EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 4, 2018, 922-928
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On the Bessel operator $\odot_B^t$ related to the Bessel-Helmholtz and Bessel Klein-Gordon operator

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Abstract. In this paper, we study the Bessel operator $\odot_B^t$, iterated t-times and denote by

$$\odot_B^t = \left(\left(B_{a_1} + \cdots + B_{a_p} + m^2 \right)^2 - \left(B_{a_{p+1}} + \cdots + B_{a_{p+q}} \right)^2 \right)^t$$

where $p+q=n, B_{a_i} = \frac{\partial^2}{\partial a_i^2} + \frac{2v_i}{a_i}\frac{\partial}{\partial a_i}, 2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, a_i > 0$, $t \in \mathbb{Z}^+ \cup \{0\}$, $m \in \mathbb{R}^+ \cup \{0\}$ and $p+q=n$ is the dimension of $\mathbb{R}_n^+ = \{a : a = (a_1, \ldots, a_n), a_1 > 0, \ldots, a_n > 0\}$.

2010 Mathematics Subject Classifications: 46F10

Key Words and Phrases: Bessel Helmholtz operator, Bessel Klein-Gordon operator, Bessel diamond operator

1. Introduction

Yildirim, Sarikaya and Ozturk [7] have showed that $(-1)^t S_{2t}(a) * R_{2t}(a)$ is the solution of the $\diamondsuit_B^t \left((-1)^t S_{2t}(a) * R_{2t}(a)\right) = \delta$, where

$$\diamondsuit_B^t = \left(\left(\sum_{i=1}^{p} B_{a_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{a_j} \right)^2 \right)^t . \tag{1}$$

Here $p+q=n, B_{a_i} = \frac{\partial^2}{\partial a_i^2} + \frac{2v_i}{a_i}\frac{\partial}{\partial a_i}, 2v_i = 2\alpha_i + 1, \alpha_i > -\frac{1}{2}, a_i > 0, i = 1, 2, \ldots, n$, $t \in \mathbb{Z}^+ \cup \{0\}$ and n is the dimension of the $\mathbb{R}_n^+ = \{a : a = (a_1, \ldots, a_n), a_1 > 0, \ldots, a_n > 0\}$. Otherwise, the operator $\diamondsuit_B^k$ can also be expressed in the form $\diamondsuit_B^t = \Box_B^t \triangle_B^t = \triangle_B^t \Box_B^t$, where $\Box_B^t$ denote by

$$\Box_B^t = \left(B_{a_1} + B_{a_2} + \cdots + B_{a_p} - B_{a_{p+1}} - B_{a_{p+2}} - \cdots - B_{a_{p+q}} \right)^t , \tag{2}$$

DOI: https://doi.org/10.29020/nybg.ejpam.v11i4.3319

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and $\triangle_B^t$ denote by

$$\triangle_B^t = \left(B_{a_1} + B_{a_2} + \cdots + B_{a_n}\right)^t. \tag{3}$$

Now in this paper,

$$\odot_B^t = \left(\left(\sum_{i=1}^{p} B_{a_i} - \sum_{j=p+1}^{p+q} B_{a_j}\right) + m^2\right)^t \left(\sum_{i=1}^{n} B_{a_i} + m^2\right)^t, p+q=n. \tag{4}$$

Thus

$$\odot_B^t = \left(\Box_B + m^2\right)^t \left(\triangle_B + m^2\right)^t = \left(\triangle_B + m^2\right)^t \left(\Box_B + m^2\right)^t, \tag{5}$$

where

$$\left(\triangle_B + m^2\right)^t = \left(B_{a_1} + B_{a_2} + \cdots + B_{a_n} + m^2\right)^t \tag{6}$$

and

$$\left(\Box_B + m^2\right)^t = \left(B_{a_1} + B_{a_2} + \cdots + B_{a_p} - B_{a_{p+1}} - \cdots - B_{a_{p+q}} + m^2\right)^t \tag{7}$$

and from (4) with $q = 0$ and $t = 1$, we obtain

$$\odot_B = \left(\triangle_{B,p} + m^2\right)^2,$$

where

$$\left(\triangle_{B,p} + m^2\right) = \left(B_{a_1} + B_{a_2} + \cdots + B_{a_p} + m^2\right). \tag{8}$$

Moreover for $m = 0$, then we obtain Bessel diamond operator and defined by (1).

2. Preliminaries

Denoted by T_a^b the generalized shift operator acting according to the law [2]

$$T_a^b\varphi(a) = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi\left(\sqrt{a_1^2 + b_1^2 - 2a_1b_1\cos\theta_1}, \ldots, \sqrt{a_n^2 + b_n^2 - 2a_nb_n\cos\theta_n}\right)$$
$$\times \left(\Pi_{i=1}^n \sin^{2v_i-1}\right) d\theta_1 \ldots d\theta_n,$$

where $a, b \in \mathbb{R}_n^+, C_v^* = \Pi_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [2].

$$\frac{d^2U}{da^2} + \frac{2v}{a}\frac{dU}{da} = \frac{d^2U}{db^2} + \frac{2v}{b}\frac{dU}{db}$$
$$U(a,0) = f(a),$$
$$U_b(a,0) = 0.$$

The convolution operator determined by T_a^b is as follow:

$$(f * \varphi) = \int_{\mathbb{R}_n^+} f(b) T_a^b \varphi(a) \left(\Pi_{i=1}^n b_i^{2v_i}\right) db. \tag{9}$$

Convolution (9) is known as a B-convolution. We note the following properties for the B-convolution and the generalized shift operator:

(a) $T_a^b \cdot 1 = 1$.

(b) $T_a^0 \cdot f(a) = f(a)$.

(c) If $f(a), g(a) \in C(\mathbb{R}_n^+), g(a)$ is a bounded function, $a > 0$ and

$$\int_0^\infty |f(a)| \left(\Pi_{i=1}^n a_i^{2v_i}\right) da < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_a^b f(a) g(b) \left(\Pi_{i=1}^n b_i^{2v_i}\right) db = \int_{\mathbb{R}_n^+} f(b) T_a^b g(a) \left(\Pi_{i=1}^n b_i^{2v_i}\right) db.$$

(d) From (c), we have the following equality for $g(a) = 1$,

$$\int_{\mathbb{R}_n^+} T_a^b f(a) \left(\Pi_{i=1}^n b_i^{2v_i}\right) db = \int_{\mathbb{R}_n^+} f(b) \left(\Pi_{i=1}^n b_i^{2v_i}\right) db$$

(e) $(f * g)(a) = (g * f)(a)$.

Definition 1. *([6]) A distribution E is said to be a fundamental solution or an elementary solution for the differential operator L if*

$$LE = \delta$$

, where δ is Dirac-delta distribution. Let $L(D)$ be a differential operator with constant coefficients. We say that a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ is a fundamental solution or the elementary solution of the differential operator $L(D)$ if E satisfies $L(D)E = \delta$ in $\mathcal{D}'(\mathbb{R}^n)$.

Lemma 1. *If $\Box_B^t u(a) = \delta$ for $a \in \Gamma_+ = \{a \in \mathbb{R}^n : a_1 > 0, a_2 > 0, \ldots, a_n > 0$ and $U > 0\}$, where $\Box_B^t$ is the Bessel ultra-hyperbolic operator iterated t-times defined by (2). Then $u(a) = R_{2t}(a)$ is the unique elementary solution of the operator $\Box_B^t$ where*

$$R_{2t}(a) = \frac{U^{\left(\frac{2t-n-2|v|}{2}\right)}}{y_n(2t)} = \frac{\left(\sum_{i=1}^p a_i^2 - \sum_{j=p+1}^{p+q} a_j^2\right)^{\left(\frac{2t-n-2|v|}{2}\right)}}{y_n(2t)} \tag{10}$$

for

$$y_n(2t) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma\left(\frac{2+2t-n-2|v|}{2}\right) \Gamma\left(\frac{1-2t}{2}\right) \Gamma(2t)}{\Gamma\left(\frac{2+2t-p-2|v|}{2}\right) \Gamma\left(\frac{p-2t}{2}\right)}, |v| = \sum_{i=1}^n v_i. \tag{11}$$

Lemma 2. *Given the equation $\triangle_B^t u(a) = \delta$ for $a \in \mathbb{R}_n^+$, where $\triangle_B^t$ is the Laplace-Bessel operator iterated t-times defined by (3). Then $u(a) = (-1)^t S_{2t}(a)$ is an elementary solution of the operator $\triangle_B^t$ where*

$$S_{2t}(a) = \frac{|a|^{2t-n-2|v|}}{z_n(2t)} \tag{12}$$

for

$$z_n(2t) = \frac{\Pi_{i=1}^n 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \Gamma(t)}{2^{n+2|v|-4t} \Gamma\left(\frac{n+2|v|-2t}{2}\right)}.$$

Proof. The proofs of Lemma 1 and Lemma 2 are given in [7].

Lemma 3. *Given the equation* $\left(\Box_B + m^2\right)^t u(a) = \delta$ *for* $a \in \mathbb{R}_n^+$, *where* $\left(\Box_B + m^2\right)^t$ *is the Bessel Klein-Gordon operator iterated t-times defined by equation* (7), δ *is the Dirac-delta distribution,* $a \in \mathbb{R}_n^+$ *and* $t \in \mathbb{Z}^+ \cup \{0\}$, *then* $u(a) = F_{B,2t}(a,m)$, *where*

$$F_{B,2t}(a,m) = \sum_{r=0}^{\infty} \binom{-t}{r} m^{2r} R_{2t+2r}(a), \tag{13}$$

$R_{2t}(a)$ *is defined by* (10).

Proof. See [5].

Lemma 4. *Let* $\Box_B$ *be the Bessel ultra-hyperbolic operator, defined by* (2) *and* δ *is the Dirac delta distribution for* $a \in \mathbb{R}_n^+$, *then*

$$\left(\Box_B + m^2\right)^t \delta = F_{B,-2t}(a,m),$$

where $F_{B,-2t}(a,m)$ *is the inverse of* $F_{B,2t}(a,m)$ *in the convolution algebra.*

Proof. Let

$$D(a) = \left(\Box_B + m^2\right)^t \delta,$$

convolving both sides by $F_{B,2t}(a,m)$, then

$$\begin{aligned} F_{B,2t}(a,m) * D(a) &= F_{B,2t}(a,m) * \left(\Box_B + m^2\right)^t \delta \\ &= \left(\Box_B + m^2\right)^t F_{B,2t}(a,m) * \delta \\ &= \delta. \end{aligned} \tag{14}$$

Since $F_{B,2t}(a,m)$ is lie in S', where S' is a space of tempered distribution, choose $S' \subset D'_R$, where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $F_{B,2t}(a,m) \in D'_R$, it follow that $F_{B,2t}(a,m)$ is an element of convolution algebra, thus by ([4], p.150-151), we have that the equation (14) has a unique solution

$$D(a) = F_{B,-2t}(a,m) * \delta = F_{B,-2t}(a,m). \tag{15}$$

That complete the proof.

Lemma 5. *Given the equation* $\left(\triangle_B + m^2\right)^t u(a) = \delta$ *for* $a \in \mathbb{R}_n^+$, *where* $\left(\triangle_B + m^2\right)^t$ *is the Bessel-Helmholtz operator iterated t-times defined by equation* (6), δ *is the Dirac-delta distribution,* $a \in \mathbb{R}_n^+$ *and* $t \in \mathbb{Z}^+ \cup \{0\}$, *then* $u(a) = H_{B,2t}(a,m)$ *is an elementary solution of the operator* $\left(\triangle_B + m^2\right)^t$, *where*

$$H_{B,2t}(a,m) = \sum_{r=0}^{\infty} \binom{-t}{r} m^{2r} (-1)^{t+r} S_{2t+2r}(a), \tag{16}$$

$S_{2t}(a)$ *is defined by* (12).

Proof. See [9].

Lemma 6. *The convolution* $F_{B,2t}(a,m) * H_{B,2t}(a,m)$ *exists and is a tempered distribution where* $F_{B,2t}(a,m)$ *and* $H_{B,2t}(a,m)$ *be defined by* (13) *and* (16)*, respectively.*

Proof. From (13) and (16), we have

$$\begin{aligned} F_{B,2t}(a,m) * H_{B,2t}(a,m) &= \left(\sum_{r=0}^{\infty}\binom{-t}{r}m^{2r}R_{2t+2r}(a)\right) \\ &\quad * \left(\sum_{r=0}^{\infty}\binom{-t}{r}m^{2r}(-1)^{t+r}S_{2t+2r}(a)\right) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\binom{-t}{r}\binom{-t}{s}m^{2r+2s}(-1)^{t+r}S_{2t+2r}(a) * R_{2t+2s}(a). \end{aligned}$$

Since the function $S_{2t+2r}(a)$ and $R_{2t+2s}(a)$ are tempered distributions, see([3], p.302 and [1], p.97). From ([10], p.152), the convolution of functions

$$(-1)^{t+r}S_{2t+2r}(a) * R_{2t+2s}(a),$$

exists and is also a tempered distribution. Thus, $F_{B,2t}(a,m) * H_{B,2t}(a,m)$ exists and also is a tempered distribution.

3. Main results

Theorem 1. *Given the equation*

$$\odot_B^t T(a,m) = \delta \tag{17}$$

for $a \in \mathbb{R}_n^+$*, where* $\odot_B^t$ *is the Bessel operator iterated t-times defined by* (5)*, then*

$$T(a,m) = F_{B,2t}(a,m) * H_{B,2t}(a,m) \tag{18}$$

is an elementary solution of (17)*, where* $F_{B,2t}(a,m)$ *and* $H_{B,2t}(a,m)$ *are defined by* (13) *and* (16)*, respectively,* $t \in \mathbb{Z}^+ \cup \{0\}$ *and* $m \in \mathbb{R}^+ \cup \{0\}$*. Moreover, from* (18) *we obtain*

$$F_{B,-2t}(a,m) * T(a,m) = H_{B,2t}(a,m) \tag{19}$$

as an elementary solution of the Bessel-Helmholtz operator $(\triangle_B + m^2)^t$ *iterated t-times defined by* (6) *and in particular, for* $q = 0$ *then* $\odot_B^t$ *reduces to the Bessel-Helmhotz operator* $\left(\triangle_{B,p} + m^2\right)^{2t}$ *of p-dimension iterated 2t-times and is defined by* (8)*, where*

$$\triangle_{B,p} = B_{a_1} + B_{a_2} + \cdots + B_{a_p},$$

thus (17) *becomes*

$$\left(\triangle_{B,p} + m^2\right)^{2t} T(a,m) = \delta \tag{20}$$

we obtain

$$T(a,m) = H_{B,4t}(a,m) \tag{21}$$

is an elementary solution of (20).

Proof. From (5) and (17) we have

$$\odot_B^t T(a,m) = \left(\left(\Box_B + m^2\right)^t \left(\triangle_B + m^2\right)^t\right) T(a,m) = \delta.$$

Convolution of the above equation by $F_{B,2t}(a,m) * H_{B,2t}(a,m)$ and the properties of convolution with derivatives, we obtain

$$\begin{aligned} &\left(\Box_B + m^2\right)^t F_{B,2t}(a,m) * \left(\triangle_B + m^2\right)^t H_{B,2t}(a,m) * T(a,m) \\ &= F_{B,2t}(a,m) * H_{B,2t}(a,m) * \delta. \end{aligned} \tag{22}$$

Thus

$$T(a,m) = \delta * \delta * T(a,m) = F_{B,2t}(a,m) * H_{B,2t}(a,m) \tag{23}$$

by Lemma 3 and Lemma 5. Now from (18) and by Lemma 3 and Lemma 4 and properties of inverses in the convolution algebra, we obtain

$$F_{B,-2t}(a,m) * T(a,m) = \delta * H_{B,2t}(a,m) = H_{B,2t}(a,m)$$

is an elementary solution of the Bessel-Helmhotz operator iterated t-times defined by (6). In particular, for $q = 0$ then (17) becomes

$$\left(\triangle_{B,p} + m^2\right)^{2t} T(a,m) = \delta \tag{24}$$

where $\left(\triangle_{B,p} + m^2\right)^{2t}$ is the Bessel-Helmholtz operator of p-dimension, iterated $2t$-times and is defined by (8). By Lemma 5, we have

$$T(a,m) = H_{B,4t}(a,m) \tag{25}$$

is an elementary solution of (17). This completes the proof.

Corollary 1. *Given the equation*

$$\odot_B^t T(a,0) = \delta \tag{26}$$

for $a \in \mathbb{R}_n^+$, *where* $\odot_B^t$ *is the Bessel operator iterated* t*-times defined by* (5), *then*

$$T(a,0) = (-1)^t S_{2t}(a) * R_{2t}(a) \tag{27}$$

is an elementary solution of Bessel diamond operator, where $R_{2t}(a)$ *and* $S_{2t}(a)$ *are defined by* (10) *and* (12), *respectively.*

Proof. If $m = 0$, then we have $T(a,0) = (-1)^t S_{2t}(a) * R_{2t}(a)$ yielding the result,, see [7].

Acknowledgements

The author would like to thank the referee for his suggestions which enhanced the presentation of the paper. The author was supported by Sakon Nakhon Rajabhat University
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