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On Hoehnke ideal in ordered semigroups

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Abstract. For a proper subset A of an ordered semigroup S, we denote by $H_A(S)$ the subset of S defined by $H_A(S) := \{h \in S$ such that if $s \in S \backslash A$, then $s \notin (shS]\}$. We prove, among others, that if A is a right ideal of S and the set $H_A(S)$ is nonempty, then $H_A(S)$ is an ideal of S; in particular it is a semiprime ideal of S. Moreover, if A is an ideal of S, then $A \subseteq H_A(S)$. Finally, we prove that if A and I are right ideals of S, then $I \subseteq H_A(S)$ if and only if $s \notin (sI]$ for every $s \in S \backslash A$. We give some examples that illustrate our results. Our results generalize the Theorem 2.4 in Semigroup Forum 96 (2018), 523–535.

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1. Introduction and prerequisites

Regarding the prime ideals, Clifford uses the term “prime” while Petrich the term “completely prime”. Clifford uses the term “semiprime ideal” and Petrich the term “completely semiprime ideal (subset)”. For ordered semigroups I adopted the terminology due to Clifford; the authors in [2] the terminology by Petrich. Since in the present paper we refer to [2], for the sake of completeness, in particular for this paper, we will use the terms prime, semiprime, completely prime, completely semiprime. For an ordered semigroup S the zero of S, denoted by 0, is an element of S such that $0x = x0 = 0$ and $0 \leq x$ for every $x \in S$ [1, 3]. In an ordered semigroup, the order plays an essential role and a relation between the multiplication and the order is needed.

Let us first give the following definitions.

Definition 1.1. [5; Definition 2] Let S be an ordered semigroup. A subset A of S is called *completely prime* if for any subsets B, C of S such that $BC \subseteq A$, we have $B \subseteq A$ or $C \subseteq A$.
Equivalent Definition: if $x, y \in S$ such that $xy \in A$, then $x \in A$ of $y \in A$.

Definition 1.2. [5; Definition 3] Let S be an ordered semigroup. A subset A of S is called *prime* if for any ideals B, C of S such that $BC \subseteq A$, we have $B \subseteq A$ or $C \subseteq A$.

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Definition 1.3. [5; Definition 4] Let S be an ordered semigroup. A subset A of S is called *completely semiprime* if for any subset B of S such that $B^2 \subseteq A$, we have $B \subseteq A$. Equivalent Definition: for every $x \in S$ such that $x^2 \in A$, we have $x \in A$.

Definition 1.4. [4; Remark 4] Let S be an ordered semigroup. A subset A of S is called *semiprime* if for any ideal B of S such that $B^2 \subseteq A$, we have $B \subseteq A$.

Clearly, every completely prime (resp. completely semiprime) subset of S is a prime (resp. semiprime) subset.

The authors in [2] call a right ideal I of an ordered semigroup S prime if it is proper and for any right ideals A, B of S, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. They call a right ideal of S semiprime if it is proper and for any right ideal A of S, $A^2 \subseteq I$ implies $A \subseteq I$. Their definition regarding the completely semiprime right ideal is the same with the usual one in rings, semigroups, ordered semigroups (see, for example [8]) with the only difference that they defined it as "proper". Since every ordered semigroup (ring, semigroup) is itself a completely prime (prime) or completely semiprime (semiprime) subset of itself, these concepts have been never defined as "proper" in the existed bibliography (so in proofs, as well).

For an ordered semigroup $(S, \cdot, \leq)$ possessing a zero 0 and an identity e of $(S, \cdot)$ such that $e \neq 0$ it has been proved in [2] that if A is proper right ideal of S, then the set $H_A(S) := \{h \in S$ such that if $s \in S \backslash A$ then $s \notin (shS]\}$ is an interior ideal of S; since S possess an identity, the interior ideal $H_A(S)$ is also an ideal of S, but this should be emphasized in [2] since an interior ideal is not an ideal in general. Then the authors proved that if A is a proper right ideal of S then, for any right ideal I of S we have $I \subseteq H_A(S)$ if and only if $s \notin (sI]$ for all $s \in S \backslash A$ (property (2) in [2; Theorem 2.4]) and using this property they proved that if A is a proper ideal of S, then $A \subseteq H_A(S)$ (property (3) in [2; Theorem 2.4]) and that $H_A(S)$ is a semiprime ideal of S (property (1) in the same theorem) in the sense that $H_A(S)$ is a proper ideal of S and for any right ideal I of S such that $I^2 \subseteq H_A(S)$ we have $I \subseteq H_A(S)$. That is, it has been proved that (2) $\Rightarrow$ (1) and (3). The set $H_A(S)$ has been called "Hoehnke ideal" in [2].

In the present paper we define the $H_A(S)$ for any proper subset A of an ordered semigroup S. We keep the definitions of semiprime and prime subsets of ordered semigroups given above, and we first prove that the set $H_A(S)$ is a semiprime subset of S. Then we prove that, if A is a proper ideal of S, then A is a subset of $H_A(S)$. We show, among others, that if A is a right ideal of S and the set $H_A(S)$ is nonempty, then $H_A(S)$ is an ideal of S; and hence a semiprime ideal of S. Finally, we prove that if A and I are right ideals of an ordered semigroup S, then we have $I \subseteq H_A(S)$ if and only if $s \notin (sI]$ for every $s \in S \backslash A$. Unlike in [2], we have not used this last property to prove that $H_A(S)$ is semiprime and that $A \subseteq H_A(S)$; each of the three properties have been proved independently and the proof of Theorem 2.4 in [2] can be also given in the same way.

In [2] only the definition of semiprime right ideal is given, there is no the definition of semiprime ideal in the paper. However, according to the proof of Theorem 2.4, the authors call an ideal A of an ordered semigroup S semiprime if it is proper and for any right ideal I of S, $I^2 \subseteq A$ implies $I \subseteq A$. In case of ideals, this definition is equivalent to our Definition 1.4. So the results of the present paper generalize the Theorem 2.4 in [2]. On this occasion some information concerning the associate prime ideal has been also given. We give some examples that illustrate our results.

2. Main results

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *right* (resp. *left*) *ideal* of S if (1) $AS \subseteq A$ (resp. $SA \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$ [4, 5]. For a subset A of an ordered semigroup $(S, \cdot, \leq)$, we denote by $(A]$ the subset of S defined by $(A] := \{t \in S \mid t \leq a \text{ for some } a \in A\}$ [4, 5]. If S is a right ideal, left ideal or an ideal of an ordered semigroup S, then $(A] = A$. For a proper subset A of S we denote by $H_A(S)$ the subset of S defined by

$$H_A(S) := \{h \in S \text{ such that if } s \in S\backslash A, \text{ then } s \notin (shS]\} \text{ [2].}$$

Clearly, $H_A(S) = \emptyset$ or $H_A(S) \neq \emptyset$. Let us give an example for which $H_A(S) = \emptyset$.

Example 2.1. For the ordered semigroup $S = \{a, b, c\}$ defined by Table 2 and Figure 2 and the subset $A = \{a, b\}$ of S, we have $H_A(S) = \emptyset$. For $A = \{b, c\}$ we also have $H_A(S) = \emptyset$.

$\cdot$	a	b	c
a	a	b	a
b	a	b	a
c	a	b	c

Table 2.

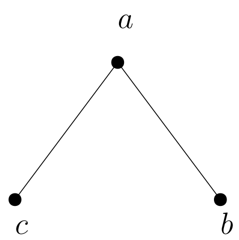

Figure 2.

Proposition 2.2. (see also [2; Theorem 2.4(1)]) *If S is an ordered semigroup, then the set $H_A(S)$ is a semiprime subset of S.*

Proof. Let I be an ideal of S such that $I^2 \subseteq H_A(S)$. Then $I \subseteq H_A(S)$. Indeed: Let $h \in I$. We have to prove that $h \in H_A(S)$ that is, if $s \in S\backslash A$, then $s \notin (shS]$. Suppose $s \in S\backslash A$ and $s \in (shS]$. Since $h \in I$ and I is an ideal of S, we have

$$s \in (s(IS)] \subseteq (sI] \subseteq (SI] \subseteq (I] = I,$$

then $s^2 \in I^2 \subseteq H_A(S)$. Since $s^2 \in H_A(S)$ and $s \in S \backslash A$, we have $s \notin (ss^2S] = (S] = S$ which is impossible. □

Proposition 2.3. (see also [2; Theorem 2.4(3)] *Let S be an ordered semigroup. Then we have the following:*

If A is a (proper) ideal of S, then $A \subseteq H_A(S)$.

Proof. Let $h \in A$. Then $h \in H_A(S)$. Indeed: First of all, since $A \subseteq S$, we have $h \in S$. Let now $s \in S \backslash A$. Then $s \notin (shS]$. In fact: If $s \in (shS]$ then, since A is an ideal of S, we have

$$s \in (s(hS)] \subseteq (s(AS)] \subseteq (sA] \subseteq (SA] \subseteq (A] = A$$

which is impossible. Since $h \in S$, $s \in S \backslash A$ and $s \notin (shS]$, we have $h \in H_A(S)$. □

Corollary 2.4. *If A is a (proper) ideal of S, then $H_A(S) \neq \emptyset$.*

In Proposition 2.3 and Corollary 2.4 is not necessary to assume that the ideal A is a "proper" ideal of S; this is because, by writing $H_A(S)$, we already accepted that A is a proper subset of S.

Proposition 2.5. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then*

If $H_A(S) \neq \emptyset$, then $H_A(S)$ is a right ideal of S.

Proof. By hypothesis, $H_A(S)$ is a nonempty subset of S. Let $h \in H_A(S)$ and $t \in S$. Then $ht \in H_A(S)$. In fact: First of all, $ht \in S$. Let now $s \in S \backslash A$. Then $s \notin (shtS]$. Indeed: if $s \in (sh(tS)]$, then $s \in (shS]$. On the other hand, since $h \in H_A(S)$ and $s \in S \backslash A$, we have $s \notin (shS]$, we get a contradiction. Let now $h \in H_A(S)$ and $S \ni g \leq h$. Then $g \in H_A(S)$. Indeed: Let $s \in S \backslash A$. Since $h \in H_A(S)$ and $s \in S \backslash A$, we have $s \notin (shS]$. Since $g \leq h$, we have $(sgS] \subseteq (shS]$. Then we get $s \notin (sgS]$. Since $s \in S \backslash A$ and $s \notin (sgS]$, we have $g \in H_A(S)$. □

Proposition 2.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then*

If A is a right ideal of S and $H_A(S) \neq \emptyset$, then $H_A(S)$ is a left ideal of S.

Proof. By hypothesis, $H_A(S)$ is a nonempty subset of S. Let $t \in S$ and $h \in H_A(S)$. Then $th \in H_A(S)$. Indeed: Let $s \in S \backslash A$. We have to prove that $s \notin (sthS]$. Suppose $s \in (sthS]$. Then we have

$$st \in (sthS](S] \subseteq (sthS^2] \subseteq (sthS] \tag{1}$$

On the other hand, since $h \in H_A(S)$, we have $st \in A$. Indeed: Let $st \in S \backslash A$. Since $h \in H_A(S)$ and $st \in S \backslash A$, we have $st \notin (sthS]$ which is impossible by (1). Since $s \in (sthS]$ and $st \in A$, we have $s \in ((st)hS] \subseteq (AhS] \subseteq (AS] \subseteq (A] = A$, so $s \in A$ which is impossible. Finally, as in Proposition 2.5, $h \in H_A(S)$ and $S \ni g \leq h$ imply $g \in H_A(S)$; thus $H_A(S)$ is a left ideal of S. □

Corollary 2.7. *If S is an ordered semigroup, A a right ideal of S and $H_A(S) \neq \emptyset$, then $H_A(S)$ is an ideal of S.*

Proof. Since $H_A(S) \neq \emptyset$, by Proposition 2.5, $H_A(S)$ is a right ideal of S. Since A a right ideal of S and $H_A(S) \neq \emptyset$, by Proposition 2.6, $H_A(S)$ is a left ideal of S; and so $H_A(S)$ is an ideal of S. □

Corollary 2.8. (see also [2; Theorem 2.4(1)]) *If S is an ordered semigroup, A a right ideal of S and $H_A(S) \neq \emptyset$, then $H_A(S)$ is a semiprime ideal of S.*

Proof. Since A is a right ideal of S and $H_A(S) \neq \emptyset$, by Corollary 2.7, $H_A(S)$ is an ideal of S. On the other hand, by Proposition 2.2, $H_A(S)$ is a semiprime subset of S. Thus $H_A(S)$ is a semiprime ideal of S. □

Proposition 2.9. *Let $(S, \cdot, \leq)$ be an ordered semigroup and I a right ideal of the semigroup $(S, \cdot)$. If $s \notin (sI]$ for every $s \in S \backslash A$, then $I \subseteq H_A(S)$.*

Proof. Let $h \in I$. Then $h \in H_A(S)$, that is if $s \in S \backslash A$, then $s \notin (shS]$. Indeed: Let $s \in S \backslash A$ and $s \in (shS]$. Then we have $s \in (s(IS)] \subseteq (sI]$, we get a contradiction. □

Proposition 2.10. *Let S be an ordered semigroup, $H_A(S)$ a left ideal of S and I be a subset of S such that $I \subseteq H_A(S)$. Then $s \notin (sI]$ for every $s \in S \backslash A$.*

Proof. Let $s \in S \backslash A$ such that $s \in (sI]$. Then

$$s \in (sI] \subseteq (sH_A(S)] \subseteq (SH_A(S)] \subseteq (H_A(S)] = H_A(S)$$

since $H_A(S)$ is a left ideal of S. We have $s \in S \backslash A$ and $s \in H_A(S)$, so $s \notin (ssS] = (S] = S$ which is impossible. □

Corollary 2.11. *Let S be an ordered semigroup, A a right ideal of S, $H_A(S) \neq \emptyset$ and I a subset of S such that $I \subseteq H_A(S)$. Then $s \notin (sI]$ for every $s \in S \backslash A$.*

Proof. Since A is a right ideal of S and $H_A(S) \neq \emptyset$, by Proposition 2.6, $H_A(S)$ is a left ideal of S. Since $H_A(S)$ is a left ideal of S and I a subset of S such that $I \subseteq H_A(S)$, by Proposition 2.10, $s \notin (sI]$ for every $s \in S \backslash A$. □

Corollary 2.12. (see also [2; Theorem 2.4(2)]) *Let $(S, \cdot, \leq)$ be an ordered semigroup and A, I right ideals of $(S, \cdot, \leq)$. Then $I \subseteq H_A(S)$ if and only if $s \notin (sI]$ for every $s \in S \backslash A$.*

Proof. $\Longrightarrow$. Since I is a right ideal of S and $I \subseteq H_A(S)$, we have $H_A(S) \neq \emptyset$. Since A is a right ideal of S, $H_A(S) \neq \emptyset$ and I is a subset of S such that $I \subseteq H_A(S)$, by Corollary 2.11, $s \notin (sI]$ for every $s \in S \backslash A$.
$\Longleftarrow$. Since I is a right ideal of the ordered semigroup $(S, \cdot, \leq)$, it is a right ideal of the semigroup $(S, \cdot)$ as well. Since I is a right ideal of $(S, \cdot)$ and $s \notin (sI]$ for every $s \in S \backslash A$, by Proposition 2.9, we have $I \subseteq H_A(S)$. □

Summarizing, from Proposition 2.3, Corollary 2.8 and Corollary 2.12 we have the following theorem

Theorem 2.13. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then we have the following:*

(1) *If A is a (proper) ideal of S, then $A \subseteq H_A(S)$.*

(2) *If A a right ideal of S and $H_A(S) \neq \emptyset$, then $H_A(S)$ is a semiprime ideal of S.*

(3) *If A and I are right ideals of S, then $I \subseteq H_A(S)$ if and only if $s \notin (sI]$ for every $s \in S \backslash A$.*

Again in property (1) the assumption "proper" can be omitted.

Theorem 2.13 generalizes the Theorem 2.4 in [2]. It is enough to observe that if S has a zero and A is a proper right ideal of S, then $0 \in H_A(S)$ and so $H_A(S) \neq \emptyset$.

We apply the above results to the following examples. The first two examples are on ordered semigroups in general; the third one is an example of an ordered semigroup $(S, \cdot, \leq)$ that contains a zero.

Example 2.14. We consider the ordered semigroup $S = \{a, b, c, d, e, f\}$ defined by Table 3 and Figure 3.

$\cdot$	a	b	c	d	e	f
a	a	b	b	b	e	f
b	a	b	b	b	e	f
c	a	b	b	c	e	f
d	a	b	b	d	e	f
e	e	e	e	e	e	f
f	f	f	f	f	f	f

Table 3.

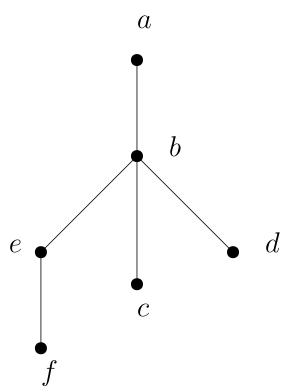

Figure 3.

For the subset $A = \{c, d, e\}$ of S, we have $H_A(S) = \emptyset$.
The sets $\{f\}$ and $\{e, f\}$ are proper subsets of S, so the sets $H_{\{f\}}(S)$ and $H_{\{e,f\}}(S)$ are defined and, by Proposition 2.2, they are semiprime subsets of S. Independently, let us prove that $H_{\{e,f\}}(S)$ is semiprime. We first prove that $H_{\{e,f\}}(S) = \{e, f\}$. Let now I be an ideal of S such that $I^2 \subseteq \{e, f\}$. Then $I \subseteq \{e, f\}$. Indeed: if $x \in I$, then $x^2 \in I^2 \subseteq \{e, f\}$, so $x^2 = e$ or $x^2 = f$. If $x^2 = e$ then, by Table 3, we have $x = e$ and so $x \in \{e, f\}$.

If $x^2 = f$, then $x = f$ and again $x \in \{e, f\}$. Independently, the set $H_{\{f\}}(S)$ is also a semiprime subset of S. Indeed, we have $H_{\{f\}}(S) = \{f\}$; and if I is an ideal of S such that $I^2 \subseteq \{f\}$ and $x \in I$, then $x^2 = f$ and, by Table 3, $x = f$; so $I \subseteq \{f\}$. The sets $\{f\}$ and $\{e, f\}$ are ideals of S. We have already seen that $\{f\} \subseteq H_{\{f\}}(S)$ and $\{e, f\} \subseteq H_{\{e,f\}}$; that is a consequence of Proposition 2.3 as well. Since $H_{\{f\}}(S) \neq \emptyset$, by Proposition 2.5, $H_{\{f\}}(S)$ is a right ideal of S; which is true. In a similar way all of the above results can be applied to this example.

Example 2.15. We consider the ordered semigroup $S = \{a, b, c, d, e, f\}$ defined by Table 4 and Figure 4.

$\cdot$	a	b	c	d	e	f
a	a	a	a	d	a	a
b	a	b	b	d	b	b
c	a	b	c	d	e	e
d	a	a	d	d	d	d
e	a	b	c	d	e	e
f	a	b	c	d	e	f

Table 4.

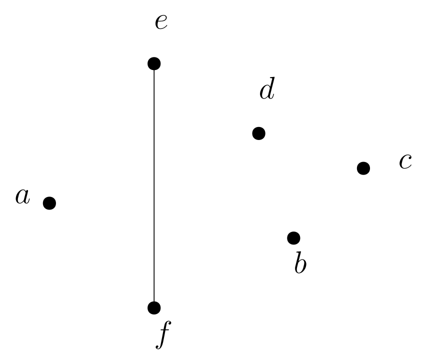

Figure 4.

The proper ideals of S are the sets $\{a, d\}$ and $\{a, b, d\}$. Moreover we have $H_{\{a,d\}}(S) = \{a, d\}$ and $H_{\{a,b,d\}}(S) = \{a, b, d\}$. Since $\{a, d\}$ (resp. $\{a, b, d\}$) is a right ideal of S and $H_{\{a,d\}}(S) \neq \emptyset$ (resp. $H_{\{a,b,d\}}(S) \neq \emptyset$), by Corollary 2.8, the sets $H_{\{a,d\}}(S)$ and $H_{\{a,b,d\}}(S)$ are semiprime ideals of S. Independently we can prove that the set $H_{\{a,d\}}(S)$ is a semiprime subset (and thus a semiprime ideal) of S, by showing that the set $I = \{a, d\}$ is the only ideal of S such that $I^2 \subseteq \{a, d\}$ or in the way indicated in Example 2.14. The sets $\{a, b, d\}$ and $\{a, d\}$ are right ideals of S and $\{a, d\} \subseteq H_{\{a,b,d\}}(S)$. So, by the $\Rightarrow$-part of Corollary 2.12, we have $s \notin (s\{a, d\}]$ for every $s \in S \backslash \{a, b, d\}$. Independently, if $s \in S \backslash \{a, b, d\}$, then $s = c$ or $s = e$ or $s = f$; $c \notin (a, d] = (c\{a, d\}]$, $e \notin (a, d] = (e\{a, d\}]$ and $f \notin (a, d] = (f\{a, d\}]$.

In addition, since $\{a, b, d\}$ and $\{a, d\}$ are right ideals of S and $s \notin (sa, sd] = (s\{a, d\}]$ for every $s \in S\backslash\{a, b, d\}$, by the $\Leftarrow$-part of Corollary 2.12, we have $\{a, d\} \subseteq H_{\{a,b,d\}}(S)$; independently, we can check that this is true. All of the results given above in a similar way can be applied.

Example 2.16. We consider the ordered semigroup $S = \{a, b, c, d, e\}$ defined by Table 5 and Figure 5. This is an ordered semigroup with zero; the element a is the zero element of S; that is $ax = xa = a$ and $a \le x$ for every $x \in S$.

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	a
d	a	a	c	c	a
e	a	a	e	e	a

Table 5.

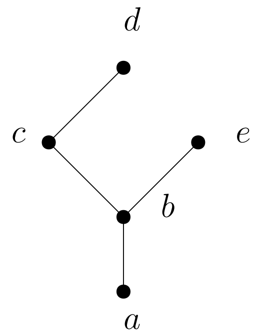

Figure 5.

The set $A = \{a, b, c, d\}$ is a right ideal of S, $S\backslash A = \{e\}$ and $H_A(S) = \{a, b, e\}$. Since $H_A(S) \neq \emptyset$, by Corollary 2.8, $H_A(S)$ is a semiprime ideal of S. Independently, by looking at Table 5 and Figure 5 we can see that this is indeed an ideal of S. Moreover, it is a semiprime subset of S (and so a semiprime ideal of S as Corollary 2.8 shows). Indeed, if I is an ideal of S such that $I^2 \subseteq \{a, b, e\}$ and $x \in I$, then $x^2 = a$ or $x^2 = b$ or $x^2 = e$. As there is no element x of S such that $x^2 = b$ or $x^2 = e$, we have $x^2 = a$ and so $x = a$ or $x = b$ or $x = e$; that is $x \in \{a, b, e\}$. Thus we have $I \subseteq \{a, b, e\}$ and $\{a, b, e\}$ is semiprime. As one can see, $A = \{a, b, c, e\}$ is an ideal of S and $H_A(S) = S$. We can check that $H_{\{a,b,e\}}(S) = \{a, b, e\}$. Since $H_{\{a,b,e\}}(S)$ is a left ideal of S and $\{a, b\}$ is a subset of S such that $\{a, b\} \subseteq H_{\{a,b,e\}}(S)$, by Proposition 2.10, for every $s \in S\backslash\{a, b, e\}$, we have $s \notin (s\{a, b\}) = (sa, sb]$, that is $c \notin (ca, cb] = (a]$ and $d \notin (da, db] = (a]$; independently we can check that this is indeed so as $c \nleq a$ and $d \nleq b$. All the above results can be applied to this example.

For a subset A of an ordered semigroup $(S, \cdot, \leq)$, we denote by $P_r(A)$ the subset of S defined by

$$P_r(A) := \{p \in S \mid \exists\ s \in S \backslash A \text{ such that } sp \in A\}$$

(cf. also [2]). Clearly $P_r(A) = \emptyset$ or $P_r(A) \neq \emptyset$. For the ordered semigroup S defined in Example 2.1 and the subset $A = \{c\}$ of S, we have $P_r(A) = \emptyset$. If A is a proper left ideal of $(S, \cdot)$, then $A \subseteq P_r(A)$, and thus $P_r(A) \neq \emptyset$. Indeed: Let $p \in A$. Take an element $s \in S \backslash A$ (A is proper). We have $sp \in (S \backslash A)A \subseteq SA \subseteq A$ and so $sp \in A$. Since $p \in S$, $s \in S \backslash A$ and $sp \in A$, we have $p \in P_r(A)$.

Example 2.17. Let us consider the ordered semigroup of the Example 2.14. For the subset $\{c, e\}$ of S, we have $P_r(\{c, e\}) = \{e\}$. For the subset $\{c, d, e\}$ of S, we also have $P_r(\{c, d, e\}) = \{e\}$. On the other hand, the sets $\{f\}$ and $\{e, f\}$ are proper left ideals of S, and we have $\{f\} = P_r(\{f\}) \subseteq P_r(\{f\})$ and $\{e, f\} = P_r(\{e, f\}) \subseteq P_r(\{e, f\})$.

In a semigroup $(S, \cdot)$ containing an identity e, if A is a proper right ideal of S, then $A \subseteq P_r(A)$. Indeed, as A is proper, we have $e \in S \backslash A$; and if $p \in A$, then $ep = p \in A$, thus $p \in P_r(A)$ (see also [2]). According to [2; Proposition 2.5], if $(S, \cdot, \leq)$ is an ordered semigroup, e an identity of $(S, \cdot)$ and A a proper right ideal of $(S, \cdot, \leq)$, then the set $P_r(A)$ is a completely prime right ideal of S and $A \subseteq P_r(A)$. The first part of this proposition can be also obtained as a corollary to the following proposition.

Proposition 2.18. *Let A be a proper right of an ordered semigroup S. If $P_r(A)$ is nonempty, then it is a completely prime right ideal of S.*

In contrast to semigroups containing identity, if S is an ordered semigroup and A is a proper right ideal of S, then the property $A \subseteq P_r(A)$ does not hold in general. Let us show it by the following

Example 2.19. Consider the ordered semigroup of the Example 2.14. As we have already seen in Example 2.17, for the subset $A = \{c, d, e\}$ of S, we have $P_r(A) = \{e\}$ and so $A \nsubseteq P_r(A)$. We observe here that the set $\{c, d, e\}$ is not an ideal of S.

In this respect, we have the following

Proposition 2.20. *Let A be a proper ideal of an ordered semigroup $(S, \cdot, \leq)$. Then $P_r(A)$ is a completely prime right ideal of S containing A.*

Proof. Since A is a proper left ideal of $(S, \cdot)$, we have $A \subseteq P_r(A)$ and so $P_r(A) \neq \emptyset$. Since A is a proper right ideal of $(S, \cdot, \leq)$ and $P_r(A) \neq \emptyset$, by Proposition 2.18, $P_r(A)$ is a completely prime right ideal of S containing A. □

We apply Proposition 2.20 to the following example

Example 2.21. Consider the ordered semigroup S of the Example 2.14. The sets $\{f\}$ and $\{e, f\}$ are the only proper ideals of S; and as we have seen in Example 2.17, for the set $A = \{e, f\}$, we have $P_r(A) = \{e, f\}$. By Proposition 2.20, $P_r(A)$ is a completely prime ideal of S. Independently, we can check that if C, D are subsets of S such that $CD \subseteq \{e, f\}$, then $C \subseteq \{e, f\}$ or $D \subseteq \{e, f\}$ (or we can check that if $x, y \in S$ such that $xy \in \{e, f\}$, then $x \in \{e, f\}$ or $y \in \{e, f\}$) which means that $\{e, f\}$ is a completely prime ideal of S. Similarly, the set $P_r(\{f\} (= \{f\})$ is a completely prime ideal of S. This being so, we add to Example 2.14 a second proof that the sets $\{e, f\}$ and $\{f\}$ are indeed semiprime ideals of S; as every completely prime ideal is a prime ideal and every prime ideal is a semiprime ideal.

According [2; p. 526, l. –9 to –7], if I and J are right ideals of an ordered semigroup S, then $(IJ]$ is a right ideal of S. The authors assume that each ordered semigroup has an identity and a zero [see p. 525, l. 22–23]. It might be noted that, more generally, if I is a nonempty subset of an ordered semigroup S and J a right ideal of S, then $(IJ]$ is a right ideal of S. If I is a left ideal of S and J a nonempty subset of S, then $(IJ]$ is a left ideal of S. As a consequence, if I and J are ideals of S, then $(IJ]$ is an ideal of S. The finite intersection of right (resp. left, two-sided) ideals of an ordered semigroup S, if it is nonempty, is a right (resp. left, two-sided) ideal of S; this generalizes the corresponding result in [2; Corollary 2.2].

Finally, it might be mentioned that the Proposition 2.1 in [2], actually a lemma used throughout the paper, is not new (see, for example [4; the Lemma] or [5; Lemma 1]). The Proposition 2.3 in [2] is also not new, it is a special case of the Proposition in [5], where has been shown that if an ideal of an ordered semigroup is completely semiprime and prime, then it is completely prime (without using the identity considered in [2]). The fact that every semigroup endowed with the order $\leq = \{x, y) \mid x = y\}$ is an ordered semigroup and, as a consequence, the notion of a right chain ordered semigroup generalizes the notion of a right chain semigroup (p. 525, l. 11–19; p. 524, l. –10 to –7] in [2]) is well known as it is known for any type of ordered semigroups –see, for example [4–7].

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