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On the existence of solution to multidimensional third order nonlinear equations

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Abstract. In this paper, we prove existence of an almost everywhere solution to mixed problem for a class of third order differential equations by non-zero rotation principle. Also studied the correctness of the formulation of the considered problem.

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1. Introduction

The paper is devoted to the problem of existence of almost everywhere solution and correctness of the formulation to the following multidimensional mixed problem for the third order nonlinear equation:

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial t} L(u(t,x)) = \mathrm{F}(u(t,x)) \quad (t \in [0,T], x \in \Omega), \tag{1}$$

$$u(0,x) = \varphi(x) \quad (x \in \Omega), \quad u_t(0,x) = \psi(x) \quad (x \in \Omega), \tag{2}$$

$$u(t,x)|_{\Gamma} = 0, \tag{3}$$

where $0 < T < +\infty$; $x = (x_1, ..., x_n)$, Ω is a bounded n dimensional domain with an enough smooth boundary S; $\Gamma = [0,T] \times S$;

$$L(u(t,x)) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u(t,x)}{\partial x_j} \right) - a(x)u(t,x), \tag{4}$$

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functions $a_{ij}(x)$ $(i,j=\overline{1,n})$ and $a(x)$ are measurable, and bounded in Ω and satisfy in Ω the following conditions:

$$a_{ij}(x)=a_{ji}(x),\ a(x)\geq 0,\ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha\cdot\sum_{i=1}^{n}\xi_i^2\ \ (\alpha=const>0)$$

where ξ_i $(i=1,2,\ldots,n)$ are arbitrary real numbers; φ, ψ are the given functions; F is some, generally speaking, nonlinear operator, and $u(t,x)$ is a sought function.

Note that the results of this work improve the results of our article [1]. There have been many works devoted to the study of mixed problems for nonlinear third order equations (see [2, 3, 5, 8, 10, 12] and references therein), where the problem of existence and uniqueness in appropriate spaces, the problem of blow up of solutions and the problems of asymptotic behavior of solutions are studied.

As well as we know the equations considered in previous publications do not cover the class of equations we study.

Considered by us equations appear in modeling dynamical processes, in elasticity theory and in modeling dynamics of shallow water waves (see [7,11]).

2. Auxiliaries

In what follows we are using the following notations and facts.

1. We denote by $\dot{D}(\Omega)$ the class of all continuously differentiable functions on Ω which vanished near the boundary of Ω. The closure of $\dot{D}(\Omega)$ with respect to the norm of $W_2^1(\Omega)$ we denote by $\overset{\circ}{D}(\Omega)$. Hence $\overset{\circ}{D}(\Omega)\subset W_2^1(\Omega)$.

Denote $\dot{D}_1(Q_T)$ $(Q_T\equiv[0,T]\times\Omega)$ the class of all continuously differentiable functions on the cylinder Q_T are equal to zero in the δ neighborhood of the lateral surface on the cylinder Q_T, having the form: $Q_{T,\delta}\equiv[0,T]\times\Omega_\delta$ where Ω_δ is a δ neighborhood of the boundary of Ω. The closure of $\dot{D}_1(Q_T)$ with respect to the norm of $W_2^1(Q_T)$ we denote by $\overset{\circ}{D}_1(Q_T)$. Hence $\overset{\circ}{D}_1(Q_T)\subset W_2^1(Q_T)$.

Definition. The function $u(t,x)\in\overset{\circ}{D}_1(Q_T)$ belonging to the space $L_2(Q_T)$ together with all its derivatives $u_t(t,x)$,$u_{x_i}(t,x)$ $(i=\overline{1,n})$, $u_{tx_i}(t,x)$ $(i=\overline{1,n})$, $u_{x_ix_j}(t,x)$ $(i,j=\overline{1,n})$, $u_{tt}(t,x)$, $u_{tx_ix_j}(t,x)$ $(i,j=\overline{1,n})$ satisfying equation (1) almost everywhere in Q_T and taking initial values (2) almost everywhere in Ω is called an almost everywhere solution of the problem (1)-(3).

2. For investigation of the problem (1)-(3) we recall one property of the operator L, generating by the differential expression (4) and boundary condition (3): there are denumerable number of negative eigenvalues

$$0>-\lambda_1^2\geq-\lambda_2^2\geq...\geq-\lambda_s^2\geq...,\quad(0<\lambda_s\to+\infty\ \ as\ s\to\infty)$$

with the corresponding generalized eigenfunctions $v_s(x)$ which are complete and orthonormal in $L_2(\Omega)$. We call function $v_s(x)\in\overset{\circ}{D}(\Omega)$ a generalized eigenfunction of the operator

L, if it is not identically zero and

$$\int\limits_{\Omega}\left\{\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial v_s(x)}{\partial x_i}\cdot\frac{\partial\Phi(x)}{\partial x_j}+a(x)v_s(x)\Phi(x)\right\}dx=\lambda_s^2\int\limits_{\Omega}v_s(x)\Phi(x)dx$$

for any function $\Phi(x)\in\overset{\circ}{D}(\Omega)$.

As the system $\{v_s(x)\}_{s=1}^{\infty}$ is complete orthonormal in $L_2(\Omega)$, then it is evident that every almost everywhere solution of problem (1)-(3) has the following form:

$$u(t,x)=\sum_{s=1}^{\infty}u_s(t)v_s(x),$$

where

$$u_s(t)=\int\limits_{\Omega}u(t,x)v_s(x)dx\quad(s=1,2,...).$$

Then, after applying the Fourier method, finding the unknown Fourier coefficients $u_s(t)$ $(s=1,2,...)$ for the almost everywhere solution $u(t,x)$ of the problem (1)-(3) is reduced to the solution of the following countable system of nonlinear integro-differential equations:

$$u_s(t)=\varphi_s+\frac{1}{\lambda_s^2}(1-e^{-\lambda_s^2t})\psi_s$$

$$+\frac{1}{\lambda_s^2}\int\limits_0^t\int\limits_{\Omega}\mathrm{F}(u(\tau,x))\cdot\left[1-e^{-\lambda_s^2(t-\tau)}\right]v_s(x)dxd\tau\quad(s=1,2,...;t\in[0,T]),\tag{5}$$

where

$$\varphi_s=\int\limits_{\Omega}\varphi(x)v_s(x)dx,\quad\psi_s=\int\limits_{\Omega}\psi(x)v_s(x)dx\quad(s=1,2,...).$$

Proceeding from the definition of almost every where solution of problem (1)-(3), it is easy to prove (see [1]) the following

Lemma. If $u(t,x)=\sum\limits_{s=1}^{\infty}u_s(t)v_s(x)$ is any almost everywhere solution of problem (1)-(3) and the generalized derivatives $\frac{\partial}{\partial x_k}a_{ij}(x)$ $(i,j,k=1,2,\ldots,n)$ are bounded on Ω, then functions $u_s(t)$ $(s=1,2,...)$ satisfy system (5).

3. We denote by $B^{\alpha_0,...,\alpha_l}_{\beta_0,...,\beta_l,T}$ a totality of all the functions of the from

$$u(t,x)=\sum_{s=1}^{\infty}u_s(t)v_s(x)$$

considered in $Q_T = [0,T] \times \Omega$, where $u_s(t) \in C^{(l)}([0,T])$ for all s and

$$N_T(u) \equiv \sum_{i=0}^{l} \left\{ \sum_{s=1}^{\infty} \left(\lambda_s^{\alpha_i} \cdot \max_{0 \le t \le T} \left| u_s^{(i)}(t) \right| \right)^{\beta_i} \right\}^{\frac{1}{\beta_i}} < +\infty,$$

with $\alpha_i \geq 0, \quad 1 \leq \beta_i \leq 2 \quad (i = 0, 1, \ldots, n)$. We define the norm in this set as $\|u\| = N_T(u)$. It is evident that all these spaces are Banach spaces ([6, p.50]).

4. Let G be class all functions $u(t,x)$ which have the properties

$$u(t,x), u_t(t,x), u_{x_i}(t,x) \ (i = \overline{1,n}), u_{tx_i}(t,x) \ (i = \overline{1,n}), u_{x_i x_j}(t,x) \ (i,j = \overline{1,n}), u_{tt}(t,x),$$

$$u_{tx_i x_j}(t,x) \ (i,j = \overline{1,n}) \in L_2(Q_T).$$

3. On the existence of almost everywhere solution

In this section, using non-zero rotation principle, the following existence theorem for the almost everywhere solution of problem (1)-(3) is proved for n:

Theorem 1. Let

(i) $a_{ij}(x) \in C^{(2)}(\bar{\Omega}) \ (i,j = \overline{1,n})$; $a(x) \in C^{(1)}(\bar{\Omega})$; $S \in C^{(3)}$; the eigenfunctions $v_s(x)$ of the operator L under boundary condition $v_s(x)|_S = 0$ be three times continuously differentiable on $\bar{\Omega}$; $\varphi(x) \in W_2^3(\Omega)$; $\varphi(x), \ L\varphi(x) \in \overset{\circ}{D}(\Omega)$; $\psi(x) \in W_2^2(\Omega) \bigcap \overset{\circ}{D}(\Omega)$.

2. $\mathrm{F} = \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3$, where

a) the operator F_1 acts from the $B_{2,T}^2$ into the space $W_{t,x,2}^{0,1}(Q_T)$ continuously and for all $u \in B_{2,T}^2$, $t \in [0,T]$:

$$\|\mathrm{F}_1(u(t,x))\|_{W_2^1(\Omega)} \leq a_1(t) + a_2(t) \cdot \|u\|_{B_{2,T}^2}^{\gamma} + a_3(t) \cdot \|u\|_{B_{2,t}^2}, \tag{6}$$

where $a_i(t) \in L_2(0,T)$ $(i = 1,2,3)$ and $0 < \gamma < 1$;

b) the operator F_2 acts from the closed ball $K^* \left(\|u\|_{B_{2,T}^2} \leq \frac{1}{\lambda_1} a \right)$ into the space $W_{t,x,2}^{0,1}(Q_T)$ continuously, where

$$a > a_0 \equiv \max\{y : \ y^2 \leq (A_1 + A_2 \, |y|^{2\gamma}) \cdot A_3\}, \tag{7}$$

$$A_1 \equiv 2 \, \|w(t,x)\|_{B_{2,2,T}^{3,2}}^2 + 6(2T+1) \cdot C_0^2 \cdot \|a_1(t)\|_{L_2(0,T)}^2, \tag{8}$$

$$A_2 \equiv 6(2T+1) \cdot C_0^2 \cdot \frac{1}{\lambda_1^{2\gamma}} \cdot \|a_2(t)\|_{L_2(0,T)}^2, \tag{9}$$

$$A_3 \equiv \exp\left\{6(2T+1)\cdot C_0^2 \cdot \frac{1}{\lambda_1^2}\cdot \|a_3(t)\|^2_{L_2(0,T)}\right\}, \tag{10}$$

$$w(t,x) = \sum_{s=1}^{\infty}\left\{\varphi_s + \frac{1}{\lambda_s^2}\left[1-e^{-\lambda_s^2 t}\right]\psi_s\right\}\cdot v_s(x), \tag{11}$$

$$C_0 \equiv \max\left\{n\cdot \max_{i,j=\overline{1,n}}\{\|a_{ij}(x)\|_{C(\bar{\Omega})}\}, \|a(x)\|_{C(\bar{\Omega})}\right\}^{\frac{1}{2}}; \tag{12}$$

c)

$$\inf_{u\in M}\{\|u-Q_1(u)\|_{B^{3,2}_{2,2,T}} - \|Q_2(u)\|_{B^{3,2}_{2,2,T}}\} > 0, \tag{13}$$

where M the boundary of the ball $K(\|u\|_{B^{3,2}_{2,2,T}} \leq a)$, $Q_1(u) = w + P(\mathrm{F}_1(u))$, $Q_2(u) = P(\mathrm{F}_2(u))$ and

$$P(u(t,x)) \equiv \sum_{s=1}^{\infty}\frac{1}{\lambda_s^2}\int_0^t\int_\Omega u(\tau,\xi)v_s(\xi)\cdot\left[1-e^{-\lambda_s^2(t-\tau)}\right]d\xi d\tau\cdot v_s(x); \tag{14}$$

d) the operator F_3 acts from the closed ball $K_\rho(\|u\|_{B^{3,2}_{2,2,T}} \leq \rho)$ into the space $W^{0,1}_{t,x,2}(Q_T)$ and for all $u, v \in K_\rho$:

$$\|\mathrm{F}_3(u)-\mathrm{F}_3(v)\|_{W^{0,1}_{t,x,2}(Q_T)} \leq q\cdot\|u-v\|_{B^{3,2}_{2,2,T}}, \tag{15}$$

where

$$\rho \geq a, \ \ \rho \geq \rho_0 \equiv \sup_{u\in K}\{\|Q_1(u)+Q_2(u)\|_{B^{3,2}_{2,2,T}}\},$$

$$\left(\sqrt{T}+\frac{1}{\sqrt{2}}\right)\cdot C_0\cdot q \equiv q_0 \leq 1-\frac{\rho_0}{\rho}, \ \ q_0<1; \tag{16}$$

e)

$$\inf_{u\in M}\{\|u-Q_1(u)-Q_2(u)\|_{B^{3,2}_{2,2,T}} - \|Q_3(u)\|_{B^{3,2}_{2,2,T}}\} > 0, \ \ Q_3(u) = P(\mathrm{F}_3(u)); \tag{17}$$

f) $\mathrm{F}_3(0) = 0$.

3. a) For any $u \in B^{3,2}_{2,2,T}$ for almost all $t\in[0,T]$, $\mathrm{F}_1(u(t,x)) \in \overset{\circ}{D}(\Omega)$;

b) For any $u \in K$ for almost all $t\in[0,T]$, $\mathrm{F}_2(u(t,x)) \in \overset{\circ}{D}(\Omega)$;

c) For any $u \in K_\rho$ for almost all $t\in[0,T]$, $\mathrm{F}_3(u(t,x)) \in \overset{\circ}{D}(\Omega)$.

Then problem (1)-(3) has an almost everywhere solution.

Proof. Using condition 3 of this theorem, we have

$$Q_1(u(t,x)) = w(t,x) + \sum_{s=1}^{\infty} \frac{1}{\lambda_s^3} \int_0^t \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \mathrm{F}_1(u(\tau,\xi)) \cdot \frac{\partial}{\partial \xi_j} \left(\frac{v_s(\xi)}{\lambda_s} \right) \right.$$

$$\left. + a(\xi)\mathrm{F}_1(u(\tau,\xi)) \cdot \frac{v_s(\xi)}{\lambda_s} \right] \cdot \left[1 - e^{-\lambda_s^2 (t-\tau)} \right] d\xi d\tau \cdot v_s(x) \quad \forall u \in B_{2,2,T}^{3,2}, \tag{18}$$

$$Q_2(u(t,x)) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s^3} \int_0^t \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \mathrm{F}_2(u(\tau,\xi)) \cdot \frac{\partial}{\partial \xi_j} \left(\frac{v_s(\xi)}{\lambda_s} \right) \right.$$

$$\left. + a(\xi)\mathrm{F}_2(u(\tau,\xi)) \cdot \frac{v_s(\xi)}{\lambda_s} \right] \cdot \left[1 - e^{-\lambda_s^2 (t-\tau)} \right] d\xi d\tau \cdot v_s(x) \quad \forall u \in K, \tag{19}$$

$$Q_3(u(t,x)) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s^3} \int_0^t \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \mathrm{F}_3(u(\tau,\xi)) \cdot \frac{\partial}{\partial \xi_j} \left(\frac{v_s(\xi)}{\lambda_s} \right) \right.$$

$$\left. + a(\xi)\mathrm{F}_3(u(\tau,\xi)) \cdot \frac{v_s(\xi)}{\lambda_s} \right] \cdot \left[1 - e^{-\lambda_s^2 (t-\tau)} \right] d\xi d\tau \cdot v_s(x) \quad \forall u \in K_\rho. \tag{20}$$

It is easy to obtain that, for any $u, v \in B_{2,2,T}^{3,2}$

$$\|Q_1(u) - Q_1(v)\|_{B_{2,2,T}^{3,2}} \le \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \left\{ \int_0^T \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}(\xi) \frac{\partial}{\partial \xi_i} (\mathrm{F}_1(u(\tau,\xi)) \right. \right.$$

$$\left. \left. - \mathrm{F}_1(v(\tau,\xi))) \cdot \frac{\partial}{\partial \xi_j} (\mathrm{F}_1(u(\tau,\xi)) - \mathrm{F}_1(v(\tau,\xi))) + a(\xi)(\mathrm{F}_1(u(\tau,\xi)) - \mathrm{F}_1(v(\tau,\xi)))^2 \right] d\xi d\tau \right\}^{\frac{1}{2}}$$

$$\le \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot C_0 \cdot \|\mathrm{F}_1(u(t,x)) - \mathrm{F}_1(v(t,x))\|_{W_{t,x,2}^{0,1}(Q_T)} . \tag{21}$$

From (21) by virtue of the condition 2a this theorem it follows that the operator Q_1 acts continuously from the $B_{2,T}^2$ into $B_{2,2,T}^{3,2}$. Since, the space $B_{2,2,T}^{3,2}$ imbedded into the space $B_{2,T}^2$ compactly ([6, Theorem 1.1, p.51]), then the operator Q_1 acts in the $B_{2,2,T}^{3,2}$ compactly.

We consider in $B_{2,2,T}^{3,2}$ the equations

$$u = \mu Q_1(u) \quad \mu \in [0,1], \tag{22}$$

and a priori estimate their all the possible solutions $u_\mu(t,x)$. Then, using inequality (6) $\forall \mu \in [0,1]$ and $t \in [0,T]$ we have

$$\|u_\mu\|^2_{B^{3,2}_{2,2,t}} = \|\mu Q_1(u_\mu)\|^2_{B^{3,2}_{2,2,t}} \le \|Q_1(u_\mu)\|^2_{B^{3,2}_{2,2,t}} \equiv \|w + P(\mathrm{F}_1(u_\mu(t,x)))\|^2_{B^{3,2}_{2,2,t}}$$

$$\le 2\,\|w\|^2_{B^{3,2}_{2,2,T}} + 2(2T+1)\cdot C_0^2 \cdot \int\limits_0^t \|\mathrm{F}_1(u_\mu(\tau,x))\|^2_{W_2^1(\Omega)}\,d\tau$$

$$\le 2\,\|w\|^2_{B^{3,2}_{2,2,T}} + 6(2T+1)\cdot C_0^2 \cdot \left\{ \|a_1(t)\|^2_{L_2(0,T)} + \|a_2(t)\|^2_{L_2(0,T)} \cdot \frac{1}{\lambda_1^{2\gamma}} \cdot \|u_\mu\|^{2\gamma}_{B^{3,2}_{2,2,T}} \right\}$$

$$+6(2T+1)\cdot C_0^2 \cdot \frac{1}{\lambda_1^2} \int\limits_0^t a_3^2(\tau)\cdot \|u_\mu\|^2_{B^{3,2}_{2,2,\tau}}\,d\tau, \tag{23}$$

where C_0 is defined by (12).

From (23), on applying Bellman's inequality [4, pp. 188,189] and using notations (8)-(10), we obtain that $\forall \mu \in [0,1]$:

$$\|u_\mu\|^2_{B^{3,2}_{2,2,T}} \le \left(A_1 + A_2 \cdot \|u_\mu\|^{2\gamma}_{B^{3,2}_{2,2,T}}\right) \cdot A_3.$$

From here, using notation (7), we have

$$\|u_\mu(t,x)\|^2_{B^{3,2}_{2,2,T}} \le a_0 \quad \forall \mu \in [0,1], \tag{24}$$

that is, all the possible solutions u_μ of equations (22) are a priori bounded in $B^{3,2}_{2,2,T}$ and belong to the ball $K_0(\|u\|_{B^{3,2}_{2,2,T}} \le a_0)$.

From (22) and (24) we obtain that $\forall \mu \in [0,1]$ completely continuous vector field $T_\mu = J - \mu Q_1$ has no zeros on the boundary M of the ball $K(\|u\|_{B^{3,2}_{2,2,T}} \le a)$, where J is a unit vector field and a is a number appearing in the condition 2b this theorem. Consequently, completely continuous vector fields $T_0 = J$ and $T_1 = J - Q_1$ are homotopic on the sphere M. Then their rotation δ on M are the same, namely:

$$\delta(J - Q_1; M) = \delta(J; M) = 1.$$

Now, we consider the operator Q_2 in the closed ball K. Just as the completely continuity of the operator Q_1 in $B^{3,2}_{2,2,T}$, was shown it is easy to show that the operator Q_2 acts compactly from K into $B^{3,2}_{2,2,T}$. Further, on the boundary M of the ball K we consider completely continuous vector fields $F_\lambda = J - Q_1 - \lambda Q_2,\ \ \lambda \in [0,1]$. Due to of the condition 2b this theorem $\forall \lambda \in [0,1]$ and $u \in M$ we have

$$\|u - Q_1(u) - \lambda Q_2(u)\|_{B^{3,2}_{2,2,T}} \ge \|u - Q_1(u)\|_{B^{3,2}_{2,2,T}} - \|Q_2(u)\|_{B^{3,2}_{2,2,T}} > 0.$$

Hence, in particular, it follows that completely continuous vector fields $F_0 = J - Q_1$ and $F_1 = J - Q_1 - Q_2$ are homotopic on the sphere M. Consequently, on M their rotations are equal to:

$$\delta(J - Q_1 - Q_2; M) = \delta(J - Q_1; M) = \delta(J; M) = 1. \tag{25}$$

And now in the ball $K_\rho(\|u\|_{B^{3,2}_{2,2,T}} \leq \rho)$ we consider the operator Q_3. Similar to (21), $\forall u, v \in K_\rho$ we have

$$\|Q_3(u) - Q_3(v)\|_{B^{3,2}_{2,2,T}} \leq \left(\sqrt{T} + \frac{1}{\sqrt{2}}\right) \cdot C_0 \cdot \|\mathrm{F}_3(u) - \mathrm{F}_3(v)\|_{W^{0,1}_{t,x,2}(Q_T)}$$

$$\leq \left(\sqrt{T} + \frac{1}{\sqrt{2}}\right) \cdot C_0 \cdot q \cdot \|u - V\|_{B^{3,2}_{2,2,T}} = q_0 \cdot \|u - V\|_{B^{3,2}_{2,2,T}} .$$

For each fixed $V_0 \in K_{\rho_0}(\|u\|_{B^{3,2}_{2,2,T}} \leq \rho_0)$ (where the number ρ_0 is defined by (16)) and $\varepsilon \in [0, 1]$ in the ball K_ρ we consider the following equation

$$u = \varepsilon \cdot Q_3(u) + V_0. \tag{26}$$

Taking advantage fact that

$$Q_3(0) = P(\mathrm{F}_3(0)) = P(0) = 0,$$

for any $u, u_1, u_2 \in K_\rho$ we have

$$\|\varepsilon \cdot Q_3(u) + V_0\|_{B^{3,2}_{2,2,T}} \leq \|Q_3(u)\|_{B^{3,2}_{2,2,T}} + \|V_0\|_{B^{3,2}_{2,2,T}}$$

$$= \|Q_3(u) - Q_3(0)\|_{B^{3,2}_{2,2,T}} + \|V_0\|_{B^{3,2}_{2,2,T}} \leq \left(\sqrt{T} + \frac{1}{\sqrt{2}}\right) \cdot C_0 \cdot q \cdot \|u\|_{B^{3,2}_{2,2,T}} + \|V_0\|_{B^{3,2}_{2,2,T}}$$

$$= q_0 \cdot \|u\|_{B^{3,2}_{2,2,T}} + \|V_0\|_{B^{3,2}_{2,2,T}} \leq q_0 \cdot \rho + \rho_0 \leq \rho,$$

$$\|\varepsilon \cdot Q_3(u_1) + V_0 - [\varepsilon \cdot Q_3(u_2) + V_0]\|_{B^{3,2}_{2,2,T}}$$

$$\leq \|Q_3(u_1) - Q_3(u_2)\|_{B^{3,2}_{2,2,T}} \leq q_0 \|u_1 - u_2\|_{B^{3,2}_{2,2,T}} .$$

As $q_0 < 1$, then by virtue of the contracted mappings principle, the operator $A(u) = \varepsilon \cdot Q_3(u) + V_0$ has a unique fixed point u_0 in K_ρ. Comparing to each $V_0 \in K_{\rho_0}$ the unique in K_ρ solution $u_0 \in K_{\rho_0}$ of equation (26), we generate some operator R_ε, acting from K_{ρ_0} into K_ρ. Next, for any $\varepsilon \in [0, 1]$ and $V \in K_{\rho_0}$ using notation

$$R_\varepsilon(V) \equiv \varepsilon \cdot Q_3(R_\varepsilon(V)) + V,$$

it is easy to get for any $V_1, V_2 \in K_{\rho_0}$

$$\|R_\varepsilon(V_1) - R_\varepsilon(V_2)\|_{B^{3,2}_{2,2,T}} \leq \frac{1}{1-\varepsilon \cdot q_0} \cdot \|V_1 - V_2\|_{B^{3,2}_{2,2,T}}. \tag{27}$$

Due to notation (16) we have $(Q_1 + Q_2)K \subset K_{\rho_0}$. Consequently, for each $\varepsilon \in [0,1]$ the operator R_ε is defined, in particular, and on $(Q_1 + Q_2)K$. Due to (27) the operator R_ε satisfies a Lipschitz condition (and therefore continuous) on $(Q_1 + Q_2)K$. And as, the operators Q_1 and Q_2 are completely continuous on K, then for each $\varepsilon \in [0,1]$ the operator $R_\varepsilon(Q_1 + Q_2)$ compactly on K. Further, due to (17), for any $u \in M$ and $\varepsilon \in [0,1]$ we have

$$\|u - Q_1(u) - Q_2(u) - \varepsilon \cdot Q_3(u)\|_{B^{3,2}_{2,2,T}}$$

$$\geq \|u - Q_1(u) - Q_2(u)\|_{B^{3,2}_{2,2,T}} - \|Q_3(u)\|_{B^{3,2}_{2,2,T}} > 0.$$

Consequently, the completely continuous vector field $J - Q_1 - Q_2 - \varepsilon \cdot Q_3$ for any $\varepsilon \in [0,1]$ does not have zeros on M. Then does not have zeros on M same way the completely continuous vector field $J - R_\varepsilon(Q_1 + Q_2)$, because each zero on M field $J - R_\varepsilon(Q_1 + Q_2)$ is zero field $J - Q_1 - Q_2 - \varepsilon \cdot Q_3$.

Thus, the completely continuous vector fields $J - R_0(Q_1 + Q_2) = J - Q_1 - Q_2$ and $J - R_1(Q_1 + Q_2)$ are homotopic on M. Consequently, due to (25) we have

$$\delta(J - R_1(Q_1 + Q_2); M) = \delta(J - Q_1 - Q_2; M) = 1.$$

Hence, by virtue of the non-zero rotation principle [9, p. 207], the completely continuous vector field $J - R_1(Q_1 + Q_2)$ has at least one zero inside the ball of K. Since each such zero is a zero of the field $J - Q_1 - Q_2 - Q_3$, it is thus proved that there exists in K at least one fixed point $u(t,x)$ an operator $Q_1 + Q_2 + Q_3 = Q$. Further, it easy to verify (in absolutely the same way as in the proof of Theorem of [2]), that the function $u(t,x)$ is an almost everywhere solution of problem (1)-(3). The theorem is proved.

4. Correct formulation of the problem

In this section, using Bellman's inequality ([4, p. 188-189]), the following theorem on continuous dependence (in a certain sense) on initial functions $\varphi(x)$, $\psi(x)$, and nonlinear operator F for the almost everywhere solution of problem (1)-(3). Problem (1)-(3) with the data $\tilde{\varphi}$, $\tilde{\psi}$, $\tilde{\mathrm{F}}$ we let's name problem $\tilde{A}$.

Theorem 2. Let:

(i) Condition 1 of Theorem 1 be satisfied.

(ii) $\tilde{\varphi}(x) \in W_2^3(\Omega)$; $\tilde{\varphi}(x)$, $L\tilde{\varphi}(x) \in \overset{\circ}{D}(\Omega)$; $\tilde{\psi}(x) \in W_2^2(\Omega) \bigcap \overset{\circ}{D}(\Omega)$.

(iii) For each $u \in B^{3,2}_{2,2,T} \bigcup (G \bigcap B^{2,1}_{2,2,T})$ for almost all $t \in [0,T]$, $\mathrm{F}(u(t,x)) \in \overset{\circ}{D}(\Omega)$.

(iv) For each $u \in B^{3,2}_{2,2,T}$ for almost all $t \in [0,T]$, $\tilde{\mathrm{F}}(u(t,x)) \in \overset{\circ}{D}(\Omega)$.

(v) The operators F and $\tilde{\mathrm{F}}$ acts from $B^{3,2}_{2,2,T} \bigcup (G \bigcap B^{2,1}_{2,2,T})$ into $W^{0,1}_{t,x,2}(Q_T)$ so that, for all $u, v \in B^{3,2}_{2,2,T}$ and $t \in [0,T]$

$$\|\mathrm{F}(u(t,x))\|_{W^1_2(\Omega)} \le a(t) + b(t) \cdot \|u\|_{B^{3,2}_{2,2,T}}, \quad a(t),\ b(t) \in L_2(0,T),$$

$$\|\mathrm{F}(u(t,x)) - \mathrm{F}(v(t,x))\|_{W^1_2(\Omega)} \le c(t) \cdot \|u - v\|_{B^{3,2}_{2,2,T}}, \quad c(t) \in L_2(0,T),$$

$$\left\|\tilde{\mathrm{F}}(u(t,x)) - \tilde{\mathrm{F}}(v(t,x))\right\|_{W^1_2(\Omega)} \le \tilde{c}(t) \cdot \|u - v\|_{B^{3,2}_{2,2,T}}, \quad \tilde{c}(t) \in L_2(0,T),$$

$$\sup_{u \in K_1} \left\{ \left\| \mathrm{F}(u(t,x)) - \tilde{\mathrm{F}}(u(t,x)) \right\|_{W^{0,1}_{t,x,2}(Q_T)} \right\} \equiv \varepsilon < +\infty,$$

where

$$K_1 = K_1(\|u\|_{B^{3,2}_{2,2,T}} \le a_1),$$

$$a_1 \equiv \{[2\,\|w(t,x)\|^2_{B^{3,2}_{2,2,T}} + 4(2T+1) \cdot C_0^2 \cdot \|a(t)\|^2_{L_2(0,T)}] \cdot \exp[4(2T+1) \cdot C_0^2 \cdot \|b(t)\|^2_{L_2(0,T)}]\}^{\frac{1}{2}},$$

the function $w(t,x)$ is defined by (11) and the number C_0 is defined by (12).

Then for the unique almost everywhere solutions $u(t,x)$ and $\tilde{u}(t,x)$ of problems (1)-(3) and $\tilde{A}$, respectively, we have

$$\begin{aligned} \|u(t,x) - \tilde{u}(t,x)\|_{B^{3,2}_{2,2,T}} &\le \{\sqrt{3} \cdot C_0 \cdot \|L(\varphi(x) - \tilde{\varphi}(x))\|_{W^1_2(\Omega)} \\ &+ \sqrt{6} \cdot C_0 \cdot \left\|\psi(x) - \tilde{\psi}(x)\right\|_{W^1_2(\Omega)} + \sqrt{6} \cdot \left\|L(\psi(x) - \tilde{\psi}(x))\right\|_{L_2(\Omega)} \\ &+ \sqrt{6(2T+1)} \cdot C_0 \cdot \varepsilon\} \cdot \exp\{3(2T+1) \cdot C_0^2 \cdot \|\tilde{c}(t)\|^2_{L_2(0,T)}\}, \end{aligned} \tag{28}$$

where the operator L is defined by (4) and the number C_0 is defined by (12).

Proof: By Theorem 2 from work [1] each of the problems (1)-(3) and $\tilde{A}$ has a unique almost everywhere solution

$$u(t,x) = \sum_{s=1}^{\infty} u_s(t) v_s(x) \ \ \texttt{and} \ \ \tilde{u}(t,x) = \sum_{s=1}^{\infty} \tilde{u}_s(t) v_s(x),$$

respectively, so that $u \in K_1 \subset B^{3,2}_{2,2,T}$, $\tilde{u} \in B^{3,2}_{2,2,T}$. Then, by virtue of the lemma in section 2, the functions $u_s(t)$ $(s = 1, 2, \ldots)$ and $\tilde{u}_s(t)$ $(s = 1, 2, \ldots)$ satisfy system (5), so that for

$\tilde{u}_s(t)$ $(s = 1, 2, \dots)$ in the system (5) instead of φ_s, ψ_s and $\mathrm{F}(u)$ need to take $\tilde{\varphi}_s$, $\tilde{\psi}_s$ and $\tilde{\mathrm{F}}(u)$, respectively.

Using this fact, from system (5) it is easy to obtain that $\forall t \in [0, T]$:

$$\|u - \tilde{u}\|^2_{B^{3,2}_{2,2,t}} \leq 3 \left\| \sum_{s=1}^{\infty} (\varphi_s - \tilde{\varphi}_s) v_s(x) \right\|^2_{B^{3,2}_{2,2,t}} + 3 \left\| \sum_{s=1}^{\infty} \frac{1}{\lambda_s^2} (1 - e^{-\lambda_s^2 t})(\psi_s - \tilde{\psi}_s) v_s(x) \right\|^2_{B^{3,2}_{2,2,t}}$$

$$+3 \left\| \sum_{s=1}^{\infty} \frac{1}{\lambda_s^2} \int_0^t \int_\Omega [\mathrm{F}(u(\tau,\xi)) - \tilde{\mathrm{F}}(\tilde{u}(\tau,\xi))] \cdot [1 - e^{-\lambda_s^2 (t-\tau)}] v_s(\xi) d\xi d\tau \cdot v_s(x) \right\|^2_{B^{3,2}_{2,2,t}}$$

$$\leq 3 \sum_{s=1}^{\infty} [\lambda_s^3 (\varphi_s - \tilde{\varphi}_s)]^2 + 6 \sum_{s=1}^{\infty} [\lambda_s (\psi_s - \tilde{\psi}_s)]^2 + 6 \sum_{s=1}^{\infty} [\lambda_s^2 (\psi_s - \tilde{\psi}_s)]^2$$

$$+3(2T+1) C_0^2 \cdot \int_0^t \left\| \mathrm{F}(u(\tau,x)) - \tilde{\mathrm{F}}(\tilde{u}(\tau,x)) \right\|^2_{W_2^1(\Omega)} d\tau$$

$$\leq 3 \sum_{s=1}^{\infty} [\lambda_s^3 (\varphi_s - \tilde{\varphi}_s)]^2 + 6 \sum_{s=1}^{\infty} [\lambda_s (\psi_s - \tilde{\psi}_s)]^2 + 6 \sum_{s=1}^{\infty} [\lambda_s^2 (\psi_s - \tilde{\psi}_s)]^2$$

$$+6(2T+1) \cdot C_0^2 \cdot \left[\int_0^t \left\| \mathrm{F}(u(\tau,x)) - \tilde{\mathrm{F}}(u(\tau,x)) \right\|^2_{W_2^1(\Omega)} d\tau \right.$$

$$\left. + \int_0^t \left\| \tilde{\mathrm{F}}(u(\tau,x)) - \tilde{\mathrm{F}}(\tilde{u}(\tau,x)) \right\|^2_{W_2^1(\Omega)} d\tau \right]$$

$$\leq 3 \cdot C_0^2 \cdot \|L(\varphi(x) - \tilde{\varphi}(x))\|^2_{W_2^1(\Omega)} + 6 \cdot C_0^2 \cdot \left\| \psi(x) - \tilde{\psi}(x) \right\|^2_{W_2^1(\Omega)}$$

$$+6 \cdot \left\| L(\psi(x) - \tilde{\psi}(x)) \right\|^2_{L_2(\Omega)} + 6(2T+1) \cdot C_0^2 \cdot \left\| \mathrm{F}(u(\tau,x)) - \tilde{\mathrm{F}}(u(\tau,x)) \right\|^2_{W^{0,1}_{t,x,2}(Q_T)}$$

$$+6(2T+1) \cdot C_0^2 \cdot \int_0^t \tilde{c}^2(\tau) \cdot \|u - \tilde{u}\|_{B^{3,2}_{2,2,\tau}} d\tau \leq 3 \cdot C_0^2 \cdot \|L(\varphi(x) - \tilde{\varphi}(x))\|^2_{W_2^1(\Omega)}$$

$$+6 \cdot C_0^2 \cdot \left\| \psi(x) - \tilde{\psi}(x) \right\|^2_{W_2^1(\Omega)} + 6 \cdot \left\| L(\psi(x) - \tilde{\psi}(x)) \right\|^2_{L_2(\Omega)} + 6(2T+1) \cdot C_0^2 \cdot \varepsilon^2$$

$$+6(2T+1)\cdot C_0^2\cdot\int_0^t \tilde{c}^2(\tau)\cdot\|u-\tilde{u}\|^2_{B^{3,2}_{2,2,\tau}}\,d\tau.$$

From here, on applying Bellman's inequality ([4, p.188, 189]), we obtain the estimate (28).

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References

[1] S.Aliyev , A.Aliyeva , G.Abdullayeva; The study of a mixed problem for one class of third order differential equations, Advances in Difference Equations **208** (2018), 1-10.

[2] S.Aliyev, A.Aliyeva; On the existence for almost everywhere solution of multidimensional mixed problem for one class third order differential equations with nonlinear operator in the right-hand side, International Journal of Pure and Applied Matematics **115** (3), (2017), 549-560.

[3] S.Aliyev, A.Aliyeva; The study of multidimensional mixed problem for one class of third order semilinear psevdohyperbolic equations, European Journal of Pure and Applied Mathematics **10** (5), (2017), 1078-1091.

[4] E.Beckenbach, R.Bellman; Inequalities, Mir, 1965, 276p. (in Russian).

[5] G.Filippo, S.Marco; Global solutions and finite time blow up for damped semilinear wave equations, Ann. Inst. H. Poincare, Anal. Nonlineaire **23**(2), (2006), 185-207.

[6] K.Khudaverdiyev; Multidimensional mixed problem for nonlinear hyperbolic equations, Az. gostekn. University publ. Baku, 2011, 611p. (in Russian).

[7] O. Kosike; Existence of global and bounded solutions for damped sublinear wave equations, J. Math. Univ. Tokushima, **42**, (2008), 19-26.

[8] E.M. Mamedov; On stabilization of solution of the third order equation with nonlinear boundary conditions, Proc. Inst. Math. and Mech. Azerb. Acad. Sci., **16**, (2002), 75-80.

[9] Mathematical reference library; Functional analysis, Nauka, 1964, 424p. (in Russian).

[10] M.A. Salim; On the decay of solutions for a class of quasilinear hyperbolic equations with nonlinear damping and source terms, Math. Meth. Appl. Sci., **15**, (2005), 1819-1828.

[11] Z. Yang; Cauchy problem for quasilinear wave equations with nonlinear damping and source terms, J. Math. Anal. and Appl., **300**(1), (2004), 218-243.

[12] Y. Zhifeng, Q. Dehua; Energy decaying and blow up of solution for a Kirchhoff equation with strong damping, J. Math. Res. And Expo., **29**(4), (2009), 707-715.