen sets and hence are preclosed. Thus, $X = A \cup B$ is an expression of X as the union of two disjoint, non-empty, preclosed subset of (X, c), which contradicts (ii).
Statement (iii) implies statement (i): Suppose that A is a subset of X which is both preopen and preclosed, but A is neither X nor ϕ. Then $X/\ A$ is also preclosed, preopen and non-empty. Thus, X= $(X/A) \cup A$ is the expression of X as the union of two disjoint, non-empty preopen subsets, which contradicts (iii).

The following statement is evident:

Proposition 32. *Let (X, c) be a T_p-space. Then (X, c) is connected if and only if (X, c) is p-connected.*

Proof. Obvious.

Proposition 33. *Let (X, c_1) be a closure space and let $Y = \{0, 1\}$ and c_2 be a closure operator on Y defined by:*
$c_2(A) = A$, for all subset A of Y. Then the following statements are equivalent:

(i) The only contra-pre-continuous functions $f : (X, c_1) \to (Y, c_2)$ are the constant functions.

(ii) A closure space (X, c_1) is p-connected.

Proof. Statement (i) implies statement (ii): Suppose that there is a non-empty subset A of (X, c_1) such that $A \neq X$ and A is both preopen and preclosed. Then X/A is both preopen and preclosed in (X, c_1). Define a function $f : (X, c_1) \to (Y, c_2)$ by:

$$f\ (x) = \begin{cases} 0 & x \in A \\ 1 & x \in X/A \end{cases}$$

Consequently, $f^{-1}(\phi) = \phi$, $f^{-1}(\{0\}) = A$, $f^{-1}(\{1\}) = X/A = B$ and $f^{-1}(Y\ ) = X$. Since there are only four closed subsets of (Y, c_2), namely ϕ , $\{0\}$, $\{1\}$ and Y, the inverse image under $f$of any closed subset in (Y, c_2) is preopen in (X, c_1) . Thus, f is contra-pre-continuous but non-constant, which a contradiction. Therefore, (X, c_1) is p-connected.
Statement (ii) implies statement (i): Suppose that a contra-pre-continuous function f : $(X, c_1) \to (Y, c_2)$ is non-constant, where the closure operator c_2 on Y is defined by:
$c(A) = A$, for all subset A of Y.
Then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are non-empty. Further, neither $f^{-1}(\{0\})$ nor $f^{-1}(\{1\})$ are equal to X. Since $\{0\}$ and $\{1\}$ are closed subset of (Y, c_2) and f is contra pre-continuous, $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are preopen subsets of (X, c_1). But $f^{-1}(\{0\}) = X/f^{-1}(\{1\})$. Hence $f^{-1}(\{0\})$ is both preclosed and preopen. Consequently, X is not p-connected, which a contradiction.

Proposition 34. *Let $(X, c_1)$and (Y, c_2)be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a function.*

(i) If f is a contra-pre-continuous function from (X, c_1) onto (Y, c_2) and (X, c_1) is p-connected, then (Y, c_2) is connected.

(ii) If f is a pre-irresolute function from (X, c_1)onto (Y, c_2) and $(X, c_1)$is p-connected, then (Y, c_2) is p-connected.

Proof. (ii). Suppose that (Y, c_2) is not p-connected. Then there is a non-empty subset A of Y, $A \neq Y$ such that A is both preopen and preclosed. Since f is pre-irresolute, the set $f^{-1}(A)$ is both preopen and preclosed. Since f is an onto function and A is a non-empty subset of Y with $A \neq Y$, it follows that $f^{-1}(A)$ is a non-empty subset of X with $f^{-1}(A) \neq X$. Hence, (X, c_1) is not p-connected which a contradiction. Therefore, (Y, c_2) is connected. The proof of (i)is similar to that of (ii).

6. P-Compactness

As another application of preopen sets, a new kind of compactness, namely p-compactness, is introduced.

Definition 17. *A collection* $\{G_\alpha\}_{\alpha\in J}$ *of preopen sets in a closure space* (X, c_1) *is called a preopen cover of a subset* B *of* X *if* $B \subseteq \bigcup_{\alpha\in J} G_\alpha$ *holds.*

Definition 18. *A subset* A *of a closure space* (X, c)*is p-compact if every preopen cover of* A *contains a finite subcover.*

The following statements are evident:

Proposition 35. *Let* (X, c_1) *be a closure space. If* X *is p-compact and* B *is a preclosed subset of* X*, then* B *is p-compact*

Proof. Let $\{G_\alpha\}_{\alpha\in J}$ be a collection of preopen subsets of X such that $B \subseteq \bigcup_{\alpha\in J} G_\alpha$. It follows that $X{=}\bigcup_{\alpha\in J} G_\alpha \cup (X/B)$. Since B is preclosed, X/B is preopen. Consequently, $\bigcup_{\alpha\in J} G_\alpha \cup (X/B)$ is a preopen cover of X. But X is p-compact, so $\bigcup_{\alpha\in J} G_\alpha {\cup} (X/B)$ contains a finite subcover, i.e. there exits a finite subset J_0 of J such that $X{=}\bigcup_{\alpha\in J_0} G_\alpha \cup (X/B)$. Since B and X/B are disjoint, $B \subseteq \bigcup_{\alpha\in J_0} G_\alpha$. Thus, any preopen cover $\{G_\alpha\}_{\alpha\in J}$ of B contains a finite subcover. Therefore, B is p-compact.

Proposition 36. *Let* (X, c_1) *and* (Y, c_2) *be closure spaces and* $f : (X, c_1) \to (Y, c_2)$ *be a function. If* f *is pre-irresolute and a subset* B *of* X *is p-compact, then the image* $f(B) \subseteq Y$ *is p-compact.*

Proof. Let$\{G_\alpha\}_{\alpha\in J}$ be a collection of preopen subsets of Y such that $f(B) \subseteq \bigcup_{\alpha\in J} G_\alpha$ it follow that $B \subseteq f^{-1}(f(B)) \subseteq f^{-1}\left\{\bigcup_{\alpha\in J} G_\alpha\right\} = \bigcup_{\alpha\in J} f^{-1}(G_\alpha)$, but f is pre-irresolute, so $\{f^{-1}(G_\alpha)\}_{\alpha\in J}$ is a preopen cover of B. Since B is p-compact, there exists a finite subset $J_0$of J such that $B \subseteq \bigcup_{\alpha\in J_0} f^{-1}(G_\alpha)$. It follow that $f(B) \subseteq \bigcup_{\alpha\in J_0} G_\alpha$. Thus, any preopen cover $\{G_\alpha\}_{\alpha\in J}$ of $f(B)$ contains a finite subcover. Therefore, $f(B)$ is p-compact.

Proposition 37. *Let* (X, c_1) *and* (Y, c_2) *be closure spaces and* $f : (X, c_1) \to (Y, c_2)$ *be a function. If* f *is a pre-continuous surjection and* X *is p-compact, then* Y *is compact.*

Proof. Let $\{G_\alpha\}_{\alpha\in J}$ be a collection of open subsets of Y such that $Y \subseteq \bigcup_{\alpha\in J} G_\alpha$. It follows that $X = f^{-1}(Y) \subseteq f^{-1}\left(\bigcup_{\alpha\in J} G_\alpha\right) = \bigcup_{\alpha\in J} f^{-1}(G_\alpha)$. But f is pre-continuous, so $\{f^{-1}(G_\alpha)\}_{\alpha\in J}$ is a preopen cover of X. Since X is p-compact, there exists a finite subset J_0 of J such that $X = \bigcup_{\alpha\in J_0} f^{-1}(G_\alpha)$. It follow that $Y = f\left(\bigcup_{\alpha\in J_0} f^{-1}(G_\alpha)\right) = f\left(f^{-1}(\bigcup_{\alpha\in J_0}(G_\alpha))\right)$. Since f is a surjection, $Y = \bigcup_{\alpha\in J_0} G_\alpha$ Thus, any open cover $\{G_\alpha\}_{\alpha\in J}$of Y contains a finite subcover. Therefore, Y is compact.

Proposition 38. *Let (X, c_1) and (Y, c_2) be closure spaces and $f : (X, c_1) \to (Y, c_2)$ be a function. If f is a preirresolute surjection and X is p-compact, then Y is p-compact.*

Proof. Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of preopen subsets of Y such that $Y \subseteq \bigcup_{\alpha\in J} A_\alpha$. It follow that $X = f^{-1}(Y) \subseteq f^{-1}\left(\bigcup_{\alpha\in J} A_\alpha\right) = \bigcup_{\alpha\in J} f^{-1}(A_\alpha)$. But f is pre-irresolute, hence $\{f^{-1}(A_\alpha)\}_{\alpha\in J}$ is a preopen cover of X. Since X is p-compact, there exists a finite subset J_0 of J such that $X = \bigcup_{\alpha\in J_0} f^{-1}(A_\alpha)$. Hence, $(Y) = f\left(\bigcup_{\alpha\in J} f^{-1}(A_\alpha)\right) = f(f^{-1}\left(\bigcup_{\alpha\in J_0}(A_\alpha)\right)$. Since f is a surjection $Y = \bigcup_{\alpha\in J_0} A_\alpha$ of Y contains a finite subcover. Therefore, Y is p-compact.

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