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A differential Game Related to Terrorism: Stackelberg Differential Game of E-differentiable and E-convex function

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Abstract. In this work, the Stackelberg differential game of E-differentiable and E-convex function is studied in order to fight the terrorism taking into account the government's procedures such as education quality, better job opportunity, social justice, religious awareness and security arrangements. We consider Stackelberg differential game. Firstly, the government is the leader and the terrorist organization is the follower. Secondly, the terrorist organization is the leader and the government is a follower. Furthermore, we apply the necessary conditions of the Stackelberg differential game for these cases to obtain the optimal strategy of this problem.

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1. Introduction

The government's tasks are very important for counter-terror, the government must be applied some subject of mathematics to get methods for combating terrorism, particularly Operations Research.

Counter-terror measures range comes from security arrangements and the government's procedures to freezing assets of a terrorist organization or even to invade their territories for assassinating the terrorists. Since any action against terrorists must be taken into account their reactions. In this paper, the Stackelberg approach is used for studying the interaction strategies of the government and the terrorist organization. Therefore, we use the concept of E-differentiable and E-convex functions to transform a non-differentiable and non-convex function to E-differentiable and E-convex function [1, 2]. In [3] it is imposed that the success of combating terrorism depends on public opinion, while in [4] the efficiency of "water" and "fire" strategies are compared. Hsia[5, 6] introduced the fuzzy differential game to guard a territory movable and not movable, a Nash-collative

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differential game is presented in [7]. A min-max fuzzy differential game with fuzzy on the objective and control, and the large-scale differential game are discussed in [8–11]. The terrorism infighting by Stackelberg and Nash strategies are introduced in [12], the interactions strategics between R&D, defense and preemption is presented in [13], a min-max differential game approach is studied to fight the terrorism [14, 15]. The policies of anti-terrorism and economy of terrorism are introduced in [16, 17]. The study of terror support and recruitment (defense and peace economic) is presented in [18]. Megahed [19] discussed that governments must be made some procedures for fighting the terrorism such as unemployment, justice social, religious awareness, improving the education with considering the security measurements

The organization's power is measured by the terrorist's activities, the organization's resources such as weapons, financial capital, and technological expertise. The power of terror organizations changes with the time, the recruitment of terrorists is through existing terrorists. The decreasing rate of terrorists is affected by their own action and the anti-terrorist actions of the government through of the education quality, increase the chances of labor, social justice, religious awareness and security arrangements. The objectives of the government drive his utility from the loss of the terrorist resources and their activities but incur costs for combating terror and dis-utility from the terror organizations, the later tries to maximize its power both by its size and its terrorist attacks.

In this work, we study how to help the governments to counterterrorism, the Stackelberg differential game plays the main role to combat the terrorism.

2. Problem Formulation

The differential game with the state variable $z(t)$ which describe the resource of the terror organization (TO). It may also include weapons, financial capital, the network of supporters, etc. and another state variable $M(t)$ which describe the government's activities, the education quality, increasing the chances of labor, social justice, religious awareness and security arrangements, $t \in [0, \infty)$ is the time. The two players are the government and (TO) with non-negative strategy $v_1(t), v_2(t)$ respectively. The stock of resources of (TO) grows according to the growth of a linear function $g(z)$ i.e., $g(z) = rz, r > 0$, and the government's procedures grow according to the function $A(M) = \mu M$, where $\mu > 0$ is the growth rate of the government's procedures. Carrying out attacks make a reduction the growth of the resource stock as it affects negatively the number of terrorists (e.g. suicide bombing or caught and killed terrorists). Furthermore, weapons and financial means, it may even include a reduction of the network of supporters. However, the growth reduction of resources stocks does not only depend on the strength of attacks $v_2(t)$ but it is also influenced by fighting the terrorists $v_1(t)$. This effectiveness of the control variables of the two players on the growth of the resource may be denoted as "harvest function $h(v_1, v_2)$. As a consequence, the dynamic equations of the resources stocks and government's tasks $z(t), M(t)$ respectively can be written as

$$\dot{z} = rz(t) - h(v_1(t), v_2(t)), \qquad z(0) = z_0 > 0$$

$$\dot{M} = \mu M - av_1 + bv_2, \qquad M(0) = M_0 > 0$$

where $h(v_1, v_2)$is non-differentiable and not convex, z_0 denotes the initial stock of terrorist's resources, M_0 is the initial government's procedures and a, b are positive constants. Moreover, we assume that

$$z(t) \geq 0, \ M(t) \geq 0 \text{ for all } t \geq 0 \tag{1}$$

Now, we define an operator $E : R^n \to R^n$ such that $(h \circ E)(v_1, v_2)$ is differentiable and convex. Then

$$\dot{y} = rz(t) - (h \circ E)(v_1, v_2)$$

Since the increase rate of counter-terror measure and attacks leads to a reduction of growth, we assume that the partial derivatives are greater than zero: $(h \circ E)_{v_1}(v_1, v_2) > 0, (h \circ E)_{v_2}(v_1, v_2) > 0$. The counter-terror measures exhibit marginally decreasing efficiency $(h \circ E)_{v_1v_2} < 0$. Moreover the increasing rate of attacks induces disproportional higher losses of resources i.e. $(h \circ E)_{v_2v_2} > 0$. Finally, the instruments reinforce each other, i.e. $(h \circ E)_{v_1v_2} > 0$, which makes economical sense. This positive interaction means that the minor efficiency of fighting terrorism increases with the strength of terrorist's attacks. In addition to, the Inada conditions of the economy are assumed to be fulfilled

$$\lim_{v_1 \to 0} (h \circ E)_{v_1}(v_1, v_2) = \infty, \quad \lim_{v_1 \to \infty} (h \circ E)_{v_1}(v_1, v_2) = 0 \tag{2}$$

$$\lim_{v_2 \to 0} (h \circ E)_{v_2}(v_1 v_2) = 0, \quad \lim_{v_2 \to \infty} (h \circ E)_{v_2}(v_1, v_2) = \infty \tag{3}$$

This gives that the optimal strategies are nonnegative $v_1(t) \geq 0$, and $v_2(t) \geq 0, \ t > 0$ Player 1 (the government) derives its benefit from the loss of the terrorist resources and their activities$M(t)$ besides showing the dis-utility of these terrorist organization but incur costs for combating terror. For simplicity, all these terms must be linear. Thus, the objective of the government is

$$\max_{v_1 t)} \left\{ J_1 = \int_0^\infty e^{-\eta_1 t} \left[\omega(h \circ E)(v_1(t), v_2(t)) + qM(t) - cz(t) - kv_2(t) - \alpha v_1(t) \right] dt \right\} \tag{4}$$

where ω, c, k, q and α are positive constants.

The second player (TO) derives its benefit from the resource stock $z(t)$ and the terrorist actions at intensity $v_2(t)$. Then the objective of (To) problem is

$$\max_{v_2(t)} \left\{ J_2 = \int_0^\infty e^{-\eta_2 t} \left[\sigma z(t) + \beta v_2(t) - \gamma M(t) \right] dt \right\} \tag{5}$$

where $\sigma, \beta \ and \ \gamma$ are positive constants.

The decreasing rates $\eta_i, i = 1, 2$ are assumed to be greater than the growth and activity rates, r, μ

$$\eta_i > r, \quad \eta_i > \mu \text{ for } \ i = 1, 2 \tag{6}$$

In this paper, we will derive the Stackelberg equilibria. The solution procedures rely on Pontryagin's maximum [11]

3. The stackelberg differential game

The Stackelberg approach is a hierarchical solution concept, since one of the players, the leader, has a stronger position in the decision process. This solution is the result of sequential decisions: the leader announces his strategy firstly so that the other players, the followers, can only react to the leader strategy. In this paper, we consider the government plays the leader's role and the terrorist's organizations (TO) play the follower's role, and another case is vice versa.

3.1. Stackelberg game with the government is the leader and (TO) is the follower

The government chooses the rate of counter-terror measures before (TO) decides on the rate of attacks. Consider the following problem, where the government and the terrorist organizations are is the leader and the follower respectively

$$\left.\begin{array}{ll} \max_{v_1(t)} J_1 = \int_0^\infty e^{-\eta_1 t}\left[\omega(h\circ E)(v_1(t),v_2(t)) + qM(t) - cz(t) - kv_2(t) - \alpha v_1(t)\right]dt & \\ \max_{v_2(t)} J_2 = \int_0^\infty e^{-\eta_2 t}\left[\sigma z(t) + \beta v_2(t) - \gamma M(t)\right]dt & \\ y^{\cdot} = rz(t) - (h\circ E)(v_1(t),v_2(t)), & z(0)=z_0>0\ , z(t)\geq 0 \\ M^{\cdot} = \mu M(t) - av_1 + bv_2, & M(0)=M_0>0\ , M(t)\geq 0 \end{array}\right\} \tag{7}$$

3.1.1. The follower Problem

Firstly, we find the optimization of the follower (To) which consider the action of a leader is given

$$\left.\begin{array}{ll} \max_{v_2(t)} J_2 = \int_0^\infty e^{-\eta_2 t}\left[\sigma z(t) + \beta v_2(t) - \gamma\ M(t)\right]dt & \\ y^{\cdot} = rz(t) - (h\circ E)(v_1(t),v_2(t)), & z(0)=z_0 \end{array}\right\} \tag{8}$$

The Hamiltonian function of the follower (To), H_2, is defined by

$$H_2(z(t),v_1(t),v_2(t),\lambda_1(t)) = \sigma z(t) + \beta v_2(t) - \gamma\ M(t) + \ \lambda_1(t)(rz(t) - (h\circ E)(v_1(t),v_2(t))) \tag{9}$$

From the necessary conditions

$$\frac{\partial H_2}{\partial v_2} = \beta - \lambda_1(h\circ E)v_2 = 0$$

and consider the harvest function $h(v_1,v_2) = v_1^{\frac{1}{\tau}} v_2^{\frac{1}{\delta}}$, where τ,δ are positive integers

Consider the following operator $E(v_1,v_2) = ({v_1}^{n\tau}, {v_2}^{m\delta})$, m,n are positive integer, then

$$(h\circ E)(v_1,v_2) = v_1^n v_2^m$$

and

$$(h \circ E)_{v_2} = m v_1^n v_2^{m-1} = \frac{\beta}{\lambda_1}$$

and thus

$$v_2^*(v_1) = \left(\frac{\beta}{m\lambda_1}\right)^{\frac{1}{m-1}} v_1^{\frac{n}{1-m}} \tag{10}$$

and the harvest function

$$(h \circ E)(v_1, v_2^*) = \left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}} v_1^{\frac{-n}{m-1}} \tag{11}$$

The adjoint variable λ_1 satisfy the following differential equation

$$\dot{\lambda}_1 = \lambda_1 \eta_2 - \frac{\partial H_2}{\partial z} = \lambda_1(\eta_2 - r) - \sigma \tag{12}$$

Remark 1. *Since $m > 1$ (10) and (11) implies to any increase of combating measures leads to a more cautious behavior of the terrorists as well as a lower harvest of the terrorists' resources.*

3.1.2. The leader Problem (The government)

To obtain the optimal strategy v_1 for the government, the government must be taken into account the optimal strategy of the follower which is represented by an additional costate equation.

$$\left.\begin{array}{c} \max_{v_1(t)} J_1 = \int_0^\infty e^{-\eta_1 t}\left[\omega\left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}} v_1^{\frac{-n}{m-1}} + qM(t) - cz(t) - k\left(\frac{\beta}{m\lambda_1}\right)^{\frac{1}{m-1}} v_1^{\frac{n}{1-m}} - \alpha v_1(t)\right] dt \\ \dot{z} = rz(t) - \left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}} v_1^{\frac{-n}{m-1}}, \\ \dot{M} = \mu M(t) - a v_1 + b\left(\frac{\beta}{m\lambda_1}\right)^{\frac{1}{m-1}} v_1^{\frac{n}{1-m}}, \\ \dot{\lambda}_1 = \lambda_1(\eta_2 - r) - \sigma \end{array}\right\} \tag{13}$$

The Hamiltonian function of the leader

$$H_1 = \left((\omega - \psi_1)\left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}} + (b\psi_2 - k)\left(\frac{\beta}{m\lambda_1}\right)^{\frac{1}{m-1}}\right) v_1^{\frac{-n}{m-1}} + (q + \mu\psi_2)M(t)$$

$$-(\alpha + a\psi_2)v_1 + (r\psi_1 - c)z(t) + \psi_3(\lambda_1(\eta_2 - r) - \sigma)$$

from the necessary conditions,

$$\frac{\partial H_1}{\partial v_1} = \frac{n}{m-1}\left[(\psi_1 - \omega)\left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}} + (k + b\psi_2)\left(\frac{\beta}{m\lambda_1}\right)^{\frac{1}{m-1}}\right] v_1^{\frac{1-n-m}{m-1}} - (\alpha + a\psi_2) = 0$$

$$v_1 = \left[\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega - \psi_1) + nm(b\psi_2 - k)\lambda_1}\right]^{\frac{m-1}{1-n-m}} \left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m+n-1}} \tag{14}$$

The adjoint variables satisfy the following differential equations

$$\left.\begin{array}{r} \dot{\psi}_1 = \eta_1\psi_1 - \frac{\partial H_1}{\partial y} = (\eta_1 - r)\psi_1 + c \\ \dot{\psi}_2 = \eta_1\psi_2 - \frac{\partial H_1}{\partial M} = (\eta_1 - \mu)\psi_2 - q \\ \dot{\psi}_3 = \eta_1\psi_3 - \frac{\partial H_1}{\partial \lambda_1} = (\eta_1 - \eta_2 + r)\psi_3 \\ + \frac{m}{\lambda_1(m-1)}\left[\omega - \psi_1 + \frac{(b\psi_2 - k)\lambda_1}{\beta}\right]\left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}} v_1^{\frac{-m}{m-1+}} \end{array}\right\} \tag{15}$$

Remark 2. *The adjoint variable ψ_3 of the leader with respect to the adjoint variable λ_1 of the follower has no influence on the optimization of the leader*

Proposition 1. *A feasible solution of the game with the government is the leader and the (TO) is the follower is exists if and only if*

$$k < b\psi_2 + \frac{\beta(\omega - \psi_1)}{m\lambda_1} \tag{16}$$

Proof. . The optimal solution of the leader and follower is feasible if and only if

$$\frac{(\alpha + a\psi_2)m(m-1)\lambda_1}{n\beta(\psi_1 - \omega) + nm(k + b\psi_2)\lambda_1} > 0$$

then, $n\beta(\omega - \psi_1) + nm(b\psi_2 - k)\lambda_1 > 0$ and thus

$$k < b\psi_2 + \frac{\beta(\omega - \psi_1)}{m\lambda_1}$$

The optimal strategies are given by

$$v_1 = \left[\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega - \psi_1) + nm(b\psi_2 - k)\lambda_1}\right]^{\frac{m-1}{1-n-m}} \left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m+n-1}} \tag{17}$$

$$v_2 = \left[\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega - \psi_1) + nm(b\psi_2 - k)}\right]^{\frac{n}{n+m-1}} \left(\frac{\beta}{m\lambda_1}\right)^{\frac{1-n}{n+m-1}} \tag{18}$$

and the harvest function with the operator E is

$$(h \circ E)(v_1, v_2) = \left[\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega - \psi_1) + nm(b\psi_2 - k)}\right]^{\frac{n}{n+m-1}} \left(\frac{\beta}{m\lambda_1}\right)^{\frac{n}{n+m-1}} \tag{19}$$

Proposition 2. *The values of the steady state for the inventory resources and the governments's procedures are given by*

$$z^{\infty} = \frac{1}{r}\left[\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega - \psi_1) + nm(b\psi_2 - k)}\right]^{\frac{n}{n+m-1}} \left(\frac{\beta}{m\lambda_1}\right)^{\frac{n}{n+m-1}}$$

$$M^{\infty} = \frac{1}{\mu}\left[a\left(\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)\lambda_1}\right)^{\frac{m-1}{1-n-m}}\left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m+n-1}}\right]$$
$$-\frac{1}{\mu}\left[b\left(\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)}\right)^{\frac{n}{n+m-1}}\left(\frac{\beta}{m\lambda_1}\right)^{\frac{1-n}{n+m-1}}\right]$$

Proof. . The solution of the differential equation $\dot{z(t)} = rz(t) - (h \circ E)(v_1, v_2)$ is

$$z(t)\ e^{-rt} = \frac{(h \circ E)(v_1, v_2)}{r}\ e^{-rt} + c_1$$

where c_1 is the constant, for $t \to \infty$, then $c_1 = 0$, and

$$z^{\infty} = \frac{1}{r}\left[\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)}\right]^{\frac{n}{n+m-1}}\left(\frac{\beta}{m\lambda_1}\right)^{\frac{n}{n+m-1}}$$

Also, the solution of the differential equation $\dot{M} = \mu M + bv_2 - av_1$ is

$$Me^{-\mu t} = \frac{1}{\mu}e^{-\mu t}(av_1 - bv_2) + c_0(constant)$$

For $t \to \infty$ then $c_0 = 0$ and

$$M^{\infty} = \frac{1}{\mu}\left[a\left(\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)\lambda_1}\right)^{\frac{m-1}{1-n-m}}\left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m+n-1}}\right]$$
$$-\frac{1}{\mu}\left[b\left(\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)}\right)^{\frac{n}{n+m-1}}\left(\frac{\beta}{m\lambda_1}\right)^{\frac{1-n}{n+m-1}}\right]$$

Proposition 3. *(i) The government as the leader is more active but the (TO) as the follower is more cautiously if and only if*

$$\frac{a\beta}{m\lambda_1} > b\left(\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)\lambda_1}\right)\left(\frac{\beta}{m\lambda_1}\right)$$

(ii) The government as the leader is more cautiously and the (TO) as the follower is more aggressively if and only if

$$\frac{a\beta}{m\lambda_1} < b\left(\frac{(\alpha + a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)\lambda_1}\right)\left(\frac{\beta}{m\lambda_1}\right)$$

Proof.

(i) *The government as the leader is more active if and only if $M(t) > 0$, from Proposition 2 we have*

$$a\left(\frac{(\alpha+a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)\lambda_1}\right)^{\frac{m-1}{1-n-m}}\left(\frac{\beta}{m\lambda_1}\right)^{\frac{m}{m+n-1}}$$
$$-b\left(\frac{(\alpha+a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)}\right)^{\frac{n}{n+m-1}}\left(\frac{\beta}{m\lambda_1}\right)^{\frac{1-n}{n+m-1}}>0$$

Then

$$\frac{a\beta}{m\lambda_1}>b\left(\frac{(\alpha+a\psi_2)(m-1)\beta}{n\beta(\omega-\psi_1)+nm(b\psi_2-k)\lambda_1}\right)\left(\frac{\beta}{m\lambda_1}\right)$$

(ii) *The proof of 2 is similar to 1 with less than sign instead of greater than sign*

Lemma 1. *The objectives of the government and (TO) are given by*

$$J_1=\tfrac{1}{\eta_1}\left[\omega\left(\tfrac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}}-k\left(\tfrac{\beta}{m\lambda_1}\right)^{\frac{1}{m-1}}\right]v_1^{\frac{-n}{m-1}}+\tfrac{q}{\eta_1-\mu}\left(M_0-\tfrac{av_1-bv_2}{\eta_1\mu}\right)$$
$$+\tfrac{q}{\mu\eta_1}(av_1-bv_2)-\tfrac{c}{\eta_1-r}\left(y_0-\tfrac{h\circ E(v_1,v_2)}{r}\right)+\tfrac{(h\circ E)(v_1,v_2)}{r\eta_1}$$

$$J_2=\frac{\sigma}{\eta_2-r}\left(z_0-\frac{h\circ E(v_1,v_2)}{r}\right)+\frac{\sigma h\circ E(v_1,v_2)}{\eta_2}-\frac{\gamma}{\eta_2-\mu}\left(M_0-\frac{av_1-bv_2}{\mu}\right)-\frac{\gamma(av_1-bv_2)}{\mu\eta_2}+\frac{\beta v_2}{\eta_2}$$

Proof. . The solution of the differential equation $\dot{M}(t)=\mu M-av_1+bv_2$ is

$$Me^{-\mu(t)}=\frac{1}{\mu}(av_1-bv_2)e^{-\mu t}+c_4(constant)$$

For $t\to 0$ then $c_4=M_0-\frac{av_1-bv_2}{\mu}$,and thus

$$M(t)=\left(M_0-\frac{av_1-bv_2}{\mu}\right)e^{\mu t}+\frac{av_1-bv_2}{\mu} \tag{20}$$

similarly the solution of the differential equation $\dot{z}(t)=rz(t)-h(v_1,v_2)$

$$z(t)=\left(z_0-\frac{h(v_1,v_2)}{r}\right)e^{rt}+\frac{h(v_1.v_2)}{r} \tag{21}$$

from (20) and (21) in the objectives of the leader and follower,and by integration we have

$$J_1=\tfrac{1}{\eta_1}\left[\omega\left(\tfrac{\beta}{m\lambda_1}\right)^{\frac{m}{m-1}}-k\left(\tfrac{\beta}{m\lambda_1}\right)^{\frac{1}{m-1}}\right]v_1^{\frac{-n}{m-1}}+\tfrac{q}{\eta_1-\mu}\left(M_0-\tfrac{av_1-bv_2}{\eta_1\mu}\right)$$
$$+\tfrac{q}{\mu\eta_1}(av_1-bv_2)-\tfrac{c}{\eta_1-r}\left(y_0-\tfrac{h\circ E(v_1,v_2)}{r}\right)+\tfrac{(h\circ E)(v_1,v_2)}{r\eta_1}$$

$$J_2 = \frac{\sigma}{\eta_2 - r}\left(z_0 - \frac{h \circ E(v_1, v_2)}{r}\right) + \frac{\sigma h \circ E(v_1, v_2)}{\eta_2} - \frac{\gamma}{\eta_2 - \mu}\left(M_0 - \frac{av_1 - bv_2}{\mu}\right) - \frac{\gamma(av_1 - bv_2)}{\mu\eta_2} + \frac{\beta v_2}{\eta_2}$$

where v_1, v_2 and $h(v_1.v_2)$ are defined in (17), (18), and (19)

Remark 3. *As shown in (20), (21) and according to the condition (1), we must be assume that,* $M_0 > \frac{av_1 - bv_2}{\mu}$*,*
and $z_0 > \frac{h \circ E(v_1.v_2)}{r}$

3.2. The Stackelberg with the (TO) is the leader and the government is the follower

In this case, the (TO) attacks the government before the government makes counter-terror measures. Consider the following problem, where the(TO) is the leader and government is the follower.

3.2.1. The follower problem (The government problem)

Firstly, we consider the optimization of the follower (government) which consider the action of leader is given

$$\left.\begin{array}{ll} \max_{v_1(t)} J_1 = \int_0^\infty e^{-\eta_1 t}\left[\omega(h \circ E)(v_1(t), v_2(t)) + qM(t) - cz(t) - kv_2(t) - \alpha v_1(t)\right] dt & \\ \dot{z} = rz(t) - (h \circ E)(v_1(t), v_2(t)), \qquad z(0) = z_0 > 0\ , z(t) \geq 0 & \\ \dot{M} = \mu M(t) - av_1 + bv_2, \qquad M(0) = M_0 > 0\ , M(t) \geq 0 & \end{array}\right\} \tag{22}$$

The Hamiltonian function of the follower H_1 is defined by

$$H_1 = \omega(h \circ E)(v_1.v_2) + qM - cz(t) - kv_2 - \alpha v_1 + \lambda_1(ry - h) + \lambda_2(\mu M - av_1 + bv_2)$$

From the necessary conditions $\frac{\partial H_1}{\partial v_1} = \omega(h \circ E)_{v_1} - \alpha - \lambda_1(h \circ E)_{v_1} - \lambda_2 a = 0$

$$(h \circ E)_{v_1} = \frac{\alpha + \lambda_2 a}{\omega - \lambda_1}$$

and consider the harvest function $h(v_1, v_2) = v_1^{\frac{1}{\tau}} v_2^{\frac{1}{\delta}}$, where τ, δ are positive integers

Consider the following operator $E(v_1, v_2) = ({v_1}^{n\tau}, {v_2}^{m\delta})$, m, n are positive integers, then

$$(h \circ E)(v_1, v_2) = v_1^n v_2^m$$

Then

$$(h \circ E)_{v_1} = n v_1^{n-1} v_2^m = \frac{\alpha + \lambda_2 a}{\omega - \lambda_1}$$

and thus

$$v_1^*(v_2) = \left(\frac{\alpha + \lambda_2 a}{n(\omega - \lambda_1)}\right)^{\frac{1}{n-1}} v_2^{\frac{m}{1-n}} \tag{23}$$

and the harvest function

$$(h \circ E)(v_1^*(v_2), v_2) = \left(\frac{\alpha + \lambda_2 a}{n(\omega - \lambda_1)}\right)^{\frac{n}{n-1}} v_2^{\frac{m}{1-n}} \tag{24}$$

Remark 4. *: The follower (government) reacts with a higher strength of counter-terror measures in case (TO) intensifies its rate attacks, as shown in (23) which implies to a higher harvest*

(i) If $\frac{m}{1-n} = 1$ see Fig.1(An increasing of counter-terror measures and the terrorists is more cautious),

(ii) If $\frac{m}{1-n} < 1$ see Fig.2 (The (TO) is more aggressively when the government increases their counter-terror measures),

(iii) If $\frac{m}{1-n} > 1$ see Fig.3 (The government is more aggressively with any incrossing of (TO) attacks).

The adjoint variables λ_1, λ_2 satisfy the following differential equations

$$\dot{\lambda_1} = \eta_1 \lambda_1 - \frac{\partial H_1}{\partial y} = (\eta_1 - r)\lambda_1 + c$$

$$\dot{\lambda_2} = \eta_1 \lambda_2 - \frac{\partial H_1}{\partial M} = (\eta_1 - \mu)\lambda_2 - q$$

3.2.2. The leader problem (TO)

To obtain on the optimal strategy v_2 for the (TO), it must be taken into account the optimal strategy of the follower which is represented by an additional costate equation

$$\left.\begin{array}{c} \max_{v_2(t)} J_2 = \int_0^\infty e^{-\eta_2 t} \left[\sigma z(t) + \beta v_2(t) - \gamma\, M(t)\right] dt \\ \dot{z} = r z(t) - \left(\frac{\alpha+\lambda_2 a}{n(\omega-\lambda_1)}\right)^{\frac{n}{n-1}} v_2^{\frac{m}{1-n}}, \\ \dot{\lambda_1} = (\eta_1 - r)\lambda_1 + c \\ \dot{\lambda_2} = (\eta_1 - \mu)\lambda_2 - q \end{array}\right\} \tag{25}$$

The Hamiltonian function of the leader

$$\begin{aligned} H_2 &= \sigma z(t) + \beta v_2(t) - \gamma\, M(t) + \Psi_1 \left(r z(t) - \left(\frac{\alpha + \lambda_2 a}{n(\omega - \lambda_1)}\right)^{\frac{n}{n-1}} v_2^{\frac{m}{1-n}} \right) \\ &+ \Psi_2((\eta_1 - r)\lambda_1 + c) + \Psi_3((\eta_1 - \mu)\lambda_2 - q) \end{aligned} \tag{26}$$

From the necessary conditions

$$\frac{\partial H_2}{\partial v_2} = \beta - \Psi_1 \frac{m}{1-n}\left(\frac{\alpha+\lambda_2 a}{n(\omega-\lambda_1)}\right)^{\frac{n}{n-1}} v_2^{\frac{m+n-1}{1-n}} = 0$$

and thus

$$v_2 = \left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{1-n}{n+m-1}} \left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{n+m-1}} \tag{27}$$

The adjoint variables satisfy the differential equations

$$\left.\begin{array}{c} \dot{\Psi}_1 = \eta_1\Psi_1 - \frac{\partial H_2}{\partial y} = (\eta_1 - r) \\ psi_1 - \sigma \\ \dot{\Psi}_2 = \eta_1\Psi_2 - \frac{\partial H_2}{\partial \lambda_1} = r\Psi_2 + \frac{\psi_1 n}{(n-1)(\omega-\lambda_1)}\left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{1-n}} v_2^{\frac{m}{1-n}} \\ \dot{\Psi}_3 = \eta_1\psi_3 - \frac{\partial H_2}{\partial \lambda_2} = \mu\Psi_3 + \frac{a\Psi_1}{(n-1)(\omega-\lambda_1)}\left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{1}{1-n}} v_2^{\frac{m}{1-n}} \end{array}\right\} \tag{28}$$

Proposition 4. *The optimal strategies, the harvest function and the steady state values of state variables $z(t)$ and $M(t)$ with the (TO) is the leader and the government is follower are given by*

$$v_1 = \left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{m}{n+m-1}} \left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{1-m}{n+m-1}}$$

$$v_2 = \left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{1-n}{n+m-1}} \left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{n+m-1}}$$

$$(h\circ E)(v_1,v_2) = \left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{m}{n+m-1}} \left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{n+m-1}}$$

$$z^{\infty} = \frac{1}{r}\left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{m}{n+m-1}} \left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{n+m-1}}$$

$$M^{\infty} = \frac{1}{\mu}\left[a\left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{m}{n+m-1}} \left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{1-n}{n+m-1}} - b\left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{1-n}{n+m-1}} \left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{n+m-1}}\right]$$

Proof. from (23), (24), and (27) we get on v_1, v_2, and $(h\circ E)(v_1,v_2)$
The solution of the differential equation $\dot{M}(t) = \mu M - av_1 + bv_2$ is

$$Me^{-\mu(t)} = \frac{1}{\mu}(av_1 - bv_2)e^{-\mu t} + c_5(constant)$$

For $t\to\infty$ then $c_4 = 0$, then $M(t) = \frac{av_1 - bv_2}{\mu}$

Similarly the solution of the differential equation $\dot{z} = rz(t) - \left(\frac{\alpha+\lambda_2 a}{n(\omega-\lambda_1)}\right)^{\frac{n}{n-1}} v_2^{\frac{m}{1-n}}$ is

$$z(t) = \frac{1}{r}\left(\frac{\alpha+\lambda_2 a}{m(\omega-\lambda_1)}\right)^{\frac{n}{n-1}} v_2^{\frac{m}{1-n}}$$

From (23),(24) and (27), we have

$$M^{\infty}=\frac{1}{\mu}\left[a\left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{\delta}{n+m-1}}\left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{1-m}{n+m-1}}-b\left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{1-n}{n+m-1}}\left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{n+m-1}}\right]$$

$$z^{\infty}=\frac{1}{r}\left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{m}{n+m-1}}\left(\frac{\alpha+a\lambda_2}{n(\omega-\lambda_1)}\right)^{\frac{n}{n+m-1}}$$

Lemma 2. *The objectives functional of the leader (TO),* J_2*, and the follower,* J_1*, are given by*

$$\begin{aligned} J_1=&\frac{\omega(h\circ E)(v_1,v_2)}{\eta_1}+\frac{q}{\eta_1-\mu}\left(M_0-\frac{av_1-bv_2}{\mu}\right)+\frac{av_1-bv_2}{\mu\eta_1}\\ &-\frac{c}{\eta_1-r}\left(z_0-\frac{(h\circ E)(v_1,v_2)}{r}\right)-\frac{c\ (h\circ E)(v_1,v_2)}{r\eta_1}\end{aligned}$$

$$J_2=\frac{\sigma}{\eta_2-r}\left(z_0-\frac{(h\circ E)(v_1,v_2)}{r}\right)+\frac{\sigma(h\circ E)(v_1,v_2)}{r\eta_2}+\frac{\beta v_2}{\eta_2}-\frac{\gamma}{\eta_2-\mu}\left(M_0-\frac{av_1-bv_2}{\mu}\right)-\frac{\gamma(av_1-bv_2)}{\mu\eta_2}$$

Proof. . From (20) and (21) in the follower's objective, J_1, and the leader's objective, J_2, then

$$\begin{aligned} J_1 &= \int_0^{\infty} e^{-\eta_1 t}\left[\omega(h\circ E)(v_1,v_2)+q\left(M_0-\frac{av_1-bv_2}{\mu}\right)e^{\mu t}+\frac{q(av_1-bv_2)}{\mu}\right]\\ &- c\ e^{-\eta_1 t}\left[\left(z_0-\frac{(h\circ E)(v_1,v_2)}{r}\right)e^{rt}-\frac{c(h\circ E)(v_1,v_2)}{r}-kv_2-\alpha v_1\right]dt\\ &= \frac{\omega(h\circ E)(v_1,v_2)}{\eta_1}+\frac{q}{\eta_1-\mu}\left(M_0-\frac{av_1-bv_2}{\mu}\right)+\frac{q(av_1-bv_2)}{\mu\eta_1}\\ &- \frac{c}{\eta_1-r}\left(z_0-\frac{(h\circ E)(v_1,v_2)}{r}\right)-\frac{c(h\circ E)(v_1,v_2)}{r\eta_1}-\frac{kv_2}{\eta_1}-\frac{\alpha v_1}{\eta_1}\end{aligned}$$

and

$$\begin{aligned} J_2 &= \int_0^{\infty} e^{-\eta_2 t}\left[\sigma\left(z_0-\frac{(h\circ E)(v_1,v_2)}{r}\right)e^{rt}+\frac{\sigma(h\circ E)(v_1,v_2)}{r}\right]\\ &- e^{-\eta_2 t}\left[\gamma\left(M_0-\frac{av_1-bv_2}{\mu}\right)e^{\mu t}+\frac{\gamma(av_1-bv_2)}{\mu}+\beta v_2\right]dt\\ &= \frac{\sigma}{\eta_2-r}\left(z_0-\frac{(h\circ E)(v_1,v_2)}{r}+\right)+\frac{\sigma(h\circ E)(v_1,v_2)}{r\eta_2}\\ &- \frac{\gamma}{\eta_2-\mu}\left(M_0-\frac{av_1-bv_2}{\mu}\right)-\frac{\gamma(av_1-bv_2)}{\mu\eta_2}+\frac{\beta v_2}{\eta_2}\end{aligned}$$

Proposition 5. *(i) The government as the follower is more cautiously and the (TO) as leader is more aggressively if and only if*

$$a\left(\frac{\beta(1-n)}{Psi_1 m}\right) < b\left(\frac{\alpha + a\lambda_2}{n(\omega - \lambda_1)}\right)$$

(ii) The government as the follower is more active and the (TO) as leader is more cautiously if and only if

$$a\left(\frac{\beta(1-n)}{Psi_1 m}\right) > b\left(\frac{\alpha + a\lambda_2}{n(\omega - \lambda_1)}\right)$$

Proof. .

(i) The government as the follower is more cautiously and the (TO) as leader is more aggressively if $M(t) < 0$, Since $M(t) = \frac{av_1 - bv_2}{\mu}$ then, $av_1 < bv_2$. From the values v_1 and v_2 of Proposition 4, we find that

$$a\left(\frac{\beta(1-n)}{\Psi_1 m}\right)^{\frac{m}{n+m-1}}\left(\frac{\alpha + a\lambda_2}{n(\omega - \lambda_1)}\right)^{\frac{1-m}{n+m-1}} < b\left(\frac{\beta(1-n)}{\Psi_1 \delta}\right)^{\frac{1-n}{n+m-1}}\left(\frac{\alpha + a\lambda_2}{n(\omega - \lambda_1)}\right)^{\frac{n}{n+m-1}}$$

and thus

$$a\left(\frac{\beta(1-n)}{\Psi_1 m}\right) < b\left(\frac{\alpha + a\lambda_2}{n(\omega - \lambda_1)}\right)$$

(ii) The proof of 2 is similar to 1 with greater than sign instead of less than sign

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References

[1] Youness, E. A. : E-convex sets, E-convex functions and E-convex programming, J. Optim. Theory Appl. 102(**3)**(1999) 439-450

[2] Megahed et al.,Optimality conditions of E-convex programming for an E-differentiable function, Journal of Inequalities and Applications (2013) 2013:246

[3] JP. Caulkins, G. Feichtinger, D.Grass, G.Tragler, Optimal control of terrorism and global reputation: A case study with novel threshold behavior, Oper. Res. Lett. **3**(2009) 387-391.

[4] JP. Caulkins, G.Feichtinger, D.Grass, G.Tragler, Optimizing Counterterror operations: Should one fight with "Fire "or "water"?Comput. Oper. Res. **35**, (2008)1874-1855

[5] K.H.Hsia, and J. G.Hsie, A first approach to fuzzy differential Game problem: guarding Territory, Fuzzy set.Syst. , **No.55**, (1993), pp. 157-167.

[6] I.C.Hung, K.H.Hsia, and L.W.Chen, Fuzzy Differential Game of Guarding a Movable Territory, Inform. Sciences **No.91**, (1993)pp.113-131.

[7] E.Youness, J.B.Hughes, and El-kholy, Parametric Nash Collative Differential Games, Math. Comput.Model.,**Vol.26**, No.2, (1997), pp. 97-105.

[8] E.Youness and A.A.Megahed, A study on Fuzzy Differential Game, Le Matematche, **Vol. VI**, Fasc.1, (2001), pp.97-107.

[9] E.Youness, A.A.Megahed, A Study on Large Scale Continuous Differential Games, Bull. Cul. Math. Soc.,**94(5)**, (2002) 359-368.

[10] S .Hegazy, A.A Megahed,E.Youness and A. Elbanna , Min-Max Zero-Sum two Persons Fuzzy Continuous Differential Games, IJAM, **Vol.21**, No.1, (2008), pp. 1-16.

[11] A.A. Megahed, S. Hegazy, Min-Max Zero two Persons Continuous Differential Game with Fuzzy Control, AJCEM **2**:, 2 March (2013) p.86-98

[12] A.J. Nova,G.Feichtinger,G.Leitmann, A differential game related to terrorism: Nash and Stackelberg strategies, J. Optim. Theory Appl. **144**, (2010), 533-555

[13] Abhra Roy, Jomon Aliyas Paul, Terrorism deterrence in a two country framework: strategic interactions between R&D, defense and pre-emption, Ann.Oper. Res., **V.211**,Issue 1 (December 2013), pp.399-432.

[14] Abd El-Monem.A. Megahed, A differential game related to terrorism: Min-Max Zero-Sum two persons differential game, NCA, **published at 28-11-2016** ,pp.1-6

[15] Abd El-Monem.A. Megahed (2017) The development of A differential Game Related to Terrorism:Min-Max Differential Game,JEMS,**25**, Issue 3, pp. 308-312

[16] Enders, W. and T. Sandler (2012) Political Economy of Terrorism, Cambridge University Press. Cambridge.

[17] Enders, W. and T. Sandler (1993) The effectiveness of anti-terrorism policies: A vectorautoregression-intervention analysis, American Political Science Review 87, 829-844.

[18] Faria, J.R. and D. Arce (2005) Terror support and recruitment, Defence and Peace Economics 16, 263-273.

[19] Abd El-Monem A. Megahed, The Stackelberg diferential game forcounter-terrorism, Quality & Quantity **published at 19-3-2018** pp.1-14