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Aluthge Transformation Of quasi n-class Q and quasi n-class Q^* operators

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Abstract. In this paper, a new class of operators called quasi n-class Q and quasi n-class Q^* operators are introduced and studied some properties. Quasi n-class Q and quasi n-class Q^* composition and weighted composition operators on $L^2(\lambda)$ and $H^2(\beta)$ are characterized. Also we discuss quasi n-class Q and quasi n-class Q^* composite multiplication operator on L^2 space and Aluthge transformation of these class of operators are obtained.

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1. Introduction

Let H be an infinite dimensional separable Complex Hilbert space. Let $B(H)$ be the algebra of all bounded linear operators acting on H. Let T be an operator on H. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of (T^*T). If U is determined uniquely by the kernel condition $N(U) = N(|T|)$, then this decomposition is called the polar decomposition, which is one of the most important results in operator theory.

Recall that an operator T is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in H$ [7]. An operator T is said to be n-paranormal if $\|Tx\|^{n+1} \leq \|T^{n+1}x\|\|x\|^n$ for every $x \in H$ [16] and normaloid if $r(T) = \|T\|$, where $r(T)$ denotes the spectral radius of T. An operator T is of class Q [3], if $T^{*2}T^2 - 2T^*T + I \geq 0$. Equivalently $T \in Q$ if $\|Tx\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$. Class Q operators are introduced and studied by B. P. Duggal et al and it is well known that every class Q operator is not necessarily

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normaloid and every paranormal operator is a normaloid of class Q. ie $P \subseteq Q \cap N$, where P and N denotes the class of paranormal and normaloid operators respectively. Also he showed that the restiction of T to an invariant subspace is again a class Q operator.

Devika, Suresh [5], introduced a new class of operators which we call the quasi class Q operators and it is defined as, for $T \in B(H)$

$$\|T^2x\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2) \text{ for every } x \in H$$

In [8], A k-quasi class Q operator is defined as follows,
An operator T is of k-quasi class Q if

$$\|T^{k+1}x\|^2 \leq \frac{1}{2}(\|T^{k+2}x\|^2 + \|T^kx\|^2) \text{ for every } x \in H$$

and k is a natural number. D. Senthil Kumar, Prasad. T in [11], has defined the new class of operators which we call M-class Q operators. An operator T is of M class Q if for a fixed real number $M \geq 1$, T satisfies $M^2T^{*2}T^2 - 2T^*T + I \geq 0$ or equivalently $\|Tx\|^2 \leq \frac{1}{2}(M^2\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$ and a fixed real number $M \geq 1$.

In [15], Youngoh Yang and Cheoul Jun Kim introduced a class Q^* operators. If

$$T^{*2}T^2 - 2TT^* + I \geq 0,$$

then T is called class Q^* operators. He also proved that if T is class Q^* if and only if $\|T^*x\|^2 \leq \frac{1}{2}(\|T^2x\|^2 + \|x\|^2)$ for every $x \in H$. In [4], D. Senthil Kumar et. al. introduced quasi class Q^* operators. If

$$T^{*3}T^3 - 2(T^*T)^2 + T^*T \geq 0,$$

then T is called quasi class Q^* operators. He also proved that if T is quasi class Q^* if and only if $\|T^*Tx\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2)$ for every $x \in H$

In this paper, we study some properties of quasi n-class Q and quasi n-class Q^* operators and we derive conditions for composition and weighted composition operators to be quasi n-class Q and quasi n-class Q^*. Aluthge transformation of quasi n-class Q and quasi n-class Q^* operators are derived. Conditions for Composite multiplication operators to be quasi n-class Q and quasi n-class Q^* are also obtained. A characterization of quasi n-class Q and quasi n-class Q^* composition and weighted composition operators on weighted Hardy space are obtained.

2. Quasi n class Q Operators

In this section, we define new class of operators called quasi n-class Q, which is a super class of n-class Q operators and studied some properties of this class of operators.

Definition 1. *An operator $T \in B(H)$ is said to be quasi n-class Q if for every positive integer n and for every $x \in H$*

$$\|T^2x\|^2 \leq \frac{1}{1+n}(\|T^{2+n}x\|^2 + n\|Tx\|^2)$$

when $n = 1$ it is of quasi class Q operators.

Theorem 1. *An operator T is of quasi n-class Q if and only if $T^{*2+n}T^{2+n} - (1+n)T^{*2}T^2 + nT^*T \geq 0$ for every positive integer n.*

Proof. Since T is quasi n class Q operator, we have
$\|T^2x\|^2 \leq \frac{1}{1+n}(\|T^{2+n}x\|^2 + n\|Tx\|^2)$
$\Leftrightarrow \|T^{2+n}x\|^2 - (1+n)\|T^2x\|^2 + n\|Tx\|^2) \geq 0$
$\Leftrightarrow \langle T^{2+n}x, T^{2+n}x\rangle - (1+n)\langle T^2x, T^2x\rangle + n\langle Tx, Tx\rangle \geq 0$
$\Leftrightarrow T^{*2+n}T^{2+n} - (1+n)T^{*2}T^2 + nT^*T \geq 0$

For example: let $x = (x_1, x_2, ...) \in l^2$, Define $T : l^2 \to l^2$ by $T(x) = (0, x_1, x_2, ...)$, $T^*(x) = (x_2, x_3, ...)$. Then $T^{*2+n}T^{2+n} - (1+n)T^{*2}T^2 + nT^*T \geq 0$. ie T is quasi n-class Q operators.

From the definition of n class Q operator we can easily say that every n class Q operator is also an operator of quasi n class Q. Hence we have the following implication

$$\text{class } Q \subset n \text{ class } Q \subset \text{quasi } n \text{ class } Q.$$

Theorem 2. *Every quasi class Q operator is quasi n class Q operator.*

Proof. By using induction principle and simple calculation we get the result.

Corollary 1. *If $T \in B(H)$ is of quasi n-class Q then T is of quasi $n+1$-class Q operator*

Corollary 2. *If $T \in B(H)$ is of quasi n-class Q then αT is of quasi n-class Q operator for any complex number α.*

Theorem 3. *Let $T \in B(H)$. If $\lambda^{\frac{-1}{2}}T$ is an operator of quasi n class Q, then T is quasi n paranormal operator for all $\lambda > 0$.*

Proof. Since $\lambda^{\frac{-1}{2}}T$ is an operator of quasi n-class Q, then

$$(\lambda^{\frac{-1}{2}}T)^{*(2+n)}(\lambda^{\frac{-1}{2}}T)^{2+n} - (1+n)(\lambda^{\frac{-1}{2}}T)^{*2}(\lambda^{\frac{-1}{2}}T)^2 + n((\lambda^{\frac{-1}{2}}T)^*(\lambda^{\frac{-1}{2}}T)) \geq 0.$$

Hence $|\lambda^{\frac{-1}{2}}|^{2(2+n)}T^{*2+n}T^{2+n} - (1+n)|\lambda^{\frac{-1}{2}}|^4T^{*2}T^2 + n|\lambda^{\frac{-1}{2}}|^2T^*T \geq 0$. By multiplying $|\lambda|^{2+n}$ and let $|\lambda| = \mu$, then

$$T^{*2+n}T^{2+n} - (1+n)\mu^nT^{*2}T^2 + n\mu^{1+n}T^*T \geq 0.$$

Hence T is quasi n-paranormal operator for all $\lambda > 0$.

Theorem 4. *If quasi n-class Q operator T doubly commutes with an isometric operator S, then TS is an operator of quasi n-class Q.*

Proof. Since T is quasi n-class Q operator, then $T^*(T^{*1+n}T^{1+n}-(1+n)T^*T+nI)T \geq 0$. Suppose T doubly commutes with an isometric operator S, then $TS = ST, S^*T = TS^*$ and $S^*S = I$. Now let $A = TS$. So we get $A^*(A^{*(1+n)}A^{(1+n)} - (1+n)A^*A + nI)A \geq 0$. Therefore TS is a quasi n-class Q operator.

Theorem 5. *If a quasi n-class Q operator $T \in B(H)$ is unitarily equivalent to operator S, then S is an operator of quasi n-class Q.*

Proof. Assume T is unitarily equivalent to operator S. Then there exists an unitary operator U such that $S = U^*TU$ and T is quasi n-class Q operator, then $S^*(S^{*1+n}S^{1+n} - (1+n)S^*S + nI)S = (U^*TU)^*((U^*TU)^{*1+n}(U^*TU)^{1+n} - (1+n)(U^*TU)^*(U^*TU) + nI)(U^*TU) \geq 0$. Therefore S is quasi n-class Q operator.

Theorem 6. *Let $T \in B(H)$ be an invertible operator and N be an operator such that N commutes with T^*T. Then operator N is quasi n class Q if and only if operator TNT^{-1} is of quasi n class Q.*

Proof. Let N be quasi n class Q operator, then $N^*(N^{*1+n}N^{1+n}-(1+n)N^*N+nI)N \geq 0$. Since operator N commutes with operator T^*T, we have
$(TNT^{-1})^*((TNT^{-1})^{*1+n}(TNT^{-1})^{1+n} - (1+n)(TNT^{-1})^*(TNT^{-1}) + nI)(TNT^{-1}) = T(N^*(N^{*1+n}N^{1+n} - (1+n)N^*N + nI)N)T^{-1}$. Since N is quasi n class Q operator, then $T(N^*(N^{*1+n}N^{1+n} - (1+n)N^*N + nI)N)T^* \geq 0$. Which implies (TT^*) commutes with $T(N^*(N^{*1+n}N^{1+n} - (1+n)N^*N + nI)N)T^*$. Also $(TT^*)^{-1}$ is also commutes with $TN^*((N^{*1+n}N^{1+n} - (1+n)N^*N + nI)N)T^*$. Then $T(N^*(N^{*1+n}N^{1+n} - (1+n)N^*N + nI)N)T^{-1} \geq 0$. Hence TNT^{-1} is quasi n class Q operator. Conversely suppose that (TNT^{-1}) is quasi n class Q operator, then $N^*(N^{*1+n}N^{1+n} - (1+n)N^*N + nI)N \geq 0$.

Corollary 3. *Let S be quasi n class Q operator and A any positive operator such that $A^{-1} = A^*$. Then $T = A^{-1}SA$ is quasi n class Q operator.*

Theorem 7. *Let T be quasi n class Q operator. Then the tensor product $T \otimes I$ and $I \otimes T$ are both quasi n class Q operators.*

Proof. By the definition of quasi n class Q and tensor product and by the simple calculation we get the result.

Theorem 8. *If $T \in B(H)$ is a quasi n-class Q operator for a positive integer n, the range of T does not have dense range then T has the following 2×2 matrix representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T)} \oplus \ker T^*$, if and only if T_1 is also quasi n-class Q operator on $\overline{ran(T)}$ and $T_3 = 0$. Further more $\sigma(T) = \sigma(T_1) \cup \{0\}$ where $\sigma(T)$ denotes the spectrum of T.*

Proof. Let P be an orthogonal projection of H onto $\overline{ran(T)}$. Then $T_1 = TP = PTP$. By Theorem 1 we have that

$$P(T^{*2+n}T^{2+n} - (1+n)T^{*2}T^2 + nT^*T)P \geq 0$$

Hence $\begin{pmatrix} T_1^{*2+n}T_1^{2+n} - (1+n)T_1^{*2}T_1^2 + nT_1^*T_1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$

This implies $T_1^{*2+n}T_1^{2+n} - (1+n)T_1^{*2}T_1^2 + nT_1^*T_1 \geq 0$

So T_1 is quasi n-class Q operator on $\overline{ran(T)}$.

Also for any $x = (x_1, x_2) \in H$,

$$\begin{aligned} \langle T_3^k x_2, x_2 \rangle &= \langle T^k(I-P)x, (I-P)x \rangle \\ &= \langle (I-P)x, T^{*k}(I-P)x \rangle \\ &= 0 \end{aligned}$$

This implies $T_3 = 0$

Since $\sigma(T) \cup \tau = \sigma(T_1) \cup \sigma(T_3)$ where τ is the union of certain holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ [by corollary 7, [10]]. $\sigma(T_3) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T)} \oplus \ker T^*$ where T_1 is quasi n-class Q operator on $\overline{ran(T)}$ and $T_3 = 0$

$$T^{2+n} = \begin{pmatrix} T_1^{2+n} & \sum_{j=0}^{1+n} T_1^j T_2 T_3^{n+1-j} \\ 0 & T_3^{2+n} \end{pmatrix} \textit{ and } T^{*2+n} = \begin{pmatrix} T_1^{*2+n} & 0 \\ (\sum_{j=0}^{n+1} T_1^j T_2 T_3^{n+1-j})^* & T_3^{*2+n} \end{pmatrix}$$

Since $T_3 = 0$

$$T^{*2+n}T^{2+n} - (1+n)T^{*2}T^2 + nT^*T = \begin{pmatrix} T_1^{*2+n}T_1^{2+n} - (1+n)T_1^{*2}T_1^2 + nT_1^*T_1 & X \\ X^* & Y \end{pmatrix} \geq 0$$

Where $X = T_1^{*2+n}T_1^{1+n}T_2 - (1+n)T_1^{*2}T_1T_2 + nT_1^*T_2$

$Y = (T_2^*T_1^{*1+n}T_1^{2+n})(T_1^{*2+n}T_1^{1+n}T_2) - (1+n)T_2^*T_1^*T_1T_2 + nT_2^*T_2$

We know that, " If A is a matrix of the form $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$ if and only if $A \geq 0$, $C \geq 0$ and $B = A^{\frac{1}{2}}WC^{\frac{1}{2}}$ for some contraction W. Since T_1 is quasi n-class Q operator, then we have $T^{*2+n}T^{2+n} - (1+n)T^{*2}T^2 + nT^*T \geq 0$. Hence T is quasi n-class Q operator.

Theorem 9. *Let M be a closed T-invariant subspace of H. Then the restriction $T|_M$ of a quasi n-class Q operator T to M is quasi n-class Q operator.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = M \oplus M^\perp$. Since T is quasi n-class Q operator then by Theorem 8, we have $T|_M$ is also quasi n-class Q operator.

Theorem 10. *Let T be a regular quasi n class Q operator, then the approximate point spectrum lies in the disc*

$$\sigma_{ap}(T) \subseteq \{\lambda \in C : \frac{(1+n)^{\frac{1}{2}}}{\|T^{-2}\|(\|T^{1+n}\|^2+n)^{\frac{1}{2}}} \leq |\lambda| \leq \|T\|$$

Proof. Suppose T is regular quasi n class Q operator, then for every unit vector x in H, we have $\|x\|^2 \leq \|T^{-2}\|^2\|T^2x\|^2 \leq \frac{\|T^{-2}\|^2}{1+n}(\|T^{1+n}\|^2\|Tx\|^2 + n\|Tx\|^2)$.

$$\text{Hence } \|Tx\|^2 \geq \frac{(1+n)\|x\|^2}{\|T^{-2}\|^2(\|T^{1+n}\|^2+n)}.$$

Now assume that $\lambda \in \sigma_{ap}(T)$. Then there exists a sequence $\{ x_m\}$, $\|x_m\| = 1$ such that $\|(T-\lambda)x_m\| \to 0$ when $m \to \infty$ we have $\|Tx_m - \lambda x_m\| \geq \|Tx_m\| - |\lambda|\|x_m\| \geq \frac{(1+n)^{1/2}}{\|T^{-2}\|(\|T^{1+n}\|^2+n)^{1/2}} - |\lambda|$. Now, when $m \to \infty$, $|\lambda| \geq \frac{(1+n)^{1/2}}{\|T^{-2}\|(\|T^{1+n}\|^2+n)^{1/2}}$

3. Quasi n-class Q^* Operators

In this section we define operators of quasi n-class Q^* and consider some basic properties and examples.

Definition 2. *An operator T is said to be quasi n-class Q^* (quasi *- n-class Q)if*

$$\|T^*Tx\|^2 \leq \frac{1}{1+n}(\|T^{2+n}x\|^2 + n\|Tx\|^2)$$

*for every $x \in H$ and every positive integer n. When $n = 1$, it is of quasi class Q^* (quasi *-class Q)operator.*

Theorem 11. *For each positive integer n, T is of quasi n-class Q^* operator if and only if $T^*(T^{*1+n}T^{1+n} - (1+n)TT^* + nI)T \geq 0$.*

For example: let $x = (x_1, x_2, ...) \in l^2$, Define $T : l^2 \to l^2$ by $T(x) = (0, x_1, x_2, ...)$, $T^*(x) = (x_2, x_3, ...)$. Then $T^{*2+n}T^{2+n} - (1+n)(T^*T)^2 + nT^*T \geq 0$. ie T is quasi n-class Q^*.

From the definition of n-class Q^* operator, we can easily say that every operator of n-class Q^* is also an operator of quasi n-class Q^*. Hence we have the following implications

$$\text{class } Q^* \subset n\text{-class } Q^* \subset \text{quasi } n\text{-class } Q^*$$

Also every quasi class Q^* is quasi n-class Q^*, but the converse is not true and every quasi n-class Q^* is quasi $n+1$-class Q^* operator. Again, if $T \in B(H)$ is quasi n-class Q^* then αT is of quasi n-class Q^* operator for any complex number α.

Theorem 12. *Let $T \in B(H)$. If $\lambda^{\frac{-1}{2}}T$ is an operator of quasi n-class Q^*, then T is quasi *-n-paranormal operator for all $\lambda > 0$.*

Proof. Since $\lambda^{\frac{-1}{2}}T$ is an operator of quasi n-class Q^* then

$$(\lambda^{\frac{-1}{2}}T)^{*(2+n)}(\lambda^{\frac{-1}{2}}T)^{2+n} - (1+n)((\lambda^{\frac{-1}{2}}T)^*(\lambda^{\frac{-1}{2}}T))^2 + n(\lambda^{\frac{-1}{2}}T)^*(\lambda^{\frac{-1}{2}}T) \geq 0$$

By multiplying $|\lambda|^{2+n}$ and letting $|\lambda| = \mu$, we have T is quasi *-n-paranormal operator for all $\lambda > 0$.

Theorem 13. *If quasi n-class Q^* operator T doubly commutes with an isometric operator S, then TS is an operator of quasi n-class Q^*.*

Theorem 14. *If a quasi n-class Q^* operator $T \in B(H)$ is unitarily equivalent to operator S, then S is an operator of quasi n-class Q^*.*

Theorem 15. *Let $T \in B(H)$ be an invertible operator and N be an operator such that N commutes with T^*T. Then operator N is quasi n class Q^* if and only if operator TNT^{-1} is quasi of n class Q^*.*

Corollary 4. *Let S be quasi n class Q^* operator and A any positive operator such that $A^{-1} = A^*$. Then $T = A^{-1}SA$ is quasi n class Q^* operator.*

Theorem 16. *Let T be quasi n class Q^* operator. Then the tensor product $T \otimes I$ and $I \otimes T$ are both quasi n class Q^* operators.*

Theorem 17. *If $T \in B(H)$ is of quasi n class Q^* operator for any positive integer n, a non zero complex number $\lambda \in \sigma_p(T)$ and T is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \ker(T-\lambda) \oplus \overline{ran(T-\lambda)}^*$, then*
1. $T_2 = 0$ and
2. T_3 is quasi n-class Q^* operator.

Proof. Let $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \ker(T-\lambda) \oplus \overline{ran(T-\lambda)}^*$. Without the loss of generality assume that $\lambda = 1$, then by Theorem 11, $T^{*2+n}T^{2+n} - (1+n)(T^*T)^2 + nT^*T \geq 0$. Now,

$$T^{2+n} = \begin{pmatrix} 1 & \sum_{j=0}^{1+n} T_2 T_3^{n+1-j} \\ 0 & T_3^{2+n} \end{pmatrix} \text{and}$$

$$T^{*2+n} = \begin{pmatrix} 1 & 0 \\ (\sum_{j=0}^{2+n} T_2 T_3^{n+1-j})^* & T_3^{*2+n} \end{pmatrix}$$

$$T^{*2+n}T^{2+n} = \begin{pmatrix} 1 & \sum_{j=0}^{1+n} T_2 T_3^{1+n-j} \\ (\sum_{j=0}^{1+n} T_2 T_3^{1+n-j})^* & 2(\sum_{j=0}^{1+n} T_2 T_3^{1+n-j})^*(\sum_{j=0}^{1+n} T_2 T_3^{1+n-j}) + T_3^{2+n} T_3^{*1+n} \end{pmatrix}$$

So, $T^{*2+n}T^{2+n} - (1+n)(T^*T)^2 + nT^*T \geq 0$ *gives*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$$

Where $A = 1 - (1+n)(1 + T_2T_2^*) + n$, $B = \sum_{j=0}^{1+n} T_2T_3^{1+n-j} - (1+n)[T_2 + T_2(T_2^*T_2 + T_3^*T_3)] + nT_2$ and $C = (\sum_{j=0}^{1+n} T_2T_3^{1+n-j})^* \sum_{j=0}^{1+n} T_2T_3^{1+n-j} + T_3^{*2+n}T_3^{2+n} - (1+n)T_2^*T_2 + T_2^*T_2 + T_3^*T_3{}^2 + n(T_2^*T_2 + T_3^*T_3)$

But,we know that, " If A is a matrix of the form $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$ if and only if $A \geq 0$,

$C \geq 0$ and $B = A^{\frac{1}{2}} W C^{\frac{1}{2}}$ for some contraction W.

Therefore $1 + n - (1+n)(1 + T_2T_2^*) + n \geq 0$, which implies that $(1+n)(-T_2T_2^*) \geq 0$. This gives $T_2 = 0$, since n is a positive integer. Also T_3 is quasi n-class Q^* operator.

Corollary 5. *If $T \in B(H)$ is of quasi n class Q^* operator for a positive integer n, then T is of the form $T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix}$ on $H = \ker(T-\lambda) \oplus \overline{ran(T-\lambda)}^*$, where T_3 is quasi n-class Q^* operator and $\ker(T-\lambda) = \{0\}$.*

Theorem 18. *If $T \in B(H)$ is a quasi n-class Q^* operator for a positive integer n, T does not have dense range and T has the following 2×2 matrix representation*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } H = \overline{ran(T)} \oplus \ker T^*,$$

if and only if $T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI \geq 0$ and $T_3 = 0$.Further more $\sigma(T) = \sigma(T_1) \cup \{0\}$ where $\sigma(T)$ denotes the spectrum of T.

Proof. Let $T \in B(H)$ be quasi n class Q^* operator and P be an orthogonal projection onto $\overline{ran(T)}$. Then $T_1 = TP = PTP$. By Theorem 11 we have that

$$P(T^{*1+n}T^{1+n} - (1+n)(TT^*) + nI)P \geq 0$$

$$T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI \geq 0$$

Also for any $x = (x_1, x_2) \in H$,

$$\begin{aligned} \langle T_3x_2, x_2 \rangle &= \langle T(I-P)x, (I-P)x \rangle \\ &= \langle (I-P)x, T^*(I-P)x \rangle = 0 \end{aligned}$$

This implies $T_3 = 0$

Since $\sigma(T) \cup \tau = \sigma(T_1) \cup \sigma(T_3)$ where τ is the union of the holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cap \sigma(T_3)$ [by corollary 7, [10]]. $\sigma(T_3) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T)} \oplus \ker T^*$, $T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI \geq 0$ and $T_3 = 0$. Then we have

$$\begin{aligned} T^{*2+n}T^{2+n} &- (1+n)(T^*T)^2 + nT^*T \\ &= \begin{pmatrix} T_1^{*2+n} & 0 \\ (T_2^*T_1^{*1+n}) & 0 \end{pmatrix} \begin{pmatrix} T_1^{2+n} & T_1^{1+n}T_2 \\ 0 & 0 \end{pmatrix} \\ &- (1+n) \begin{pmatrix} (T_1^*T_1)^2 + T_1^*T_2T_2^*T_1 & T_1^*T_1T_1^*T_2 + T_1^*T_2T_2^*T_2 \\ T_2^*T_1T_1^*T_1 + T_2^*T_2T_2^*T_1 & T_2^*T_1T_1^*T_2 + (T_2^*T_2)^2 \end{pmatrix} \end{aligned}$$

$$+ n \begin{pmatrix} T_1^*T_1 & T_1^*T_2 \\ T_2^*T_1 & T_2^*T_2 \end{pmatrix}$$
$$= \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$$

Where $A = T_1^*(T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI)T_1$
$B = T_1^*(T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI)T_2$
$C = T_2^*(T_1^{*1+n}T_1^{1+n} - (1+n)(T_1T_1^* + T_2T_2^*) + nI)T_2$.
Hence T is quasi n class Q^* operator.

Theorem 19. *Let M be a closed T-invariant subspace of H. Then the restriction $T|_M$ of a quasi n class Q^* operator T to M is quasi n class Q^* operator.*

Proof. By Theorem 18, $T|_M$ is also quasi n class Q^* operator.

Theorem 20. *Let T be a regular quasi n class Q^* operator, then the approximate point spectrum lies in the disc*

$$\sigma_{ap}(T) \subseteq \{\lambda \in C : \frac{(1+n)^{(\frac{1}{2})}}{\|T^{-1}\|\|T^{*-1}\|(\|T^{1+n}\|^2+n)^{\frac{1}{2}}} \leq |\lambda| \leq \|T\|$$

Proof. Suppose T is regular quasi n class Q^* operator, then for every unit vector x in H, we have

$$\|Tx\|^2 \geq \frac{(1+n)\|x\|^2}{\|T^{-1}\|^2\|T^{*-1}\|^2(\|T^{1+n}\|^2+n)}$$

Now assume that $\lambda \in \sigma_{ap}(T)$. Then there exists a sequence $\{ x_m\}$, $\|x_m\| = 1$ such that $\|(T-\lambda)x_m\| \to 0$ when $m \to \infty$ we have

$$\begin{aligned} \|Tx_m - \lambda x_m\| &\geq \|Tx_m\| - |\lambda|\|x_m\| \\ &\geq \|T\| - |\lambda| \\ &\geq \frac{(1+n)^{1/2}}{\|T^{*-1}\|\|T^{-1}\|(\|T^{1+n}\|^2+n)^{1/2}} - |\lambda| \end{aligned}$$

Now when $m \to \infty$, $|\lambda| \geq \frac{(1+n)^{1/2}}{\|T^{-1}\|\|T^{*-1}\|(\|T^{1+n}\|^2+n)^{1/2}}$

4. Quasi n-class Q and Quasi n-class Q^* Composition Operators

Let $L^2(\lambda) = L^2(X, \Sigma, \lambda)$, where (X, Σ, λ) be a sigma-finite measure space. A bounded linear operator $C_Tf = f \circ T$ on $L^2(X, \Sigma, \lambda)$ is said to be a composition operator induced by T, a non-singular measurable transformation from X into itself, when the measure λT^{-1} is absolutely continuous with respect to the measure λ and the Radon-Nikodym derivative $\frac{d\lambda T^{-1}}{d\lambda} = f_0$ is essentially bounded. The Radon-Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by $f_0^{(k)}$, where T^k is obtained by composing T- k times. Every essentially bounded complex-valued measurable function f_0 induces the bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_0}f = f_0 f$ for every $f \in L^2(\lambda)$. Further $C_T^* C_T = M_{f_0}$, $C_T^{*2}C_T^2 = M_{f_0{(2)}}$ and $C_T^{*1+n}C^{1+n}2_T = M_{f_0{(1+n)}}$.

The following lemma due to Harrington and Whitley [9] is well known.

Lemma 1. *Let P denote the projection of L^2 on $\overline{R(C)}$*

(i) $C_T^* C_T f = f_0 f$ and $C_T C_T^* f = (f_0 \circ T)Pf$ for all $f \in L^2$, where P is the projection of L^2 onto $\overline{R(C)}$.

(ii) $\overline{R(C)} = \{f \in L^2 : f$ is $T^{-1}\Sigma$ measurable$\}$.

In this section quasi n-class Q and quasi n-class Q^* composition operator on L^2 space are characterized as follows.

Theorem 21. *Let $C_T \in B(L^2(\lambda))$. Then C_T is of quasi n-class Q if and only if $f_0^{(2+n)} - (1+n)f_0^{(2)} + nf_0 \geq 0$ a.e.*

Proof. Let $C_T \in B(L^2(\lambda))$ is of quasi n-class Q if and only if $C_T^{*2+n}C_T^{2+n} - (1 + n)C_T^{*2}C_T^2 + nC_T^* C_T \geq 0$. By Theorem 1

Thus $\langle (C_T^{*2+n}C_T^{2+n} - (1+n)C_T^{*2}C_T^2 + nC_T^* C_T)\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $C_T^* C_T = M_{f_0}$ and $C_T^{*2+n}C_T^{2+n} = M_{f_0^{(2+n)}}$, then $\langle (M_{f_0^{(2+n)}} - (1+n)M_{f_0^{(2)}} + nM_{f_0})\chi_E, \chi_E\rangle \geq 0$. Hence $\int_E (f_0^{(2+n)} - (1+n)f_0^{(2)} + nf_0)d\lambda \geq 0$ for every E in Σ. Hence C_T is of quasi n class Q if and only if $f_0^{(2+n)} - (1+n)f_0^{(2)} + nf_0 \geq 0$ a.e.

Example 1. *Let $X = N$, the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1) = 1$, $T(4p + q - 2) = p + 1$ for $q = 0, 1, 2, 3$ and $p \in N$. We have $f_0(p) = f_0^{(2)}(p) = ... = f_0^{(n)}(p) = 1$ for $p = 1$. $f_0(p) = 4, f_0^{(2)}(p) = 16, ... = f_0^{(2+n)}(p) = 4^{2+n}$ for $p \in N - \{1\}$. Since $f_0^{(2+n)}(p) - (1+n)f_0^{(2)}(p) + nf_0(p) \geq 0$ for every p, Hence C_T is of quasi n class Q operator.*

Theorem 22. *[14] If $C_T \in B(L^2(\lambda))$ has dense range then $f_0 = g_0 \circ T$ a.e.*

Corollary 6. *If C_T is quasi n-class Q with dense range on $L^2(\lambda)$ then $(g_0 \circ T)^{(2+n)} - (1+n)(g_0 \circ T)^{(2)} + n(g_0 \circ T) \geq 0$ a.e.*

Proof. By Theorem 21 and Theorem 22, we obtain the result.

Theorem 23. *Let $C_T \in B(L^2(\lambda))$. Then C_T^* is of quasi n-class Q operator if and only if $(f_0^{(2+n)} \circ T^{2+n})P_{2+n} - (1+n)(f_0^{(2)} \circ T^{(2)})P_2 + n(f_0 \circ T)P_1 \geq 0$ a.e, where $P_1, P_2, ..., P_{2+n}$ are the projections of L^2 onto $\overline{R(C)}, \overline{R(C^2)}, ..., \overline{R(C^{2+n})}$ respectively.*

Proof. Suppose $C_T \in B(L^2(\lambda))$ and C_T^* is of quasi n-class Q if and only if $C_T^{2+n}C_T^{*2+n} - (1+n)C_T^2C_T^{*2} + nC_TC_T^* \geq 0$. By Theorem 1. then $\langle(C_T^{2+n}C_T^{*2+n} - (1+n)C_T^2C_T^{*2} + nC_TC_T^*)f, f\rangle \geq 0$ for every $f \in L^2(\lambda)$. Since $\langle CC^*f, f\rangle = \langle(f_0 \circ T)P_1f, f\rangle$ By [9]. Hence $\langle(f_0^{(2+n)} \circ T^{2+n})P_{2+n}f, f\rangle - (1+n)\langle(f_0^{(2)} \circ T^2)P_2f, f\rangle + n\langle(f_0 \circ T)P_1f, f\rangle \geq 0$ for every $f \in L^2(\lambda)$. Hence $\langle((f_0^{(2+n)} \circ T^{2+n})P_{2+n} - (1+n)(f_0^{(2)} \circ T^2)P_2 + n(f_0 \circ T)P_1)f, f\rangle \geq 0 \Leftrightarrow (f_0^{(2+n)} \circ T^{2+n})P_{2+n} - (1+n)(f_0^{(2)} \circ T^2)P_2 + n(f_0 \circ T)P_1 \geq 0$ a.e.

Corollary 7. *Let $C_T \in B(L^2(\lambda))$ with dense range. Then C_T^* is of quasi n-class Q operator if and only if $(f_0^{(2+n)} \circ T^{2+n}) - (1+n)(f_0^{(2)} \circ T^2) + n(f_0 \circ T) \geq 0$ a.e.*

Theorem 24. *Let $C_T \in B(L^2(\lambda))$. Then C_T is of quasi n-class Q^* if and only if $f_0^{(2+n)} - (1+n)(f_0)^2P + nf_0 \geq 0$ a.e.*

Proof. Let $C_T \in B(L^2(\lambda))$ is of quasi n-class Q^* if and only if $C_T^{*2+n}C_T^{2+n} - (1+n)(C_T^*C_T)^2 + nC_T^*C_T \geq 0$.

Thus $\langle(C_T^{*2+n}C_T^{2+n} - (1+n)(C_T^*C_T)^2 + nC_T^*C_T)\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $C_T^*C_T = M_{f_0}$, $C_T^{*2+n}C_T^{2+n} = M_{f_0^{(2+n)}}$, then $\langle(M_{f_0^{(2+n)}} - (1+n)(M_{f_0})^2 + nM_{f_0})\chi_E, \chi_E\rangle \geq 0$. Hence $\int_E(f_0^{(2+n)} - (1+n)(f_0)^2 + nf_0)d\lambda \geq 0$ for every E in Σ. Hence C_T is quasi n class Q^* if and only if $f_0^{(2+n)} - (1+n)(f_0)^2 + nf_0 \geq 0$ a.e.

Example 2. *Let $X = N$, the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1) = T(2) = T(3) = 1$, $T(4p+q) = p+1$ for $q = 0, 1, 2, 3$ and $p \in N$. Since $f_0^{(2+n)} - (1+n)(f_0)^2 + nf_0 \geq 0$ for every p, Hence C_T is of quasi n class Q^* operator.*

Corollary 8. *If C_T is quasi n-class Q^* with dense range on $L^2(\lambda)$ if and only if $f_0^{(2+n)} - (1+n)(f_0)^2 + nf_0 \geq 0$ a.e.*

Theorem 25. *Let $C_T \in B(L^2(\lambda))$. Then C_T^* is quasi n-class Q^* if and only if $(f_0^{(2+n)} \circ T^{2+n})P_{2+n} - (1+n)(f_0 \circ T)^2P_1 + n(f_0 \circ T)P_1 \geq 0$ a.e. where P_i's are the projections of L^2 onto $\overline{R(C^i)}$, respectively.*

Proof. Let $C_T^* \in B(L^2(\lambda))$ is of quasi n-class Q^* if and only if $C_T^{2+n}C_T^{*2+n} - (1+n)(C_TC_T^*)^2 + nC_TC_T^* \geq 0$.

Thus $\langle(C_T^{2+n}C_T^{*2+n} - (1+n)(C_TC_T^*)^2 + nC_TC_T^*)\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $C_T^*C_T = M_{f_0}$, $C_T^{*1+n}C_T^{1+n} = M_{f_0^{(1+n)}}$ and $C_TC_T^* = (f_0 \circ T)P$, then $\int_E((f_0^{(2+n)} \circ T^{2+n})P_{2+n} - (1+n)(f_0 \circ T)^2P_1 + n(f_0 \circ T)P_1)d\lambda \geq 0$ for every E in Σ. Hence C_T is of quasi n class Q^* if and only if $(f_0^{(2+n)} \circ T^{2+n})P_{2+n} - (1+n)(f_0 \circ T)^2P_1 + n(f_0 \circ T)P_1 \geq 0$ a.e.

Corollary 9. *Let $C_T \in B(L^2(\lambda))$ with dense range. Then C_T^* is quasi n-class Q^* if and only if $(f_0^{(2+n)} \circ T^{2+n}) - (1+n)(f_0 \circ T)^2 + n(f_0 \circ T) \geq 0$ a.e.*

5. Quasi n-class Q and quasi n-class Q^* Weighted Composition Operators

A weighted composition operator is a linear transformation acting on the set of complex valued Σ measurable functions f of the form $W_T f = w(f \circ T)$, where w is a complex valued Σ measurable function. In the case that $w = 1$ a.e., we say that W_T is a composition operator. Let w_k denote $w(w \circ T)(w \circ T^2)...(w \circ T^{k-1})$ so that $W_T^k f = w_k(f \circ T)^k$ [13].

To examine the weighted composition operators efficiently, Alan Lambert [12], associated conditional expectation operator E with each transformation T as $E(\bullet|T^1\Sigma) = E(\bullet)$.

$E(f)$ is defined for each non-negative measurable function $f \in L^p(1 \leq p)$ and is uniquely determined by the conditions

(i) $E(f)$ is $T^{-1}\Sigma$ measurable and

(ii) If B is any $T^{-1}\Sigma$ measurable set for which $\int_B f d\lambda$ converges, then we have $\int_B f d\lambda = \int_B E(f) d\lambda$.

As an operator on L^p, E is the projection onto the closure range of C. E_n the identity on L^p if and only if $T^{-1}\sigma = \sigma$. Now we are ready to derive the characterization of quasi n-class Q and quasi n-class Q^* weighted composition operator as follows.

Theorem 26. *Let W_T be a weighted composition operator on $B(L^2(\lambda))$. Then W_T is of quasi n-class Q if and only if $(f_0^{(2+n)} E(w_{2+n}^2) \circ T^{-(2+n)}) - (1+n)(f_0^{(2)} E(w_2^2) \circ T^{-2}) + n f_0 E(w^2) \circ T^{-1} \geq 0$ a.e.*

Proof. Since $W_T \in B(L^2(\lambda))$ is of quasi n-class Q if and only if

$$W_T^{*2+n} W_T^{2+n} - (1+n) W_T^{*2} W_T^2 + n W_T^* W_T \geq 0. \text{ By Theorem 1.}$$

Thus $\langle (W_T^{*2+n} W_T^{2+n} - (1+n) W_T^{*2} W_T^2 + n W_T^* W_T)\chi_E, \chi_E \rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $W_T^* W_T = f_0 E(w^2) \circ T^{-1} f$, $W_T^k f = w_k (f \circ T)^k, W_T^{*k} f = f_0^{(k)} E(w_k f) \circ T^{-k}$ and $W_T^{*k} W_T^k f = f_0^{(k)} E(w_k^2) \circ T^{-k} f$. Hence $\langle (f_0^{(2+n)} E(w_{2+n}^2) \circ T^{-(2+n)} - (1+n) f_0^{(2)} E(w_2^2) \circ T^{-2} + n f_0 E(w^2) \circ T^{-1})\chi_E, \chi_E \rangle \geq 0$. Hence $\int_E (f_0^{(2+n)} E(w_{2+n}^2) \circ T^{-(2+n)} - (1+n) f_0^{(2)} E(w^2)^2 \circ T^{-2} + n f_0 E(w^2) \circ T^{-1}) d\lambda \geq 0$ for every E in Σ. Hence W is of quasi n class Q if and only if $f_0^{(2+n)} E(w_{2+n}^2) \circ T^{-(2+n)} - (1+n) f_0^{(2)} E(w_2^2) \circ T^{-2} + n f_0 E(w^2) \circ T^{-1} \geq 0$ a.e.

Corollary 10. *Let W_T be a weighted composition operator in $B(L^2(\lambda))$ and assume that $T^{-1}\Sigma = \Sigma$. Then W_T is of quasi n-class Q if and only if $f_0^{(2+n)} w_{2+n}^2 \circ T^{-(2+n)} - (1+n) f_0^{(2)} w_2^2 \circ T^{-2} + n f_0 (w^2) \circ T^{-1} \geq 0$ a.e.*

Theorem 27. *Let W_T be a weighted composition operator in $B(L^2(\lambda))$. Then W_T^* is of quasi n-class Q if and only if $w_{2+n}(f_0^{(2+n)} \circ T^{2+n}) E(w_{2+n}) - (1+n) w_2 (f_0^{(2)} \circ T^2) E(w_2) + n w (f_0 \circ T) E(w) \geq 0$ a.e.*

Proof. Since $W_T^* \in B(L^2(\lambda))$ is of quasi n-class Q if and only if

$W_T^{2+n}W_T^{*2+n} - (1+n)W_T^2W_T^{*2} + nW_TW_T^* \geq 0$. By Theorem 1

Thus $\langle (W_T^{2+n}W_T^{*2+n} - (1+n)W_T^2W_T^{*2} + nW_TW_T^*)\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $W_TW_T^*f = w(f_0 \circ T)E(wf)$, $W_T^kf = w_k(f\circ T)^k, W_T^{*k}f = f_0^kE(w_kf)\circ T^{-k}$ and $W_T^kW_T^{*k}f = w_k(f_0^{(k)}\circ T^{(k)})E(w_kf)$. Then $\langle (w_{2+n}(f_0^{(2+n)}\circ T^{2+n})E(w_{2+n}) - (1+n)w_2(f_0^{(2)}\circ T^2)E(w_2) + nw(f_0\circ T)E(w))\chi_E, \chi_E\rangle \geq 0$. which gives $\int_E (w_{2+n}(f_0^{(2+n)}\circ T^{2+n})E(w_{2+n}) - (1+n)w_2(f_0^{(2)}\circ T^2)E(w_2) + nw(f_0 \circ T)E(w))d\lambda \geq 0$ for every E in Σ. Hence W_T^* is quasi n class Q if and only if $(w_{2+n}(f_0^{(2+n)}\circ T^{2+n})E(w_{2+n}) - (1+n)w_2(f_0^{(2)}\circ T^2)E(w_2) + nw(f_0\circ T)E(w)) \geq 0$ a.e.

Corollary 11. *Let W_T be a weighted composition operator in $B(L^2(\lambda))$ and $T^{-1}(\Sigma) = \Sigma$. Then W_T^* is of n-class Q if and only if*
$w_{2+n}^2(f_0^{(2+n)}\circ T^{2+n}) - (1+n)w_2^2(f_0^{(2)}\circ T^2) + nw^2(f_0\circ T) \geq 0$ *a.e.*

Theorem 28. *Let W_T be a weighted composition operator on $B(L^2(\lambda))$. Then W_T is quasi n-class Q^* if and only if $(f_0^{(2+n)}E(w_{2+n}^2)\circ T^{-(2+n)}) - (1+n)w(f_0E(w^2)\circ T^{-1})^2 + nf_0E(w^2)\circ T^{-1} \geq 0$ a.e.*

Proof. Since $W_T \in B(L^2(\lambda))$ is quasi n-class Q^* if and only if

$$W_T^{*2+n}W_T^{2+n} - (1+n)(W_T^*W_T)^2 + nW_T^*W_T \geq 0 \text{ a.e.}$$

Thus $\langle (W_T^{*2+n}W_T^{2+n} - (1+n)(W_T^*W_T)^2 + nW_T^*W_T)\chi_E, \chi_E\rangle \geq 0$ for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$. Since $W_T^*W_T = f_0E(w^2)\circ T^{-1}f$, $W_T^kf = w_k(f\circ T)^k, W_T^{*k}f = f_0^{(k)}E(w_kf)\circ T^{-k}$ and $W_T^{*k}W_T^kf = f_0^{(k)}E(w_k^2)\circ T^{-k}f$. Then $\langle (f_0^{(2+n)}E(w_{2+n}^2)\circ T^{-(2+n)} - (1+n)(f_0E(w^2)\circ T^{-1})^2 + nf_0E(w^2)\circ T^{-1})\chi_E, \chi_E\rangle \geq 0$. Which implies $\int_E (f_0^{(2+n)}E(w_{2+n}^2)\circ T^{-(2+n)} - (1+n)(f_0E(w^2)\circ T^{-1})^2 + nf_0E(w^2)\circ T^{-1}d\lambda \geq 0$ for every E in Σ. Hence W is quasi n class Q^* if and only if $(f_0^{(2+n)}E(w_{2+n}^2)\circ T^{-(2+n)} - (1+n)(f_0E(w^2)\circ T^{-1})^2 + nf_0E(w^2)\circ T^{-1}) \geq 0$ a.e.

Corollary 12. *Let W_T be a weighted composition operator in $B(L^2(\lambda))$ and assume that $T^{-1}\Sigma = \Sigma$. Then W_T is quasi n-class Q^* if and only if*
$(f_0^{(2+n)}(w_{2+n}^2)\circ T^{-(2+n)} - (1+n)(f_0(w^2)\circ T^{-1})^2 + nf_0(w^2)\circ T^{-1}) \geq 0$ *a.e.*

Theorem 29. *Let W_T be a weighted composition operator on $B(L^2(\lambda))$. Then W_T^* is quasi n-class Q^* if and only if*
$w_{2+n}(f_0^{(2+n)}\circ T^{2+n})E(w_{2+n}) - (1+n)[w(f_0\circ T)E(w)]^2 + nw(f_0\circ T)E(w) \geq 0$ *a.e.*

Corollary 13. *If W_T is a weighted composition operator in $B(L^2(\lambda))$ and assume that $T^{-1}\Sigma = \Sigma$. Then W_T^* is quasi n-class Q^* if and only if*
$w_{2+n}^2(f_0^{(2+n)}\circ T^{2+n}) - (1+n)w^4(f_0\circ T)^2 + nw^2f_0\circ T \geq 0$ *a.e.*

The Aluthge transform of T is the operator $\tilde{T}$ given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was introduced in [1] by Aluthge. The idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties but it is closed to being a normal operator. More generally we may form the family of operators $T_r : 0 < r \leq 1$ where $T_r = |T|^r U|T|^{1-r}$ [2]. For a composition operator C, the polar decomposition is given by $C = U|C|$ where $|C|f = \sqrt{f_0}f$ and $Uf = \frac{1}{\sqrt{f_0 \circ T}} f \circ T$.

In [12] Lambert has given more general Aluthge transformation for composition operators as $C_r = |C|^r U|C|^{1-r}$ and $C_r f = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}} f \circ T$. That is C_r is weighted composition operator with weight $\pi = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}}$ where $0 < r < 1$. Since C_r is a weighted composition operator it is easy to show that $|C_r|f = \sqrt{f_0(E(\pi)^2 \circ T^{-1})}f$ and $|C_r^*|f = vE(vf)$ where $v = \frac{\pi\sqrt{f_0 \circ T}}{(E(\pi\sqrt{f_0 \circ T})^2)^{\frac{1}{4}}}$. Also we have
$C_r^k f = \pi_k (f \circ T)^k$
$C_r^{*k} f = f_0^k E(\pi_k f) \circ T^{-k}$
$C_r^{*k} C_r^k f = f_0^k E(\pi_k^2) \circ T^{-k} f$

Theorem 30. *Let $C_r \in B(L^2(\lambda))$. Then C_r is of quasi n-class Q if and only if $(f_0^{(2+n)} E(\pi_{2+n}^2) \circ T^{-(2+n)}) - (1+n)(f_0^{(2)} E(\pi_2^2) \circ T^{-2}) + n(f_0 E(\pi^2) \circ T^{-1}) \geq 0$ a.e.*

Proof. Since C_r is a weighted composition operator with weight $\pi = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}}$, it follows from Theorem 26 that C_r is quasi n-class Q if and only if $(f_0^{(2+n)} E(\pi_{2+n}^2) \circ T^{-(2+n)}) - (1+n)(f_0^{(2)} E(\pi_2^2) \circ T^{-2}) + n(f_0 E(\pi^2) \circ T^{-1}) \geq 0$ a.e.

Corollary 14. *If $T^{-1}\Sigma = \Sigma$ and $C_r \in B(L^2(\lambda))$. Then C_r is of quasi n-class Q if and only if $(f_0^{(2+n)} (\pi_{2+n}^2) \circ T^{-(2+n)}) - (1+n)(f_0^{(2)} (\pi_2^2) \circ T^{-2}) + n(f_0 (\pi^2) \circ T^{-1}) \geq 0$ a.e.*

Theorem 31. *Let $C_r \in B(L^2(\lambda))$. Then C_r^* is of quasi n-class Q if and only if $\pi_{2+n}(f_0^{(2+n)} \circ T^{2+n}) E(\pi_{2+n}) - (1+n)\pi_2(f_0^{(2)} \circ T^2) E(\pi_2) + n\pi(f_0 \circ T) E(\pi) \geq 0$ a.e.*

Proof. Since C_r^* is a weighted composition operator with weight $\pi = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}}$, it follows from Theorem 27 that C_r^* is of quasi n-class Q if and only if $\pi_{2+n}(f_0^{(2+n)} \circ T^{2+n}) E(\pi_{2+n}) - (1+n)\pi_2(f_0^{(2)} \circ T^2) E(\pi_2) + n\pi(f_0 \circ T) E(\pi) \geq 0$ a.e.

Corollary 15. *Let $C_r \in B(L^2(\lambda))$ and $T^{-1}\Sigma = \Sigma$. Then C_r^* is of quasi n-class Q if and only if $\pi_{2+n}^2 (f_0^{(2+n)} \circ T^{2+n}) - (1+n)\pi_2^2 (f_0^{(2)} \circ T^2) + n\pi^2 (f_0 \circ T) \geq 0$ a.e.*

Theorem 32. *Let $C_r \in B(L^2(\lambda))$. Then C_r is of quasi n-class Q^* if and only if $(f_0^{(2+n)} E(\pi_{1+n}^2) \circ T^{-(2+n)}) - (1+n)(f_0 E(\pi^2) \circ T^{-1})^2 + n(f_0 E(\pi^2) \circ T^{-1}) \geq 0$ a.e.*

Proof. Since C_r is a weighted composition operator with weight $\pi = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}}$, it follows from Theorem 46 that C_r is of quasi n-class Q^* if and only if $(f_0^{(2+n)} E(\pi_{1+n}^2) \circ T^{-(2+n)}) - (1+n)(f_0 E(\pi^2) \circ T^{-1})^2 + n(f_0 E(\pi^2) \circ T^{-1}) \geq 0$ a.e.

Corollary 16. *If* $T^{-1}\Sigma = \Sigma$ *and* $C_r \in B(L^2(\lambda))$. *Then* C_r *is of quasi n-class* Q^* *if and only if* $(f_0^{(2+n)}(\pi^2_{1+n}) \circ T^{-(2+n)}) - (1+n)(f_0(\pi^2) \circ T^{-1})^2 + n(f_0(\pi^2) \circ T^{-1}) \geq 0$ *a.e.*

Theorem 33. *Let* $C_r \in B(L^2(\lambda))$. *Then* C_r^* *is of quasi n-class* Q^* *if and only if* $\pi_{2+n}(f_0^{(2+n)} \circ T^{2+n})E(\pi_{2+n}) - (1+n)(\pi(f_0 \circ T)E(\pi))^2 + n\pi(f_0 \circ T)E(\pi) \geq 0$ *a.e.*

Corollary 17. *If* $T^{-1}\Sigma = \Sigma$ *and* $C_r^* \in B(L^2(\lambda))$ *is quasi n-class* Q^* *if and only if* $\pi^2_{2+n}(f_0^{(2+n)} \circ T^{2+n}) - (1+n)(\pi^2(f_0 \circ T))^2 + n\pi^2(f_0 \circ T) \geq 0$ *a.e.*

B. P Duggal [6] described the second Aluthge Transformation of T by $\tilde{T} = |\widehat{T}|^{\frac{1}{2}} V |\widehat{T}|^{\frac{1}{2}}$, where $\widehat{T} = V|\widehat{T}|$ is the polar decomposition of $\widehat{T}$. Now we consider $\tilde{C} = |C_r|^{\frac{1}{2}} V |C_r|^{\frac{1}{2}}$, where $C_r = V|C_r|$ is the polar decomposition of the generalized Aluthge transformation $C_r : 0 < r < 1$. We have $|C_r|f = \sqrt{J}f$, where $J = f_0E(\pi^2) \circ T^{-1}$.

$$\tilde{C} = |C_r|^{\frac{1}{2}} V |C_r|^{\frac{1}{2}} = \sqrt{J}^{\frac{1}{2}} V(\sqrt{J}^{\frac{1}{2}} f)| = \sqrt{J}^{\frac{1}{2}} \pi(\frac{\chi_{supJ}}{\sqrt{J}} J^{\frac{1}{4}} f) \circ T = J^{\frac{1}{4}} \pi((\frac{\chi_{supJ}}{J^{\frac{1}{4}}}) \circ T)(f \circ T).$$

We see then that $\tilde{C}$ is a weighted composition operator with weight $w' = J^{\frac{1}{4}} \pi((\frac{\chi_{supJ}}{J^{\frac{1}{4}}}) \circ T)$.

Theorem 34. *If* $\tilde{C}$ *is of quasi n-class Q if and only if* $f_0^{(2+n)} E(w'^2_{2+n}) \circ T^{-(2+n)} - (1+n)(f_0^{(2)} E(w'^2_2) \circ T^{-2}) + n(f_0 E(w'^2) \circ T^{-1}) \geq 0$ *a.e.*

Proof. Since $\tilde{C}$ is a weighted composition operator with weight $w' = J^{\frac{1}{4}} \pi((\frac{\chi_{supJ}}{J^{\frac{1}{4}}}) \circ T)$, then by Theorem 26 we obtain the result.

Corollary 18. *If* $T^{-1}\Sigma = \Sigma$ *and* $\tilde{C} \in B(L^2(\lambda))$ *is of quasi n-class Q if and only if* $f_0^{(2+n)}(w'^2_{2+n}) \circ T^{-(2+n)} - (1+n)(f_0^{(2)}(w'^2_2) \circ T^{-2}) + n(f_0(w'^2) \circ T^{-1}) \geq 0$ *a.e.*

Theorem 35. *Let* $\tilde{C} \in B(L^2(\lambda))$. *Then* $\tilde{C}^*$ *is of quasi n-class Q if and only if* $w'_{2+n}(f_0^{(2+n)} \circ T^{2+n})E(w'_{2+n}) - (1+n)w'_2(f_0^{(2)} \circ T^2)E(w'_2) + nw'(f_0 \circ T)E(w') \geq 0$ *a.e.*

Proof. Since $\tilde{C}^*$ is a weighted composition operator with weight $w' = J^{\frac{1}{4}} \pi((\frac{\chi_{supJ}}{J^{\frac{1}{4}}}) \circ T)$, then by from Theorem 27 we obtain the result.

Corollary 19. *Let* $\tilde{C} \in B(L^2(\lambda))$ *and* $T^{-1}\Sigma = \Sigma$. *Then* $\tilde{C}^*$ *is quasi n-class Q if and only if* $w'^2_{2+n}(f_0^{(2+n)} \circ T^{2+n}) - (1+n)w'^2_2(f_0^{(2)} \circ T^2) + nw'^2(f_0 \circ T) \geq 0$ *a.e.*

Theorem 36. *If* $\tilde{C}$ *is quasi n-class* Q^* *if and only if* $f_0^{(2+n)} E(w'^2_{2+n}) \circ T^{-(2+n)} - (1+n)(f_0 E(w'^2) \circ T^{-1})^2 + n(f_0 E(w'^2) \circ T^{-1}) \geq 0$ *a.e.*

Corollary 20. *If* $T^{-1}\Sigma = \Sigma$ *and* $\tilde{C} \in B(L^2(\lambda))$ *is of n-class* Q^* *if and only if* $f_0^{(2+n)}(w'^2_{2+n}) \circ T^{-(2+n)} - (1+n)(f_0(w'^2) \circ T^{-1})^2 + n(f_0(w'^2) \circ T^{-1}) \geq 0$ *a.e.*

Theorem 37. *Let* $\tilde{C} \in B(L^2(\lambda))$. *Then* $\tilde{C}^*$ *is of quasi n-class* Q^* *if and only if* $w'_{2+n}(f_0^{(2+n)} \circ T^{2+n})E(w'_{2+n}) - (1+n)(w'(f_0 \circ T)E(w'))^2 + n(w'(f_0 \circ T)E(w')) \geq 0$ *a.e.*

Corollary 21. *Let* $\tilde{C} \in B(L^2(\lambda))$ *and* $T^{-1}\Sigma = \Sigma$. *Then* $\tilde{C}^*$ *is of n-class* Q^* *if and only if* $w'^2_{2+n}(f_0^{(2+n)} \circ T^{2+n}) - (1+n)(w'^2(f_0 \circ T))^2 + n(w'^2(f_0 \circ T)) \geq 0$ *a.e.*

6. Quasi n-class Q and Quasi n-class Q^* Weighted Composition Operators on Weighted Hardy Space

The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{m=0}^{\infty} a_m z^m$ such that $\|f\|_\beta^2 = \sum_{m=0}^{\infty} |a_m|^2 \beta_m^2 < \infty$ is a Hilbert space of functions analytic in the unit disc with the inner product.

$\langle f, g\rangle_\beta = \sum_{m=0}^{\infty} a_m \overline{b_m} \beta_m^2$ for an analytic map f on the open unit disc D and $g(z) = \sum_{m=0}^{\infty} b_m z^m$.

Let $\phi : D \to D$ be an analytic self map of the unit disc and consider the corresponding composition operator C_ϕ acting on $H^2(\beta)$. That is $C_\phi(f) = f \circ \phi$ for $f \in H^2(\beta)$. The operators C_ϕ are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined in some special cases on the classical Hardy Space H^2 (the case when $\beta_n = 1$ for all n) and on a general space $H^2(\beta)$ if the function ϕ is analytic on some open set containing the closed unit disc having supremum norm strictly smaller than one. The weighted composition operator W_ϕ is defined as $(W_\phi f)(z) = \pi f(\phi(z))$ and $(W_\phi^* f)(z) = \bar{\pi} f(\phi(z))$ for every $z \in D$

Let w be a point on the open disc. Define $k_w^\beta(z) = \sum_{m=0}^{\infty} \frac{z^m \overline{w}^{-m}}{\beta_m^2}$. Then the function k_w^β is a point evaluation for $H^2(\beta)$.Then k_w^β is in $H^2(\beta)$ and $\|k_w^\beta\|^2 = \sum_{m=0}^{\infty} \frac{|w|^{2m}}{\beta_m^2}$. Thus $\|k_w\|$ is an increasing function of $|w|$. If $f(z) = \sum_{m=0}^{\infty} a_m z^m$ then $\langle f, k_w^\beta\rangle = f(w)$ for all f and k_w^β. Hence we can easily seen that $C_\phi^* k_w^\beta = k_{\phi(w)}^\beta$, $W_\phi^* k_w^\beta = \bar{\pi} k_{\phi(w)}^\beta$ and $k_0^\beta = 1$ (the function identically equal to 1).

Now we characterize quasi n class Q and quasi n-class Q^* composition operators on this space as follows.

Theorem 38. *If C_ϕ is of quasi n-class Q operator in $H^2(\beta)$, then* $C_\phi^{*2+n} C_\phi^{2+n} - (1 + n) C_\phi^{*2} C_\phi^2 + n C_\phi^* C_\phi \geq 0$

Proof. For $f \in H^2(\beta)$, consider

$$\begin{aligned}
\langle (C_\phi^{*2+n} C_\phi^{2+n} &- (1+n) C_\phi^{*2} C_\phi^2 + n C_\phi^* C_\phi) f, f\rangle \\
&= \langle C_\phi^{*2+n} C_\phi^{2+n} f, f\rangle - (1+n)\langle C_\phi^{*2} C_\phi^2 f, f\rangle + n\langle C_\phi^* C_\phi f, f\rangle \\
&= \langle C_\phi^{2+n} f, C_\phi^{2+n} f\rangle - (1+n)\langle C_\phi^2 f, C_\phi^2 f\rangle + n\langle C_\phi f, C_\phi f\rangle \\
&= \|C_\phi^{2+n} f\|^2 - (1+n)\|C_\phi^2 f\|^2 + n\|C_\phi f\|^2
\end{aligned}$$

Let $f = k_0^\beta$ then

$$\begin{aligned}
\langle (C_\phi^{*2+n} C_\phi^{2+n} - (1+n) C_\phi^{*2} C_\phi^2 + n C_\phi^* C_\phi) f, f\rangle& \\
&= \|C_\phi^{2+n} k_0^\beta\|^2 - (1+n)\|C_\phi^2 k_0^\beta\|^2 + n\|C_\phi k_0^\beta\|^2 \\
&= \|k_0^\beta\|^2 - (1+n)\|k_0^\beta\|^2 + n\|k_0^\beta\|^2 = 0
\end{aligned}$$

Hence C_ϕ is quasi n-class Q operator.

Theorem 39. *If* C_ϕ^* *is quasi* n*-class* Q *operator in* $H^2(\beta)$*, then* $C_\phi^{2+n}C_\phi^{*2+n} - (1+n)C_\phi^2 C_\phi^{*2} + nC_\phi C_\phi^* \geq 0$

Proof. For $f \in H^2(\beta)$, consider

$$\begin{aligned}\langle (C_\phi^{2+n}C_\phi^{*2+n} &- (1+n)C_\phi^2 C_\phi^{*2} + nC_\phi C_\phi^*)f, f\rangle \\ &= \langle C_\phi^{2+n}C_\phi^{*2+n}f, f\rangle - (1+n)\langle C_\phi^2 C_\phi^{*2}f, f\rangle + n\langle C_\phi C_\phi^* f, f\rangle \\ &= \langle C_\phi^{*2+n}f, C_\phi^{*2+n}f\rangle - (1+n)\langle C_\phi^{*2}f, C_\phi^{*2}f\rangle + n\langle C_\phi^* f, C_\phi^* f\rangle \\ &= \|C_\phi^{*2+n}f\|^2 - (1+n)\|C_\phi^{*2}f\|^2 + n\|C_\phi^* f\|^2\end{aligned}$$

Let $f = k_0^\beta$ and $\phi(0) = 0$ then we have

$$\begin{aligned}\langle (C_\phi^{2+n}C_\phi^{*2+n} &- (1+n)C_\phi^2 C_\phi^{*2} + nC_\phi C_\phi^*)f, f\rangle \\ &= \|C_\phi^{*2+n}k_0^\beta\|^2 - (1+n)\|C_\phi^{*2}k_0^\beta\|^2 + n\|C_\phi^* k_0^\beta\|^2 \\ &= \|k_0^\beta\|^2 - (1+n)\|k_0^\beta\|^2 + n\|k_0^\beta\|^2 = 0\end{aligned}$$

Hence C_ϕ^* is quasi n-class Q operator.

Theorem 40. *If* C_ϕ *is quasi* n*-class* Q^* *operator in* $H^2(\beta)$ *if and only if* $\|k_0^\beta\|^2 \geq \|k_{\phi(0)}^\beta\|^2$.

Theorem 41. *If* C_ϕ^* *is of quasi* n*-class* Q^* *operator in* $H^2(\beta)$ *if and only if* $\|k_{\phi^{2+n}(0)}^\beta\|^2 \geq \|k_{\phi(0)}^\beta\|^2$.

Next we characterize the quasi n class Q and quasi n class Q^* weighted composition operator on weighted hardy space as follows

Theorem 42. *An operator* $W_\phi \in H^2(\beta)$ *is quasi* n *class* Q *if and only if* $\|\pi^{2+n}\|^2 - (1+n)\|\pi^2\|^2 + n\|\pi\|^2 \geq 0$.

Proof. Since W_ϕ is quasi n class Q operator, then for any $f \in H^2(\beta)$, we have

$$\begin{aligned}&\langle (W_\phi^{*2+n}W_\phi^{2+n} - (1+n)W_\phi^{*2}W_\phi^2 + nW_\phi^* W_\phi)f, f\rangle \geq 0 \\ &\Leftrightarrow \|W_\phi^{2+n}f\|^2 - (1+n)\|W_\phi^2 f\|^2 + n\|W_\phi f\|^2 \geq 0 \\ &\Leftrightarrow \|W_\phi^{2+n}k_0^\beta\|^2 - (1+n)\|W_\phi^2 k_0^\beta\|^2 + n\|W_\phi k_0^\beta\|^2 \geq 0 \quad \text{when } f = k_0^\beta \\ &\Leftrightarrow \|\pi^{2+n}k_0^\beta\|^2 - (1+n)\|\pi^2 k_0^\beta\|^2 + n\|\pi k_0^\beta\|^2 \geq 0 \\ &\Leftrightarrow \|\pi^{2+n}\|^2\|k_0^\beta\|^2 - (1+n)\|\pi^2\|^2\|k_0^\beta\|^2 + n\|\pi\|^2\|k_0^\beta\|^2 \geq 0 \\ &\Leftrightarrow \|\pi^{2+n}\|^2 - (1+n)\|\pi^2\|^2 + n\|\pi\|^2 \geq 0\end{aligned}$$

Theorem 43. *An operator* $W_\phi^* \in H^2(\beta)$ *is quasi* n *class* Q *if and only if* $\|\overline{\pi}^{2+n}\|^2 - (1+n)\|\pi^2\|^2 + n\|\overline{\pi}\|^2 \geq 0$.

Proof. Since W_ϕ^* is quasi n class Q operator, we have

$$\langle (W_\phi^{2+n} W_\phi^{*2+n} - (1+n)(W_\phi W_\phi^*)^2 + nW_\phi W_\phi^*)f, f\rangle \geq 0 \text{ for any } f \in H^2(\beta)$$

$$\begin{aligned}
&\langle (W_\phi^{2+n} W_\phi^{*2+n} - (1+n)(W_\phi W_\phi^*)^2 + nW_\phi W_\phi^*)f, f\rangle \geq 0 \\
&\Leftrightarrow \|W_\phi^{*2+n} f\|^2 - (1+n)\|W_\phi^{*2} f\|^2 + n\|W_\phi^* f\|^2 \geq 0 \\
&\Leftrightarrow \|\overline{\pi}^{2+n} k_0^\beta\|^2 - (1+n)\|\pi^2 k_0^\beta\|^2 + n\|\overline{\pi} k_0^\beta\|^2 \geq 0 \quad for\ f = k_0^\beta and \phi(0) = 0 \\
&\Leftrightarrow \|\overline{\pi}^{2+n}\|^2 - (1+n)\|\pi^2\|^2 + n\|\overline{\pi}\|^2 \geq 0
\end{aligned}$$

Hence the theorem.

Theorem 44. *An operator W_ϕ is of quasi n-class Q^* operator in $H^2(\beta)$ if and only if $(\|\pi^{2+n}\|^2 + n\|\pi\|^2)\|k_0^\beta\|^2 \geq (1+n)|\pi|^2\|k_{\phi(0)}^\beta\|^2$.*

Theorem 45. *An operator $W_\phi^* \in H^2(\beta)$ is of quasi n-class Q^* if and only if $\|\overline{\pi}^{2+n}\|^2 \geq (1+n)|\pi|^2 - n\|\overline{\pi}\|^2$.*

7. quasi n-class Q and quasi n-class Q^* Composite Multiplication operator

As composite multiplication operator to a linear transformation acting on a set of complex value Σ measurable functions f of the form $M_{u,T}(f) = C_T M_u f = u \circ T f \circ T$ where u is a complex valued Σ measurable function. In the case $u = 1$ a.e, $M_{u,T}$ becomes a composition operator denoted by C_T.

Proposition 1. *Let the composite multiplication operator $M_{u,T}(f) \in B(L^2(\lambda))$ then for $u \geq 0$*

(i) $M_{u,T}^* M_{u,T} f = u^2 f_0 f$.

(ii) $M_{u,T} M_{u,T}^* f = (u^2 \circ T)(f_0 \circ T).E(f)$.

Since $M_{u,T}(f) = C_T M_u f = u \circ T f \circ T$ $M_{u,T}^n(f) = (C_T M_u)^n(f) = u^n (f \circ T)^2$ and $M_{u,T}^*(f) = u f_0.E(f) \circ T^{-1}$ $M_{u,T}^{*n}(f) = u f_0.E(u f_0) \circ T^{-(n-1)}.E(f) \circ T^{-n}$ where $E(uf_0) \circ T^{-(n-1)} = E(uf_0) \circ T^{-1}, E(uf_0) \circ T^{-2}, ..., E(uf_0) \circ T^{-(n-1)}$
$E(uf_0) \circ T^{n-1} = E(uf_0) \circ T^1, E(uf_0) \circ T^2, ..., E(uf_0) \circ T^{n-1}$

In this section, we study quasi n-class Q and quasi n-class Q^* composite multiplication operator as follows.

Theorem 46. *Let the composite multiplication operator $M_{u,T} \in B(L^2(\lambda))$. Then $M_{u,T}$ is quasi n class Q if and only if $uf_0.E(uf_0) \circ T^{-(1+n)}.E(u_{2+n}) \circ T^{-(2+n)} - (1+n)uf_0.E(uf_0) \circ T^{-1}.E(u_2) \circ T^{-2} + nu^2 f_0 \geq 0$. a.e.*

Proof. Suppose $M_{u,T}$ is quasi n class Q operator, then $M_{u,T}^{*2+n}M_{u,T}^{2+n}-(1+n)M_{u,T}^{*2}M_{u,T}^{2}+nM_{u,T}^{*}M_{u,T}\geq 0$ Then for any $f\in L^2(\lambda)$, we have

$$\langle (M_{u,T}^{*2+n}M_{u,T}^{2+n}-(1+n)M_{u,T}^{*2}M_{u,T}^{2}+nM_{u,T}^{*}M_{u,T})f,f\rangle\geq 0$$
$$\langle M_{u,T}^{*2+n}M_{u,T}^{2+n}f,f\rangle-(1+n)\langle M_{u,T}^{*2}M_{u,T}^{2}f,f\rangle+n\langle M_{u,T}^{*}M_{u,T}f,f\rangle\geq 0$$

Since $M_{u,T}^{*k}M_{u,T}^{k}=uf_0.E(uf_0)\circ T^{-(k-1)}.E(f)\circ T^{-n}$
$M_{u,T}^{k}M_{u,T}^{*k}=u_k.u\circ T^k.f_0\circ T^k.E(uf_0)\circ T^{k-1}.E(f)$
where $u_k=u\circ T.u\circ T^2...u\circ T^k$

$$\Leftrightarrow \langle (uf_0.E(uf_0)\circ T^{-(1+n)}.E(u_{2+n})\circ T^{-(2+n)})f,f\rangle-$$
$$(1+n)\langle (uf_0.E(uf_0)\circ T^{-1}.E(u_2)\circ T^{-2})f,f\rangle+n\langle (u^2f_0)f,f\rangle\geq 0$$
$$\Leftrightarrow \int_E (uf_0.E(uf_0)\circ T^{-(1+n)}.E(u_{2+n})\circ T^{-(2+n)}-(1+n)uf_0.E(uf_0)\circ T^{-1}$$
$$.E(u_2)\circ T^{-2}+nu^2f_0)d\lambda\geq 0$$
$$\Leftrightarrow uf_0.E(uf_0)\circ T^{-(1+n)}.E(u_{2+n})\circ T^{-(2+n)}-(1+n)uf_0.E(uf_0)\circ T^{-1}$$
$$.E(u_2)\circ T^{-2}+nu^2f_0\geq 0\ a.e$$

Corollary 22. *If the composition operator $C_T\in B(L^2(\lambda))$ then C_T is quasi n class Q if and only if $f_0.E(f_0)\circ T^{-(1+n)}-(1+n)f_0.E(f_0)\circ T^{-1}+nf_0\geq 0$. a.e.*

Proof. By putting $u=1$ in Theorem 46, we get the result.

Theorem 47. *Let the composite multiplication operator $M_{u,T}\in B(L^2(\lambda))$. Then $M_{u,T}^{*}$ is quasi n class Q if and only if $u_{2+n}.u\circ T^{2+n}.f_0\circ T^{2+n}.E(uf_0)\circ T^{1+n}.E(f)-(1+n)u_2(u\circ T^2)(f_0\circ T^2).E(uf_0)\circ TE(f)+n(u^2\circ T)(f_0\circ T).E(f)\geq 0$. a.e.*

Proof. Suppose $M_{u,T}^{*}$ is quasi n class Q operator, then $M_{u,T}^{2+n}M_{u,T}^{*2+n}-(1+n)M_{u,T}^{2}M_{u,T}^{*2}+nM_{u,T}M_{u,T}^{*}\geq 0$ Then for any $f\in L^2(\lambda)$, we have

$$\langle (M_{u,T}^{2+n}M_{u,T}^{*2+n}-(1+n)M_{u,T}^{2}M_{u,T}^{*2}+nM_{u,T}M_{u,T}^{*})f,f\rangle\geq 0$$
$$\Leftrightarrow \int_E (u_{2+n}.u\circ T^{2+n}.f_0\circ T^{2+n}.E(uf_0)\circ T^{1+n}.E(f)-(1+n)$$
$$u_2(u\circ T^2)(f_0\circ T^2).E(uf_0)\circ TE(f)+n(u^2\circ T)(f_0\circ T).E(f))d\lambda\geq 0$$
$$\Leftrightarrow u_{2+n}.u\circ T^{2+n}.f_0\circ T^{2+n}.E(uf_0)\circ T^{1+n}.E(f)-(1+n)$$
$$u_2(u\circ T^2)(f_0\circ T^2).E(uf_0)\circ TE(f)+n(u^2\circ T)(f_0\circ T).E(f)\geq 0\ a.e$$

Corollary 23. *If the composition operator $C_T\in B(L^2(\lambda))$ then C_T^{*} is quasi n class Q if and only if $f_0\circ T^{2+n}.E(f_0)\circ T^{1+n}.E(f)-(1+n)(f_0\circ T^2).E(f_0)\circ TE(f)+n(f_0\circ T).E(f)\geq 0$. a.e.*

Theorem 48. *Let the composite multiplication operator* $M_{u,T} \in B(L^2(\lambda))$. *Then* $M_{u,T}$ *is quasi n class* Q^* *if and only if* $uf_0.E(uf_0) \circ T^{-(1+n)}.E(u_{2+n}) \circ T^{-(2+n)} - (1+n)(u^4)(f_0^2) + n(u^2)(f_0) \geq 0$. *a.e.*

Corollary 24. *If the composition operator* $C_T \in B(L^2(\lambda))$. *Then* C_T *is quasi n class* Q^* *if and only if* $f_0.E(f_0) \circ T^{-(1+n)} - (1+n)(f_0^2) + n(f_0) \geq 0$. *a.e.*

Theorem 49. *Let the composite multiplication operator* $M_{u,T} \in B(L^2(\lambda))$. *Then* $M^*_{u,T}$ *is quasi n class* Q^* *if and only if* $u_{2+n}u \circ T^{2+n} f_0 \circ T^{2+n}.E(uf_0) \circ T^{1+n}.E(f) - (1+n)(u^2 f_0 \circ TE(f))^2 + n(u^2 f_0 \circ T)E(f) \geq 0$. *a.e.*

Corollary 25. *If the composition operator* $C_T \in B(L^2(\lambda))$. *Then* C^*_T *is quasi n class* Q^* *if and only if* $f_0 \circ T^{2+n}.E(f_0) \circ T^{1+n}.E(f) - (1+n)(f_0 \circ TE(f))^2 + n(f_0 \circ T)E(f) \geq 0$. *a.e.*

8. Aluthge transformation of quasi n-class Q and quasi n class Q^* operator

Let $T = U|T|$ be the polar decomposition of T. Then the Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was introduced by Aluthge[1]. An operator T is called w hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$ and he defined $\tilde{\tilde{T}} = |\tilde{T}|^{\frac{1}{2}}\tilde{T}|\tilde{T}|^{\frac{1}{2}}$ where $\tilde{T} = \tilde{U}|\tilde{T}|$. Also the adjoint of aluthge transformation is defined as $\tilde{T}^* = |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}$, *-Aluthge transformation is $\tilde{T}^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$ and adjoint of *-Aluthge transformation is given by $\tilde{T^*}^* = |T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}}$.

Theorem 50. *An operator* T *is quasi* n *class* Q *if and only if* $(1+n)T^*|T|^2T \leq T^*|T^{(1+n)}|^2T + nT^*T$ *for all* $x \in H$ *and for every positive integer* n.

Proof. Since T is quasi n class Q operator, then $T^*(T^{*1+n}T^{1+n} - (1+n)T^*T + nI)T \geq 0$ for every positive integer n. By simple calculation we get the result.

Theorem 51. *If* $T = U|T|$ *is the polar decomposition of quasi* n *class* Q *operator* T, *then* T *is quasi* n *class* Q *operator.*

Theorem 52. *If* T *is quasi* n *class* Q *operator* T *and* S *is unitary such that* $TS = ST$ *then* $A = TS$ *is also quasi* n *class* Q *operator.*

Theorem 53. *Let* $T = U|T|$ *be the polar decomposition of quasi* n *class* Q *operator* T, *where* U *is unitary if and only if* $\tilde{T}$ *is quasi* n *class* Q *operator.*

Proof. Suppose we assume that T is quasi n class Q operator and $T = U|T|$ is the polar decomposition of T, then we have that
$T^*(T^{*1+n}T^{1+n} - (1+n)T^*T + nI)T \geq 0$ for every positive integer n.

$$\Leftrightarrow (U|T|)^*((U|T|)^{*1+n}(U|T|)^{1+n} - (1+n)(U|T|)^*(U|T|) + nI)(U|T|) \geq 0.$$
$$\Leftrightarrow |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}(|T^{(1+n)}|^{\frac{1}{2}}U^{*(1+n)}|T^{*(1+n)}|U^{(1+n)}|T^{(1+n)}|^{\frac{1}{2}} - (1+n)$$

$$|T|^{\frac{1}{2}}U^*|T^*|U|T|^{\frac{1}{2}} + nI)|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \geq 0.$$
$$\Leftrightarrow \tilde{T}^*(\tilde{T}^{*1+n}\tilde{T}^{1+n} - (1+n)\tilde{T}^*\tilde{T} + nI)\tilde{T} \geq 0$$

for every positive integer n. Hence $\tilde{T}$ is quasi n class Q operator.

Theorem 54. *Let $T = U|T|$ be the polar decomposition of quasi n class Q operator T and U is unitary, then T is quasi n class Q if and only if $\tilde{T}^*$ is quasi n class Q operator.*

Proof. Suppose we assume that T is quasi n class Q operator and $T = U|T|$ is the polar decomposition of T, then we have that
$T^*(T^{*1+n}T^{1+n} - (1+n)T^*T + nI)T \geq 0$ for every positive integer n.

$$\Leftrightarrow (U|T|)^*[(U|T|)^{*1+n}(U|T|)^{1+n} - (1+n)(U|T|)^*(U|T|) + nI](U|T|) \geq 0.$$
$$\Leftrightarrow |T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}(|T^{(1+n)}|^{\frac{1}{2}}U^{(1+n)}|T^{*(1+n)}|U^{*(1+n)}|T^{(1+n)}|^{\frac{1}{2}} - (1+n)$$
$$|T|^{\frac{1}{2}}U|T^*|U^*|T|^{\frac{1}{2}} + nI)|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \geq 0.$$
$$\Leftrightarrow \tilde{T}^*(\tilde{T}^{1+n}\tilde{T}^{*1+n} - (1+n)\tilde{T}\tilde{T}^* + nI)\tilde{T} \geq 0$$

for every positive integer n. Hence $\tilde{T}^*$ is quasi n class Q operator.

Corollary 26. *If $\tilde{T}$ is quasi n class Q if and only if $\tilde{T}^*$ is quasi n class Q operator.*

Theorem 55. *Let $T = U|T|$ be the polar decomposition of quasi n class Q operator T and U is unitary, then T is quasi n class Q if and only if $\tilde{T^*}^*$ is quasi n class Q operator.*

Theorem 56. *Let $T = U|T|$ be the polar decomposition of quasi n class Q operator T and U is unitary, then $\tilde{T}^*$ is quasi n class Q if and only if $\tilde{T^*}^*$ is quasi n class Q operator.*

Theorem 57. *An operator T is quasi n class Q^* if and only if $(1+n)T^*|T^*|^2T \leq T^*|T^{(1+n)}|^2T + nT^*T$ for all $x \in H$ and for every positive integer n.*

Theorem 58. *If $T = U|T|$ is the polar decomposition of quasi n class Q^* operator T, then T is quasi n class Q^* operator.*

Theorem 59. *If T is quasi n class Q^* operator T and S is unitary such that $TS = ST$ then $A = TS$ is also quasi n class Q^* operator.*

Theorem 60. *If $\tilde{T}$ is quasi n class Q^* if and only if $\tilde{T}^*$ is quasi n class Q^* operator.*

Theorem 61. *If $\tilde{T}^*$ is quasi n class Q^* if and only if $\tilde{T^*}^*$ is quasi n class Q^* operator.*

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