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Some estimates below the modulus of integrals of some polynomials in the complex plane

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Abstract. In this paper, we make some estimates below the modulus of some integrals in the complex plane. Our aim is to prove the Conjecture1, which we could see in [2–4]. The proof of the conjecture appears the Corollary.

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1. Introduction

In papers [2–4], we consider the Conjecture 1: If $a_k \geq 0, a_k \in \mathbb{R}$, Then we assert

$$\left| \int_0^{e^{i\varphi}} \prod_{k=1}^{n} (x + a_k)\, dx \right| \geq \frac{1}{n+1},$$

for arbitrary natural n, $\varphi \in \left[0, \frac{\pi}{2}\right]$. There exists a connection between this conjecture and Conjecture2: If $\phi_k \in \left[\frac{\pi}{2}, \pi\right]$, then

$$\left| \int_{-1}^{0} (x+1) \prod_{k=1}^{n} \left(x - e^{i\phi_k}\right) dx \right| \geq \frac{1}{n+2}.$$

Both conjectures are very important for the proofs of some famous conjectures, like Sendov's and Obreshkoff's ones. A possible connection between both conjctures appears [5]. Here we shall extend this problem (Conjecture1): what kind of set L satisfies this assertion, i.e. if a_k belongs to the set L, then the upper inequality is true. The results related with the Conjecture 1, we observe in Theorem 1 , Theorem 2. In Theorem 4 we generalize and prove the extended conjecture. We can see the results of Theorem 1 in

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[2, 4]. Such one of Theorem 2 could be seen in [3]. Many authors use some modulus of some integrals in the complex plane for various estimates in their works. For example we can see how Bojanov and Rahman in [1] use this method. These estimates are explored for the localization of the zeros of some polynomials. The results are useful in the (open) problems of [6–9].

2. Related Results

Theorem 1. *Let* $k = 1, 2, ...n, n \in \mathbb{N}, a_k \in [0, 1]\, \varphi \in \left[0, \frac{\pi}{2}\right]$. *Then the function*

$$\left| \int_{-1}^{e^{i\varphi}} x \prod_{k=1}^{n} (x + a_k)\, dx \right| \geq \frac{1}{n+2}$$

for $n = 1, 2, 3$.

Theorem 2. *let* $k \in \mathbb{N}, a \in \mathbb{R}, a \in [0, 1]$. *Then the function*

$$\left| \int_{0}^{i} x \quad (x + a)^k\, dx \right| \geq \frac{1}{n+2}.$$

3. Preliminaries

We note:
$D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$ is the open disk with center a and radius r.
$\overline{D}(0, r) = \{z \in \mathbb{C} : |z - a| \leq r\}$ is the closed disk with center a and radius r.
$A = \{z \in \mathbb{C}, Rez \leq 0\}$ is the left semiplane.

4. Main Results

Theorem 3. *We consider a polynomial* $r(z) = z^{n-1} + r_{n-1}z^{n-1} + ... + r_1 z + r_0$. *where* $r_k \in \mathbb{R}, n \geq 1, n \in \mathbb{N}, k = \overline{0, n-1}$. *The zeros* z_k *of* $r(z)$ *satisfy the condition* $Rez_k \leq 0$. *If* $a \geq 0$, *then* $I = n\int_0^a r(z)\, dz \geq a^n$.

Proof. Let $r(z) = (z + a_1)(z + a_2) ... (z + a_1)(z - b_1)\left(z - \overline{b_1}\right) ... (z - b_s)\left(z - \overline{b_s}\right)$, where $1 + 2s = n - 1, a_k \geq 0, b_m \in \mathbb{C}, k = \overline{1, l}, m = \overline{1, s}, a_k \in \mathbb{R}, k, m \in \mathbb{N}$. and $b_m = \rho_m e^{i\varphi m}, \rho_m \geq 0, \varphi_m \in \left[\frac{\pi}{2}, \pi\right], (z - b_m)\left(z - \overline{b_m}\right) = z^2 - 2\rho_m \cos\varphi_m z + \rho_m^2 \geq z^2$. Then
$n\int_0^a r(z)\, dz = n\int_0^a (z + a_1)(z + a_2) ... (z + a_1)(z - b_1)\left(z - \overline{b_1}\right) ... (z - b_s)\left(z - \overline{b_s}\right) d \geq$
$n\int_0^a z^{n-1} dz = a^n$.

Theorem 4. *We consider a polynomial* $r(z) = z^{n-1} + r_{n-1}z^{n-2} + ... + r_1 z + r_0$, *where* $r_k \in \mathbb{R}, n \geq 1, n \in \mathbb{N}, k = \overline{0, n-1}$. *The zeros* z_k *of* $r(z)$ *satisfy the condition* $z_k \in A \setminus D(z_0, a) \setminus D(\overline{z_0}, a), z_0 = ae^{i\theta_0}$, *where* $a \geq 0, \theta_0 \in \left[0, \frac{\pi}{2}\right]$. *Then*

$$I = \left| n \int_0^{z_0} r(z)\, dz \right| \geq a^n.$$

Proof. Let us put $v(\theta) = ae^{i\theta}, \theta \in [0, \theta_0], I + 2s = n - 1$,

$$r(z) = \prod_{p=1}^{l} (z + a_p) \prod_{p=1}^{s} (z - b_p)\left(z - \overline{b_p}\right),$$

$l, s \in \mathbb{N}$ (one of the factors could be not existing, i.e.,$l = 0$ or $s = 0$).
We put $f(\theta) = n \int_0^{v(\theta)} r(z)\, dz, g(\theta) = f(\theta) . \overline{f}(\theta)$.
Let us calculate

$$\frac{dg}{d\theta} = n\left[r(v(\theta)) \frac{dv}{d\theta} \overline{f}(\theta) + r(\overline{v}(\theta)) \frac{d\overline{v}}{d\theta} f(\theta)\right],$$

$$\frac{dv}{d\theta} = \frac{dae^{i\theta}}{d\theta} = iae^{i\theta},$$

and if we put

$$U_0 = v(\theta) = ae^{i\theta}, U_p = v(\theta) + a_p, p = \overline{1, l}, U_{l+2p+1} = v(\theta) - b_p,$$

$$U_{l+2p+2} = v(\theta) - \overline{b_p}, p = \overline{0, s-1}.$$

Knowing

$$\frac{df}{d\theta} . \Pi_{p=0}^{n-1} \overline{U_p} = \frac{d\overline{f}}{d\theta} . \Pi_{p=0}^{n-1} U_p,$$

we have

$$\frac{dg}{d\theta} = in\left[\overline{f}(\theta) \prod_{p=0}^{n-1} U_p - f(\theta) \prod_{p=0}^{n-1} \overline{U_p}\right],$$

$$\frac{d^2 g}{d\theta^2} = n\left[2\frac{df}{d\theta} \prod_{p=0}^{n-1} \overline{U_p} + i\frac{d\Pi_{p=0}^{n-1} U_p}{d\theta} \overline{f}(\theta) - i\frac{d\Pi_{p=0}^{n-1} \overline{U_p}}{d\theta} f(\theta)\right],$$

$$\frac{d^2 g}{d\theta^2} = n\left[2n \prod_{p=0}^{n-1} |U_p|^2 - \left(U_0 \sum_{p=0}^{n-1} \prod_{j \neq p} U_j\right) \overline{f}(\theta) - \left(\overline{U_0} \sum_{p=0}^{n-1} \prod_{j \neq p} \overline{U_j}\right) f(\theta)\right],$$

$$\frac{d^2 g}{d\theta^2} = 2n\left[n \prod_{p=0}^{n-1} |U_p|^2 - Re\left(U_0 \sum_{p=0}^{n-1} \prod_{j \neq p} U_j\right) \overline{f}(\theta)\right],$$

and consequently

$$\frac{d^2g}{d\theta^2} \geq 2n \prod_{p=0}^{n-1} |U_p| \left[n \prod_{p=0}^{n-1} |U_p| - \frac{\left| U_0 \sum_{p=0}^{n-1} \prod_{j\neq p} U_j \right| . \left| \overline{f}(\theta) \right|}{\Pi_{p=0}^{n-1} U_p} \right].$$

If we note

$$B = \{A \setminus D(z,a) \setminus D(\overline{z},a)\}, B_0 = \{A \setminus D(z_0,a) \setminus D(\overline{z_0},a)\}$$

and since

$$\theta \in [0,\theta_0] \Longrightarrow B_0 \subset B, i.e., |U_p(\theta)| \geq a, p = \overline{1,n-1}.$$

If we assume

$$|f(\theta)| = \left|\overline{f}(\theta)\right| \leq a^n,$$

then

$$\frac{d^2g}{d\theta^2} \geq 2n \prod_{p=0}^{n-1} |U_p| \left[naa^{n-1} - \left(1 + \left|\frac{U_0}{U_1}\right| + ... + \left|\frac{U_0}{U_{n-1}}\right|\right) .a^n \right]$$

$$\geq 2na.\Pi_{p=0}^{n-1} |U_p| \left[na^{n-1} - \left(1 + \frac{a(n-1)}{a}\right) .a^{n-1} \right] = 0.$$

Then

$$\frac{d^2g}{d\theta^2} \geq 0.$$

Hence

$$\frac{dg}{d\theta}(\theta) \geq \frac{dg}{d\theta}(0) = 0.$$

Consequently $g(\theta_0) > g(0)$, i.e., $|f(\theta_0)| > a^n$, according to the proof of Theorem 3. Therefore $a^n < |f(\theta_0)| \leq a^n$, which is impossible. The contradiction proves the Theorem 4.

Corollary. If in the condition of Theorem 4, we put a=1, and s=0, i.e., all the zeros of r(z) are real and negative, then we get that the Conjecture 1 is true.

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