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A Generalization of Integral Transform

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Abstract. In this paper, the generalization of integral transform (GIT) of the function $G\{f(t)\}$ is introduced for solving both the differential and interodifferential equations. This transform generalizes the integral transforms which use exponential functions as their kernels and the integral transform with polynomial function as a kernel. The generalized integral transform converts the differential equation into us domain (the transformed variables) and reconverts the result by its inverse operator. In particular, if $u = 1$, then the generalized integral transform coincides with the Laplace transform and this result can be written in another form as the polynomial integral transform.

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1. Introduction

The role of integral transforms has been increasingly recognized in the scientific world. Most of the problems in fluid mechanics, circuit design, heat transfer, population of species, are all continuous processes, which are modelled as either differential or integrodifferential equations with known initial or boundary conditions. Searching for the solutions of differential equations is a pertinent issue which concerns every scientist. Over the years, the researchers across the globe have come out with some methods for solving differential equations. The type and order of the differential equation determines the method that has to be selected to find the solution of the equation. The classical methods such as the separation of variables solves only separable differential equations. In similar vein, the use of integrating factor method solves a linear differential equation in an appropriate functional space. In addition, the differential equation has to be written in the standard form before searching for an appropriate integrating factor which transforms the differential equation into a separable form, from which the solution is obtained. This tedious and cumbersome method of searching for a solution of a differential equation is heartbreaking and undesirable.

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In order to overcome the shortcomings of the classical methods for solving differential equations Laplace transform was introduced which converts the differential equation into an algebraic equation, which is then transformed the result back by means of its inverse operator to obtain the desired solution in a suitable functional space, for example, see [1]. The authors in [2] extended the Laplace transform to double Laplace transform. In [3], the author established the convolution property of the double Laplace transform and applied this property to solve the homogeneous linear partial differential equations. The authors in [4] applied the double Laplace transform to solve nonhomogeneous linear partial differential equations. Currently, integral transform method has become reliable method for solving differential equations, integral equations and integro-differential equations. Sumudu transform was observed by [5]. The authors in [6], also introduced natural transform for solving differential equations. In [7], the authors extended the natural transform from one dimension to two dimensions and also, compared the double transform with Laplace and Sumudu transforms. The author in [8] introduced a polynomial integral transform PIT. In his work, the properties of the PIT were established and applied these properties for solving both the linear ordinary differential equations and linear partial differential equations. In [9], the authors established the relationships of the PIT among some integral transforms by equating the domain element that appears in the function $f(t)$ in PIT with other domain elements in $f(t)$ of other integral transforms.

It is not enough to introduce integral transform for solving differential equations without establishing its relationships with other integral transforms. By and large, the integral transformation of the operator of an unknown function into an algebraic equation in the same functional space is paramount and inevitably, the generalization of integral transform (GIT) is introduced in this paper. This method uses exponential function as its kernel, which converts a linear differential equations to an algebraic equation in terms of us and finally, transforms the resulting algebraic equation by means of its inverse operator to obtain the solution of the differential equation.

This paper is organized as follows. The section one contains the introduction; the GIT, its properties and applications are captured in section two and the last section of this paper contains the conclusion.

2. Main Results

In this section, the generalization of integral transform is introduced. We definitions of some integral transforms that will enable us to achieve our results.

Definition 1 (Polynomial Integral Transform). *Let $f(x)$ be a function defined for $x \geq 0$. Then the integral*

$$B(f(x)) = B(s) = \int_1^\infty f(\ln x)x^{-(s+1)}dx, \tag{1}$$

is the polynomial integral transform of $f(x)$ for all $x \in [1, \infty)$, see [8].

Definition 2 (Natural Transform)**.** *Let $f(t)$ be a function defined for $t \geq 1$. Then the integral*

$$N\{f(t)\} = \int_0^\infty f(ut)e^{-st}dt, \tag{2}$$

is the natural transform of $f(t)$ for $t \in [0, \infty)$. see [6]*.*

Definition 3 (Sumudu Transform)**.** *Let $f(t)$ be a function defined for $t \geq 0$. Then the integral*

$$S\{f(t)\} = \int_0^\infty f(ut)e^{-t}dt, \tag{3}$$

is the Sumudu transform of $f(t)$ for all $t \in [0, \infty)$. see [5]*.*

Definition 4 (Fourier Transform)**.** *Let $f(t)$ be a function defined for $t \geq 0$. Then the integral*

$$F\{f(t)\} = \int_0^\infty f(t)e^{-ist}dt \tag{4}$$

is the Fourier transform of $f(t)$ for all $t \in \mathbf{C}$, see [3]*.*

2.1. The Derivation of the Generalization of Integral Transform

In this subsection, the proof of the GIT is provided in theorem 1 below.

Theorem 1. *Let $f(t)$ be a function defined for $t \geq 0$. Then the integral*

$$G\{f(t)\} = G(s) = u\int_0^\infty f(ut)e^{-ust}dt,$$

is the generalized integral transform of $f(t)$ for all $t \in [0, \infty)$.

Proof: Equating the kernels in the polynomial integral transform and natural transform in equations (1) and (2), we obtain

$$\begin{aligned} x^{-(s+1)} &= e^{-st} \\ \Rightarrow \ln x^{-(s+1)} &= \ln e^{-st} \\ \Rightarrow -(s+1)\ln x &= -st \\ \Rightarrow x &= e^{\frac{st}{(s+1)}} \\ \Rightarrow dx &= \frac{s}{(s+1)}e^{\frac{st}{(s+1)}}dt \end{aligned} \tag{5}$$

Substituting equation (5) into equation (1) yields

$$\begin{aligned} G(f(t)) &= \frac{s}{(s+1)}\int_0^\infty f\Big(\frac{st}{s+1}\Big)e^{-\frac{s^2t}{(s+1)}}dt \\ \Rightarrow G(f(t)) &= u\int_0^\infty f(ut)e^{-sut}dt, \end{aligned} \tag{6}$$

where, $u = \frac{s}{(s+1)}$.

2.2. The Sufficient Condition for Existence of a Generalization of Integral Transform

Theorem 2. *Let $f(t)$ be a piecewise continuous function on the interval $[0,\infty)$ and of exponential order k for $t > T$, then $G\{f(t)\}$ exists for $s > k$.*

Proof: Setting $f(t) \leq M|e^{kt}|$. We see that:

$$
\begin{aligned}
G\{f(t)\} &\leq u\int_0^\infty e^{-ust}f(ut)dt \\
\|G\{f(t)\}\| &\leq \left\|u\int_0^\infty e^{-ust}f(ut)dt\right\| \\
\|G\{f(t)\}\| &\leq \|u\|\left\|\int_0^\infty e^{-ust}f(ut)dt\right\| \\
\|G\{f(t)\}\| &\leq \|u\|\int_0^\infty \left|e^{-ust}f(ut)\right| dt \\
\|G\{f(t)\}\| &\leq \|u\|\int_0^\infty e^{-ust}Me^{kut}dt, \qquad \forall t > T \\
\|G\{f(t)\}\| &\leq u\int_0^T Me^{kut}e^{-ust}dt + u\int_T^\infty Me^{kut}e^{-ust}dt \\
&= uM\int_0^T e^{-(s-k)ut}dt + uM\lim_{m\to\infty}\int_T^m e^{-(s-k)ut}dt \\
&= uM\left[-\frac{1}{(s-k)u}e^{-(s-k)ut}\right]_0^T + uM\lim_{m\to\infty}\left[-\frac{1}{(s-k)u}e^{-(s-k)ut}\right]_T^m \\
\|G\{f(t)\}\| &\leq uM\left[-\frac{1}{(s-k)u}e^{-(s-k)uT} + \frac{1}{(s-k)u}e^{-(s-k)u(0)}\right] \\
&+ uM\lim_{m\to\infty}\left[-\frac{1}{(s-k)u}e^{-(s-k)um} + \frac{1}{(s-k)u}e^{-(s-k)uT}\right] \\
\|G\{f(t)\}\| &\leq \frac{1}{(s-k)u}, \quad u > 0, s > k.
\end{aligned}
$$

We observed that $G\{f(t)\}$ exists for $u > 0, s > k$ for some $t > T$.

2.3. Properties of a Generalization of Integral Transform

Corollary 1. *(Linearity property of GIT) Let $f(t)$ and $g(t)$ be functions defined for $t \geq 0$. Then*

$$G\{\alpha f(t) + \beta g(t)\} = \alpha G\{f(t)\} + \beta G\{g(t)\},$$

where α and β are scalars.

Proof: We see from the definition of the generalized integral transform that:

$$\begin{aligned}
G\{\alpha f(t)+\beta g(t)\} &= u\int_0^\infty \Big(\alpha f(ut)+\beta g(ut)\Big)e^{-sut}dt \\
\Rightarrow G\{\alpha f(t)+\beta g(t)\} &= u\int_0^\infty \alpha f(ut)e^{-sut}dt + u\int_0^\infty \beta g(ut)e^{-sut}dt \\
\Rightarrow G\{\alpha f(t)+\beta g(t)\} &= \alpha u\int_0^\infty f(ut)e^{-sut}dt + \beta u\int_0^\infty g(ut)e^{-sut}dt \\
\Rightarrow G\{\alpha f(t)+\beta g(t)\} &= \alpha G\{f(t)\}+\beta G\{g(t)\}.
\end{aligned}$$

Corollary 2. *(First shifting theorem of a function using the GIT) If $G\{f(t)\} = G(s)$, then*

$$G\{e^{\alpha x}f(t)\} = G(s-\alpha), \quad for \;\; s>1.$$

Proof: Setting $G\{f(t)\} = G(s) = u\int_0^\infty f(ut)e^{-ust}dt$, then

$$\begin{aligned}
G\{e^{\alpha t}f(t)\} &= u\int_0^\infty e^{\alpha ut}f(ut)e^{-ust}dt \\
G\{e^{\alpha t}f(t)\} &= u\int_0^\infty f(ut)e^{-(s-\alpha)ut}dt \\
G\{e^{\alpha t}f(t)\} &= G(s-\alpha).
\end{aligned}$$

Corollary 3. *(Second shifting theorem of a function using the GIT).*
Setting

$$H_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}$$

be a unit step function. Then

$$G\{H_c f(t-c)\} = e^{-cs}G(s).$$

Proof: We see the definition of the GIT that:

$$\begin{aligned}
G\{H_c(t)f(t-c)\} &= u\int_0^\infty H_c(t)f(ut-c)e^{-ust}dt \\
\Rightarrow G\{H_c(t)f(t-c)\} &= u\lim_{m\to\infty}\int_0^m 1.f(ut-c)e^{-ust}dt \\
\Rightarrow G\{H_c(t)f(t-c)\} &= \lim_{m\to\infty}\int_0^{um-c} f(v)e^{-(s-c)v}dv \\
G\{H_c f(t-c)\} &= e^{-cs}G(s).
\end{aligned}$$

2.4. The GIT of Derivative

In this section, we derive an expression for the derivative of function which shall enable us to search for the solutions of ordinary differential equations. The result is in theorem 3 below.

Theorem 3. *If $y(t), y'(t), \ldots, y^{(n-1)}(t)$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then*

$$G\{f^{(n)}(t)\} = u^n s^n Y(s) - u^n s^{n-1} y(0) - u^{n-1} s^{n-2} y'(0) - \ldots - u y^{(n-1)}(0),$$

where $G(s) = G\{f(t)\}$.

Proof: By induction, we consider the function $y(t)$. We see from theorem (2.5) that:

$$G\{y(t)\} = Y(s) = u \int_0^\infty e^{-ust} y(ut) dt$$

Considering $G\left\{\frac{dy}{dt}\right\}$. Thus,

$$\begin{aligned} G\left\{\frac{dy}{dt}\right\} &= u \int_0^\infty e^{-ust} \frac{dy(ut)}{dt} dt \\ G\left\{\frac{dy}{dt}\right\} &= u \lim_{m \to \infty} \int_0^m e^{-ust} \frac{dy(ut)}{dt} dt. \end{aligned}$$

Using the integration by part yields

$$\begin{aligned} G\left\{\frac{dy}{dt}\right\} &= u \left\{ \lim_{m \to \infty} \left[e^{-ust} y(ut)\right]_0^m + us \lim_{m \to \infty} \int_0^m e^{-ust} y(ut) dt \right\} \\ G\left\{\frac{dy}{dt}\right\} &= u \left\{ \lim_{m \to \infty} \left[e^{-usm} y(um) - e^{-us(0)} y(0)\right] + us \lim_{m \to \infty} \int_0^m e^{-ust} y(ut) dt \right\} \\ G\left\{\frac{dy}{dt}\right\} &= u \left\{ \lim_{m \to \infty} e^{-usm} y(um) - \lim_{m \to \infty} y(0) + us \int_0^\infty e^{-ust} y(ut) dt \right\} \\ G\left\{\frac{dy}{dt}\right\} &= u \left\{ 0 - y(0) + us \int_0^\infty e^{-ust} y(ut) dt \right\} \\ G\left\{\frac{dy}{dt}\right\} &= -uy(0) + us \left[u \int_0^\infty e^{-ust} y(ut) dt \right] \end{aligned}$$

$$G\left\{\frac{dy}{dt}\right\} = usY(s) - uy(0)$$

Considering $G\left\{\frac{d^2y}{dt^2}\right\}$, we have:

$$\begin{aligned}
G\left\{\frac{d^2y}{dt^2}\right\} &= u\int_0^\infty e^{-ust}\frac{d^2y(ut)}{dt^2}dt \\
G\left\{\frac{d^2y}{dt^2}\right\} &= u\lim_{m\to\infty}\int_0^m e^{-ust}\frac{d^2y(ut)}{dt^2}dt
\end{aligned}$$

Using the integration by parts yields

$$\begin{aligned}
G\left\{\frac{d^2y}{dt^2}\right\} &= u\left\{\lim_{m\to\infty}\left[e^{-ust}\frac{dy(ut)}{dt}\right]_0^m + us\lim_{m\to\infty}\int_0^m e^{-ust}\frac{dy(ut)}{dt}dt\right\} \\
&= u\left\{\lim_{m\to\infty}\left[e^{-usm}\frac{dy(um)}{dt} - e^{-us\times 0}\frac{dy(0)}{dt}\right] + s\left(u\int_0^\infty e^{-ust}\frac{dy(ut)}{dt}dt\right)\right\} \\
&= -uy'(0) + us(usY(s) - uy(0)) \\
&= -uy'(0) + u^2s^2Y(s) - u^2sy(0) \\
G\left\{\frac{d^2y}{dt^2}\right\} &= u^2s^2Y(s) - u^2sy(0) - uy'(0)
\end{aligned}$$

Considering $G\left\{\frac{d^3y}{dt^3}\right\}$, we have:

$$\begin{aligned}
G\left\{\frac{d^3y}{dt^3}\right\} &= u\int_0^\infty e^{-ust}\frac{d^3y(ut)}{dt^3}dt \\
G\left\{\frac{d^3y}{dt^3}\right\} &= u\lim_{m\to\infty}\int_0^m e^{-ust}\frac{d^3y(ut)}{dt^3}dt.
\end{aligned}$$

Using the integration by parts, we obatin

$$\begin{aligned}
G\left\{\frac{d^3y}{dt^3}\right\} &= u\left\{\lim_{m\to\infty}\left[e^{-ust}\frac{d^2y(ut)}{dt^2}\right]_0^m + us\lim_{m\to\infty}\int_0^m e^{-ust}\frac{d^2y(ut)}{dt^2}dt\right\} \\
G\left\{\frac{d^3y}{dt^3}\right\} &= -uy''(0) + us(u^2s^2Y(s) - u^2y(0) - u^2sy'(0)) \\
G\left\{\frac{d^3y}{dt^3}\right\} &= -uy''(0) + u^3s^3Y(s) - u^3s^2y(0) - u^2sy'(0)
\end{aligned}$$

$$
\begin{aligned}
G\left\{\frac{d^3y}{dt^3}\right\} &= u^3s^3Y(s) - u^3s^2y(0) - u^2sy'(0) - uy''(0)\\
&\vdots\\
G\left\{\frac{d^ny}{dt^n}\right\} &= u^ns^nY(s) - u^ns^{n-1}y(0) - u^{n-1}s^{n-2}y'(0) - \ldots - uy^{(n-1)}(0)
\end{aligned}
$$

Corollary 4. *(Convolution Theorem for GIT) If $f(t)$ and $g(t)$ are piecewise continuous on $[0,\infty)$ and of exponential order, then*

$$
\begin{aligned}
G\{f*g\} &= G_1\{f(t)\}G_2\{g(t)\}\\
G\{f*g\} &= G_1(s)G_2(s).
\end{aligned}
$$

Proof: Setting $G_1\{f(t)\} = G_1(s) = u\int\limits_0^\infty e^{-us\varepsilon}f(u\varepsilon)d\varepsilon$ and $G_2\{g(t)\} = G_2(s) = u\int\limits_0^\infty e^{-us\tau}g(u\tau)d\tau$

$$
\begin{aligned}
G_1(s)G_2(s) &= \left(u\int_0^\infty e^{-us\varepsilon}f(u\varepsilon)d\varepsilon\right)\left(u\int_0^\infty e^{-us\tau}g(u\tau)d\tau\right)\\
G_1(s)G_2(s) &= u^2\int_0^\infty\int_0^\infty e^{-us(\varepsilon+\tau)}f(u\varepsilon)g(u\tau)d\varepsilon d\tau\\
G_1(s)G_2(s) &= u^2\int_0^\infty f(u\tau)d\tau\int_0^\infty g(u\varepsilon)e^{-us(\varepsilon+\tau)}d\varepsilon \qquad (7)
\end{aligned}
$$

$$
\left.\begin{aligned}
t &= \varepsilon+\tau\\
dt &= d\varepsilon
\end{aligned}\right\} \qquad (8)
$$

Substituting equation(8) into equation(7) yields

$$
\begin{aligned}
G_1(s)G_2(s) &= u^2\int_0^\infty f(u\tau)d\tau\int_0^\infty e^{-ust}g(u(t-\tau))dt\\
G_1(s)G_2(s) &= u^2\int_0^\infty e^{-ust}dt\int_0^t f(u\tau)g(u(t-\tau))d\tau\\
&= u\int_0^\infty e^{-ust}\left\{u\int_0^t f(u\tau)g(u(t-\tau))d\tau\right\}dt\\
G_1(s)G_2(s) &= G\{f*g\}.
\end{aligned}
$$

Corollary 5. *(Commutativity of two functions using the GIT) The convolution of functions $f(t)$ and $g(t)$ commute.*

Proof: By the convolution of $f(t)$ and $g(t)$, we have:

$$f(t) * g(t) = u\int_0^{\infty} f(u\tau)g(u(t-\tau))e^{-ust}d\tau, \tag{9}$$

$$\text{Setting}\quad \left.\begin{array}{rcl} t-\tau &=& v \\ d\tau &=& -dv. \end{array}\right\} \tag{10}$$

Substituting equation(10) into equation(9) yields

$$\begin{aligned} f(t) * g(t) &= -u\int_t^{-\infty} f(u(t-v))g(uv)e^{-us(t-v)}dv \\ f(t) * g(t) &= u\int_0^{\infty} g(uv)f(u(t-v))e^{-usv}dv \\ f(t) * g(t) &= g(t) * f(t) \end{aligned}$$

This completes the prove.

2.5. Illustration of the GIT

In this section, we show some of the areas where GIT can be used to solve problems.

Example 1.

$$\frac{dy}{dt} + 3y(t) = 2, \quad y(0) = 0$$

Taking the GIT of both sides,

$$\begin{aligned} G\left\{\frac{dy}{dt} + 3y(t)\right\} &= G\{2\} \\ G\left\{\frac{dy}{dt}\right\} + G\{3y(t)\} &= G\{2\} \\ usG(s) - uy(0) + 3G(s) &= \frac{2}{s} \\ usG(s) + 3G(s) &= \frac{2}{s} + uy(0) \\ G(s) &= \frac{2}{s(us+3)} \\ G(s) &= \frac{2}{3s} - \frac{2}{3(us+3)}. \end{aligned}$$

Taking the inverse GIT of both sides of the above equation yields

$$G^{-1}\{G(s)\} = G^{-1}\left\{\frac{2}{3s} - \frac{2}{3(us+3)}\right\}.$$

By the linearity of the inverse GIT, we get

$$\begin{aligned} y(t) &= G^{-1}\left\{\frac{2}{3s}\right\} - G^{-1}\left\{\frac{2}{3(us+3)}\right\} \\ y(t) &= \frac{2}{3}\left(1-e^{-3t}\right) \end{aligned}$$

Example 2.

$$y''(t)+16y(t)=\cos(4t), \qquad y(0)=0, \quad y'(0)=1.$$

Taking the GIT of both sides, we get

$$\begin{aligned} G\left\{y''+16y\right\} &= G\{\cos(4t)\} \\ G\left\{y''\right\}+16G\{y\} &= G\{\cos(4t)\} \\ u^2s^2Y(s)-u^2sy(0)-uy'(0)+16Y(s) &= \frac{s}{s^2+16} \\ Y(s) &= \frac{us^2+s+16}{(s^2+16)(u^2s^2+16)}. \end{aligned}$$

Taking the inverse generalized integral transform of both sides of the above equation yields

$$\begin{aligned} G^{-1}\{Y(s)\} &= G^{-1}\left\{\frac{us^2+s+16}{(s^2+16)(u^2s^2+16)}\right\} \\ y(t) &= \frac{1}{4}\sin(4t)+\frac{1}{8}t\sin(4t) \end{aligned}$$

Example 3.

$$E(t)=L\frac{di}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau \tag{1}$$

Setting $L=0.1h, \quad R=2, \ C=0.$ *If,* $E(t)=120t-120\phi(t-1), \ \ i(0)=0,$ *we obtain*

$$0.1\frac{di}{dt}+2i(t)+\frac{1}{0.1}\int_0^t i(\tau)d\tau = 120t-120\phi(t-1)$$

$$\Longrightarrow \ 0.1\frac{di}{dt}+2i(t)+10\int_0^t i(\tau)d\tau = 120t-120\phi(t-1) \tag{2}$$

Finding the GIT of the terms on both sides of the above equation yields

$$G\left\{0.1\frac{di}{dt}+2i(t)+\frac{1}{0.1}\int_0^t i(\tau)d\tau = 120t-120\phi(t-1)\right\} = G\{120t-120\phi(t-1)\}$$

By the linearly property of GIT, we get

$$0.1G\left\{\frac{di}{dt}\right\}+2G\{i(t)\}+10G\left\{\int\limits_0^t i(\tau)d\tau\right\}=120G\{t\}-120G\{t\phi(t-1)\}$$
$$\Longrightarrow\ 0.1(usI(s)-ui(0))+2I(s)+\frac{10}{s}I(s)=120\left[\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-s}\right]$$

Finding the inverse generalized integral transform on both sides of the above equation. Thus,

$$G^{-1}\{I(s)\}=G^{-1}\left\{\frac{120\left[\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-s}\right]}{\left[0.1us+2+\frac{10}{s}\right]}\right\}$$
$$i(t)=12[1-\phi(t-1)]-12[e^{-10t}-e^{-10(t-1)}\phi(t-1)]-$$
$$120te^{-10t}-1080(t-1)e^{-10(t-1}\phi(t-1).$$

$$\Longrightarrow\ i(t)=\begin{cases}12-12e^{-10t}-120te^{-10t}, & 0\le t\le 1\\ \\ -12e^{-10t}-12e^{-10(t-1)}-1080(t-1)e^{-10(t-1)}, & \forall t\ge 1\end{cases}$$

2.6. The Complex Generalized Integral Transform

In this section of the paper, we provide the complex form of the generalized integral transform. Thus, the variable $f(t)$ is transformed in a complex domain.

Corollary 6. *Let $f(t)$ be a function defined for $t\ge 0$. Then the integral*

$$G\{f(t)\}=G(is)=iu\int_0^\infty f(ut)e^{-iust}dt,$$

is the complex generalized integral transform of $f(t)$ for all $t\in\mathbf{C}^+$.

Proof: Equating the kernels in equations (1) and (8) yields

$$\begin{aligned}x^{-(s+1)} &= e^{-t}\\ \Rightarrow \ln x^{-(s+1)} &= \ln e^{-t}\\ \Rightarrow -(s+1)\ln x &= -t\\ \Rightarrow x &= e^{\frac{ist}{(s+1)}}\\ \Rightarrow dx &= \frac{is}{(s+1)}e^{\frac{ist}{(s+1)}}dt.\end{aligned}\tag{13}$$

Substituting equation (9) into equation (1) yields

$$G(f(t)) \;=\; \frac{is}{(s+1)}\int_0^\infty f\Big(\frac{ist}{s+1}\Big)e^{-\frac{is^2t}{(s+1)}}dt$$

$$G(f(t)) \quad = \quad iu\int_0^\infty f(iut)e^{-iust}dt, \tag{14}$$

where $u = \frac{s}{(s+1)}$.

3. Conclusion

We observed that, if $y(t), y'(t), \ldots, y^{(n-1)}(t)$ are continuous on $[0, \infty)$ and one of exponential order and $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$G\{f^n(t)\} = u^n s^n Y(s) - u^n s^{n-1} y(0) - u^{n-1} s^{n-2} y'(0) - \ldots - u y^{(n-1)}(0).$$

Undoubtedly, a generalization of an integral transform becomes the Laplace transform for $u = 1$. Thus,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt.$$

The GIT has some properties like the expression for derivative which is unique as compared to other integral transforms. The transformed variable u plays the role of dilating the transformed domain to obtain the desired result. On the other hand, the the transformed domain can be contracted by the transformed variable u.

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