EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 2, 2019, 469-485
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

The Quotient Inequalities

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Abstract. This paper contributes to the field of inequalities, specifically, the relationships among the norms of products of elements or vectors or functions and their quotients. Thus, we established that the norm of product of two vectors or functions is less than or equal to the norm of its quotient if the norm of the denominator is less than or equal to one. On the other hand, we prove that the norm of the quotient of two vectors or functions is less than or equal to the norm of their product. In addition, we introduce the proofs of inequalities including the norm of index power of products and their quotients, and then applied these inequalities to estabish properties of some functional spaces, as well as, extention of some of the results in these functional spaces.

2010 Mathematics Subject Classifications: 44B56, 44B57

Key Words and Phrases: Quotient inequalities, first quotient inequality, second quotient inequality, index power quotient inequalities

1. Introduction

Inequalities play a central role in mathematical analysis with numerous applications in solving ill-posed differential equations, approximation theory, optimization theory, numerical analysis, probability theory and statistics. The authors in [1] obtained estimates relating martingale difference sequences in the complex uniformly convex spaces. In [2], they obtained some estimates for geometric inequalities and compared these inequalities. The authors in [3] provided an alternative way of proving the quasi-normed linear space through binomial inequalities.

In this paper, the new inequalities; first and second quotient inequalities, for the continuous mappings or operators on a real Hilbert space, complex Hilbert space, Banach space and Hölder's spaces and Sobolev spaces are provided. These quotient inequalities are used in obtaining various new inequalities for continuous functions of both selfadjoint and adjoint operators.

The section one of this paper contains the general overview of the inequalities with emphasizes on the quotient inequalities. In section 2, the preliminary results including

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DOI: https://doi.org/10.29020/nybg.ejpam.v12i2.3384

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definitions and theorem which are necessary in establishing the quotient inequalities are provided. The quotient inequalities; the first and second quotient inequalities, are introduced in section 3 of this paper. These inequalities are used to establish the relationships among the norms of product of functions and their quotient counterparts in L^p spaces, Hilbert space, Sobolev spaces, Holder's space, Banach space and unitary space.

2. Some Preliminary Results

In this section, the definitions regarding the quotient inequalities and index power inequalities are provided.

Definition 1 (First and Second Product Inequalities). *Let a_1 and a_2 be any two positive real numbers, then*

$$\begin{aligned}(i)\|a_1\|\|a_2\| &\leq \|a_1\| + \|a_2\|, \quad \forall a_1, a_2 \in [0,2].\\ (ii)\|a_1\| + \|a_2\| &\leq \|a_1\|\|a_2\|, \quad \forall a_1, a_2 \in [2,\infty).\end{aligned}$$

See [4].

Definition 2 (Product-normed Linear Space). *Let U be a linear space over $[0,2] \subseteq \mathbf{R}$. A vector space with a product-norm $u \to \|U\|$ satisfying real-valued function $\|\cdot\|$,*

$$\|\cdot\|_{pn} : U \to [0,\infty),$$

such that for arbitrary $u, v \in U,\ \alpha \in [0,2]$, the following conditions are satisfied:

$$\begin{aligned}P1.\ \|u\| &\geq 0,\ and\ \|u\| = 0,\ if\ and\ only\ if\ u = 0\\ P2.\ \|\alpha u\| &= |\alpha|\|u\|, \quad \alpha \in [0,2], and\ u \in U\\ P3.\ \|u+v\|_{pn} &\leq \|u\|_{pn} + \|v\|_{pn}, \quad \forall u, v \in U\end{aligned}$$

We call $\|\cdot\|_{pn}$ a product norm, if in addition to $P1 - P3$, the u and v satisfy

$$\begin{aligned}P4(a).\ \|u\|_{pn}\|y\|_{pn} &\leq \|u\|_{pn} + \|v\|_{pn} \quad \forall u, v \in [0,2] \ \ (first\ product\ inequality)\\ P4(b).\|u\| + \|v\| &\leq \|u\|\|v\|, \quad \forall u, v \in [2,\infty) \qquad (second\ product\ inequality).\end{aligned}$$

See [5].

Definition 3 (Young's Inequality). *For $1 < p < \infty$, q the conjugate of p, and any two positive numbers a and b, then*

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}a^q, \ \ p, q \leq 1.$$

See [6].

Definition 4 (Contractive Mapping)**.** *Let $T : X \to X$ be a mapping from a complete normed linear space X into itself. The Lipschitz continuity on T is said to be a contraction if*

$$\|T(u) - T(v)\| \leq \lambda \|u - v\|, \quad \forall\ u,\ v \in X,\ and\ 0 < \ \lambda < 1.$$

See [7].

Definition 5. *Let $\gamma \in (0, 1]$. We say a function $T : X \to Y$ is Hölder continuous of exponent γ at $x_o \in X$, if*

$$\|u(x) - u(x_o)\| \leq L\|x - x_o\|^{\gamma},$$

where L is boundedness constant may depend on X, x_o, γ and T. See [8].

Theorem 1 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $1 \leq p < n$. Then there exists a constant $C > 0$ (depending on p and n) such that*

$$\|u\|_{p*,\mathbf{R}^n} \leq C\|\nabla u\|_{p,\mathbf{R}^n}, \quad u \in W^{1,p}(\mathbf{R}^n).$$

In particular, we have the continuous imbedding

$$W^{1,p}(\mathbf{R}^n) \hookrightarrow L^{p*}(\mathbf{R}^n).$$

For example, see authors in [9].

Definition 6. *Let A and B be selfadjoint operators with $S_p(A),\ S_p(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbf{R}$ is of $r - L-$Hölder type. Thus, for a given $r \in (0, 1]$ and $L > 0$, we have*

$$\Big|f(s) - f(t)\Big| \leq L\Big|s - t\Big|^r, \quad \forall\ \ s, t \in [m, M],$$

then we have the Ostrowski type inequality for selfadjoint operators:

$$\Big|f(s) - \langle f(A)x, x\rangle\Big| \leq \Big[\frac{1}{2}(M - m) + |s - \frac{m + M}{2}|\Big]^r, \ \ \forall\ \ s \in [m, M]\ and\ x \in H\ with\ \|x\| = 1.$$

Moreover, we have

$$\begin{aligned}
\Big|\langle f(B)y, y\rangle - \langle f(A)x, x\rangle\Big| \ &\leq\ \Big\langle\Big|f(B) - \langle f(A)x, x\rangle \cdot 1_H\Big|_y, y\Big\rangle. \\
&\leq\ L\Big[\frac{1}{2}(M - m) + \langle|B - \frac{m + M}{2} \cdot 1_H|y, y\rangle\Big]^r, \\
&\quad\ \forall\ \ x, y \in H\ \ with\ \ \|x\| = \|y\| = 1.
\end{aligned}$$

See [10].

3. Main Result

In this section, we derive both the first and second quotient inequalities in a suitable functional space.

Theorem 2 (First Quotient Inequality)**.** *Suppose that x and y are two real numbers or two real-valued vectors, then*

$$\|x\|\|y\| \leq \frac{\|x\|}{\|y\|}, \quad \forall \ 0 \leq \|x\| < \infty \ \textit{and} \ 0 < \|y\| \leq 1,$$

where the equality occurs at either $\|x\| = 0$ or $\|y\| = 1$.

Proof: Setting $f(x,y) = -\nu x^2 - 2\nu x^2 y^2$, the function attains its maximum value at zero, for all $(x,y) \in \mathbf{R}^2$ and $\nu \in [0,1]$. We can see that:

$$\begin{aligned} f(x,y) &= -\Big(\nu x^2 + 2\nu x^2 y^2\Big) \leq 0 \\ f(x,y) &= -\Big(\nu x^2 + x^2(2\nu y^2)\Big) \leq 0 \\ f(x,y) &= -\Big(\nu x^2 + x^2(2\nu y^2)\Big) \leq -\Big(x^{2\nu} + x^2 y^{4\nu}\Big) \leq 0 \end{aligned}$$

By transitivity law, we obtain

$$\begin{aligned} -\Big(x^{2\nu} + x^2 y^{4\nu}\Big) &\leq 0 \\ \Rightarrow -x^2 y^{4\nu} &\leq x^{2\nu} \\ \Rightarrow \| - x^{2\nu} y^{2\nu}\| &= \|\frac{x^{2\nu}}{x^{2(1-\nu)} y^{2\nu}}\| \\ \Rightarrow \|x\|^{2\nu}\|y\|^{2\nu} &\leq \frac{\|x\|^{2\nu}}{\|x\|^{2(1-\nu)}\|y\|^{2\nu}}. \end{aligned} \tag{1}$$

Setting $\nu = 1$ in inequality (1) yields

$$\|x\|\|y\| \leq \frac{\|x\|}{\|y\|}. \tag{2}$$

We search for the regions for which $\|x\|$ and $\|y\|$ hold. Setting $\nu = \frac{1}{4}$ in inequality (1), we obtain

$$\|x\|^3 \leq \frac{1}{\|y\|^2}. \tag{3}$$

Again, plugging $\nu = \frac{1}{2}$ into inequality (1) yields

$$\|x\| \leq \frac{1}{\|y\|^2}. \tag{4}$$

From the inequalities (3) and (4), we consider three situations for which inequality (2) holds. We observe that

$$\begin{aligned} \|x\|^3 &< \|x\| \\ \Rightarrow x(x^2-1) &< 0 \\ \Rightarrow 0 &< \|x\| < 1. \end{aligned} \tag{5}$$

Also, we can see that the two inequalities in (3) and (4) are equal if,

$$\begin{aligned} \|x\|^3 &= \|x\| \\ \Rightarrow x(x^2-1) &= 0 \\ \Rightarrow \|x\| &= 0, \ \textit{or} \ \|x\| = 1. \end{aligned} \tag{6}$$

Lastly, the norms on the left hand sides of inequalities (3) and (4) can be written as:

$$\begin{aligned} \|x\|^3 &> \|x\|. \\ \Rightarrow x(x^2-1) &> 0 \\ \Rightarrow \|x\| &> 0, \ \textit{or} \ \|x\| > 1. \end{aligned} \tag{7}$$

The region for which $\|y\|$ holds is as follows.

$$\begin{aligned} \|y\|^2 &\leq \frac{1}{\|x\|} \\ \Rightarrow \|y\|^2 &\leq 1, \quad \forall \ \|x\| = 1 \\ \Rightarrow 0 < \|y\| &\leq 1. \end{aligned} \tag{8}$$

Combining the inequalities (5), (7) and (8), and equation (6) together with inequality (2), we obtain

$$\|x\|\|y\| \leq \frac{\|x\|}{\|y\|}, \quad \forall \ 0 \leq \|x\| < \infty \ \textit{and} \ 0 < \|y\| \leq 1.$$

Theorem 3 (Second Quotient Inequality). *Suppose that x and y are any two real numbers or any two real-valued vectors, then*

$$\frac{\|x\|}{\|y\|} \leq \|x\|\|y\|, \quad \forall \ \|x\| \geq 1 \ \textit{and} \ \|y\| \geq 1,$$

where the equality occurs at either $\|x\| = \|y\| = 1$ or $\|y\| = 1$.

Proof: We can see that $f(x,y) = \nu x^2 + 2\nu x^2 y^2$ attains its minimum value at zero, for all $(x,y) \in \mathbf{R}^2$ and $\nu \in [0,1]$.

$$f(x,y) \ = \ \nu x^2 + 2\nu x^2 y^2 \geq 0$$

$$\begin{aligned}
f(x,y) &= \nu x^2 + x^2(2\nu y^2) \geq 0 \\
f(x,y) &= \nu x^2 + x^2(2\nu y^2) \geq x^{2\nu} + x^2y^{4\nu} \geq 0
\end{aligned}$$

By transitivity law, we obtain

$$\begin{aligned}
x^{2\nu} + x^2y^{4\nu} &\geq 0 \\
\Rightarrow -x^{2\nu} &\leq x^2y^{4\nu} \\
\Rightarrow \| - \frac{x^{2\nu}}{x^{2(1-\nu)}y^{2\nu}} \| &= \|x^{2\nu}y^{2\nu}\| \\
\Rightarrow \frac{\|x\|^{2\nu}}{\|x\|^{2(1-\nu)}\|y\|^{2\nu}} &\leq \|x\|^{2\nu}\|y\|^{2\nu}. \qquad (9)
\end{aligned}$$

Setting $\nu = 1$ in inequality (9) yields

$$\frac{\|x\|}{\|y\|} \leq \|x\|\|y\|.$$

The regions in which the above inequality holds are as follows. Setting $\nu = 0$ in inequality (9) yields

$$\begin{aligned}
\|x\|^2 &\geq 1 \\
\Rightarrow \|x\| &\geq 1.
\end{aligned}$$

In order to obtain the region for $\|y\|$, we set

$$\begin{aligned}
f(x,y) &= \nu x^2 + (2\nu x^2)y^2 \geq x^{2\nu} + x^2y^{4\nu} \geq 0 \\
\Rightarrow -x^{2\nu} &\leq x^{4\nu}y^2 \\
\Rightarrow \left\| \frac{-1}{y^{2(1-\nu)}} \right\| &= \left\| x^{2\nu}y^{2\nu} \right\| \\
\Rightarrow \frac{1}{\|y\|^{2(1-\nu)}} &\leq \|x\|^{2\nu}\|y\|^{2\nu}. \qquad (10)
\end{aligned}$$

Setting $\nu = 0$ in inequality (10) yields

$$\begin{aligned}
\|y\|^2 &\geq 1 \\
\Rightarrow \|y\| &\geq 1.
\end{aligned}$$

We observed that the equality occurs at either $\|x\| = \|y\| = 1$ or $\|y\| = 1$.

3.1. Illustration of the First Quotient Inequality to the Real Line

In this subsection, the illustration of the first quotient inequality is provided.

Example 1. . *Setting $a = \frac{2}{3}$ and $b = \frac{4}{5}$, then*

$$\begin{aligned}
\left\|\frac{2}{3}\right\|\left\|\frac{4}{5}\right\| &< \frac{\|\frac{2}{3}\|}{\|\frac{4}{5}\|} \\
\Rightarrow \frac{8}{15} &< \frac{5}{6}.
\end{aligned}$$

3.2. The Applications of the First Quotient Inequality to L^P Spaces

Theorem 4. *Suppose that $f(x)$ and $g(x)$ are measurable functions over the domain Ω such that*

$$\int_\Omega |f(x)|dx \leq +\infty.$$

and

$$\int_\Omega |g(x)|dx \leq 1,$$

then

$$\|f\|_p\|g\|_p \leq \frac{\|f\|_p}{\|g\|_p}.$$

Proof: We observe that:

$$\Big|\int_\Omega f(x)g(x)dx\Big| \leq \int_\Omega |f(x)||g(x)|dx.$$

Applying the first quotient inequality to the term on the right hand side of the above inequality, we obtain

$$\begin{aligned}
\Big|\int_\Omega f(x)g(x)dx\Big| &\leq \int_\Omega |f(x)||g(x)|dx \leq \int_\Omega \frac{|f(x)|}{|g(x)|}dx \\
\Rightarrow \int_\Omega |f(x)||g(x)|dx &\leq \int_\Omega \frac{|f(x)|}{|g(x)|}dx \\
\Rightarrow \int_\Omega |f(x)||g(x)|dx &\leq \frac{\int_\Omega |f(x)|dx}{\int_\Omega |g(x)|dx} \\
\Rightarrow \Big(\int_\Omega |f(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_\Omega |g(x)|^p dx\Big)^{\frac{1}{p}} &= \frac{\Big(\int_\Omega |f(x)|^p dx\Big)^{\frac{1}{p}}}{\Big(\int_\Omega |g(x)|^p dx\Big)^{\frac{1}{p}}} \\
\Rightarrow \|f\|_p\|g\|_p &\leq \frac{\|f\|_p}{\|g\|_p}.
\end{aligned}$$

3.3. The Applications of the Second Quotient Inequality to L^P Spaces

In this subsection, the second quotient inequality is used to estimate the integrals in L^p spaces.

Theorem 5. *Suppose that $f(x)$ and $g(x)$ are measurable functions over the domain Ω such that*

$$\int_\Omega |f(x)|dx \geq 1$$

and

$$\int_{\Omega} |g(x)|dx \geq 1,$$

then

$$\frac{\|f\|_p}{\|g\|_p} \leq \|f\|_p \|g\|_p.$$

Proof: We can see that:

$$\begin{aligned}
\Big|\int_{\Omega} \frac{f(x)}{g(x)} dx\Big| &\leq \int_{\Omega} \frac{|f(x)|}{|g(x)|} dx \\
\Big|\int_{\Omega} \frac{f(x)}{g(x)} dx\Big| &\leq \frac{\int_{\Omega} |f(x)|dx}{\int_{\Omega} |g(x)|dx}
\end{aligned}$$

Applying the second quotient inequality to the term on the right hand side of the above inequality yields

$$\begin{aligned}
\Big|\int_{\Omega} \frac{f(x)}{g(x)} dx\Big| &\leq \frac{\int_{\Omega} |f(x)|dx}{\int_{\Omega} |g(x)|dx} \leq \int_{\Omega} |f(x)||g(x)|dx \\
\Rightarrow \frac{\int_{\Omega} |f(x)|dx}{\int_{\Omega} |g(x)|dx} &\leq \int_{\Omega} |f(x)||g(x)|dx \\
\Rightarrow \frac{\Big(\int_{\Omega} |f(x)|^p dx\Big)^{\frac{1}{p}}}{\Big(\int_{\Omega} |g(x)|^p dx\Big)^{\frac{1}{p}}} &= \Big(\int_{\Omega} |f(x)|^p dx\Big)^{\frac{1}{p}} \Big(\int_{\Omega} |g(x)|^p dx\Big)^{\frac{1}{p}} \\
\Rightarrow \frac{\|f\|_p}{\|g\|_p} &\leq \|f\|_p \|g\|_p.
\end{aligned}$$

Corollary 1. *Suppose that $f(x)$ and $g(x)$ are any two measurable functions defined on* $\mathbf{R}$ *with $\|f(x)\| \leq 1$ and $\|g(x)\| \leq 1$, then*

$$\int_a^b \frac{\|f(x)\|}{\|g(x)\|} dx \leq \int_a^b \|f(x)\| \|g(x)\| dx.$$

Proof: We observe that:

$$\begin{aligned}
\Big|\int_a^b \frac{f(x)}{g(x)} dx\Big| &\leq \int_a^b \Big|\frac{f(x)}{g(x)}\Big| dx \\
\Rightarrow \Big|\int_a^b \frac{f(x)}{g(x)} dx\Big| &= \int_a^b \frac{\|f(x)\|}{\|g(x)\|} dx \leq \int_a^b \|f(x)\| \|g(x)\| dx \\
\Rightarrow \int_a^b \frac{\|f(x)\|}{\|g(x)\|} dx &\leq \int_a^b \|f(x)\| \|g(x)\| dx.
\end{aligned}$$

Corollary 2 (Isotonic linear functional)**.** *Let $F(T)$ be an algebra of real functions defined on T and L a subclass of $F(T)$ satisfying the axioms:*
(i) $f, g \in L \Rightarrow f + g \in L$;
(ii) $f \in L,\ \alpha \in \mathbf{R} \Rightarrow \alpha f \in L$. A functional A defined on L is an isotonic linear functional on L provided that:

$$\begin{aligned} (a) A(\alpha f + \beta g) &= \alpha A(f) + \beta A(g), \quad \forall\ \alpha,\ \beta \in \mathbf{R} \text{ and } f, g \in L \\ (b) \qquad f(t) &\geq g(t) \Rightarrow A(f) \geq A(g), \quad t \in T. \end{aligned}$$

Proof: To prove the result in corollary $2(a)$, see for example, author in [11].
We prove the finding in corollary $2(b)$ by setting $\|f(t)g(t)\| \geq 1$, then

$$\|f(t)\| \geq \frac{1}{\|g(t)\|}.$$

Applying the second quotient inequality to the right hand side of the above inequality yields

$$\begin{aligned} \|f(t)\| &\geq \frac{1}{\|g(t)\|} \leq \|g(t)\| \\ \Rightarrow \|f(t)\| &\geq \|g(t)\|. \end{aligned}$$

This completes the proof.

3.4. Using the quotient Inequalities to obtain the estimate of Hahn-Banach contraction mapping theorem

In this section, the first product inequality is used to obtain an alternative way for proving Hahn Banach contraction mapping theorem.

Proposition 1. *A contraction mapping T, defined on a complete normed linear space, has unique fixed point.*

Proof: Setting a mapping $T : X \to X$, and let $x_o \in X$ such that

$$Tx_{n-1} = x_n, \quad \forall\ n = 1, 2, \ldots.$$

For any positive integer n, then

$$\begin{aligned} \|x_{m+1} - x_m\| &= \|Tx_m - Tx_{m-1}\| \leq \alpha \|x_m - x_{m-1}\| \\ &\vdots \\ \|x_{m+1} - x_m\| &\leq \alpha^m \|x_1 - x_o\| \\ \|x_{m+1} - x_m\| &\leq \left(\alpha^{n+k-1} + \alpha^{n+k-2} + \ldots + \alpha^n\right) \|x_1 - x_o\|. \end{aligned}$$

where, α, is the boundedness constant. Using the first quotient inequality on the right hand side of the above inequality yields

$$\begin{aligned} \|x_{m+1} - x_m\| &\leq \frac{1}{\Big(\alpha^{n+k-1} + \alpha^{n+k-2} + \ldots + \alpha^n\Big)} \|x_1 - x_o\| \\ \|x_{m+1} - x_m\| &= \frac{(1-\alpha)}{\alpha^n} \|x_1 - x_o\|. \end{aligned}$$

We can see that $\frac{(1-\alpha)}{\alpha^n} \to 0$ as $n \to \infty$ for all $\alpha \geq 1$. The sequence $\{X_n\}_{n=1}^{\infty}$ is convergent. The normed space X is complete since $\{X_n\}_{n=1}^{\infty}$ has a limit point in X.

Let x be the element of X such that

$$\lim_{n\to\infty} X_n = x.$$

Thus,

$$\begin{aligned} Tx &= T(\lim_{n\to\infty} X_n) \\ Tx &= \lim_{n\to\infty} TX_n. \end{aligned}$$

By the continuity of T. We can see that:

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ty_{n+1} = y.$$

Suppose further that $Ty_1 = y_1$ and $Ty_2 = y_2$. Then

$$\begin{aligned} \|y_1 - y_2\| &= \|Ty_1 - Ty_2\| \\ \|y_1 - y_2\| &\leq \|y_1 - y_2\| \\ \|y_1 - y_2\| &< \|y_1 - y_2\|, \end{aligned}$$

which is a contradiction. Thus, the fixed point theorem is unique.

3.5. Using the Product and quotient Inequalities to obtain sharp Inequalities in Sobolev spaces

Lemma 1. *For any* $1 \leq p < n$, $W^{1,p}(\mathbf{R}^n) \hookrightarrow L^r(\mathbf{R}^n)$ *is continuously imbedded,* $\forall\, \alpha \in (0,1]$ *and* $\|u\| \geq 2$.

Proof:Setting $\alpha \in (0,1]$ *and* $u \in W^{1,p}(\mathbf{R}^n)$, we see that $u \in L^{p*}(\mathbf{R}^n)$. Then

$$\begin{aligned} \|u\|_r^r &\leq \alpha \int_{\mathbf{R}^n} (|u|^r + |u|^r) dx \\ \|u\|_r^r &= \alpha\Big(\int_{\mathbf{R}^n} |u|^r dx + \int_{\mathbf{R}^n} |u|^r dx \Big) \\ \|u\|_r^r &\leq \alpha\Big((\int_{\mathbf{R}^n} |u|^p dx)^{\frac{r}{p}} + (\int_{\mathbf{R}^n} |u|^{p*} dx)^{\frac{r}{p*}} \Big) \end{aligned}$$

$$\|u\|_r \leq \alpha\Big(\|u\|_p + \|u\|_{p*}\Big).$$

Using the second product inequality, we obtain

$$\begin{aligned}
\|u\|_r &\leq \alpha\Big(\|u\|_p\|u\|_{p*}\Big)\\
\Rightarrow \|u\|_r &\leq \alpha\Big(c\|u\|_p\|\nabla u\|_{p*}\Big)\\
\Rightarrow \|u\|_r &\leq \alpha c\|u\|_{1,p}^2.
\end{aligned}$$

This completes the proof.

4. The Applications of the First Quotient Inequality to Unitary Space

In this section, the estimates involving the quotients of norms in the unitary space are obtained by using both the first and second product inequalities.

Definition 7. *Setting*

$$H_p^E(\epsilon) = \inf\Big\{\sup\Big(\frac{1}{2\pi}\int_0^{2\pi}\|x+e^{i\theta}y\|^p d\theta\Big)^{\frac{1}{p}} - 1 : \|y\| = 1,\ \|x\| = \epsilon\Big\},\ \ 0 < p < \infty,\ \ \textit{and}\ \ \epsilon \geq 1.$$

Theorem 6. *Suppose that $(X, \|\cdot\|_p)$ is a continuously quasi-normed space. Then for any $x \in X$, there exists $\delta > 0$ such that*

$$\frac{1}{2\pi}\int_0^{\pi}\|x+re^{i\theta}y\|d\theta \leq \frac{1}{2\pi}\int_0^{\pi}\frac{\|x\|}{\|re^{i\theta}y\|}d\theta.$$

Proof: Setting $0 < r \leq 1$, we have

$$\begin{aligned}
\Big(\frac{1}{2\pi}\int_0^{\pi}\|x+re^{i\theta}y\|^p d\theta\Big)^{\frac{1}{p}} &\leq \Big(\frac{1}{2\pi}\int_0^{\pi}(\|x\|+\|re^{i\theta}y\|)^p d\theta\Big)^{\frac{1}{p}}\\
\Rightarrow \Big(\frac{1}{2\pi}\int_0^{\pi}(\|x+re^{i\theta}y\|)^p d\theta\Big)^{\frac{1}{p}} &\leq \Big(\frac{1}{2\pi}\int_0^{\pi}(\|x\|^p+\|r\|^p)d\theta\Big)^{\frac{1}{p}} \leq \Big(\frac{1}{2\pi}\int_0^{\pi}\frac{\|x\|^p}{\|r\|^p}d\theta\Big)^{\frac{1}{p}}\\
\Rightarrow \Big(\frac{1}{2\pi}\int_0^{\pi}\|x\|^p d\theta\Big)^{\frac{1}{p}} &\leq \Big(\frac{1}{2\pi}\int_0^{\pi}\frac{\|x\|^p}{\|r\|^p}d\theta\Big)^{\frac{1}{p}}\\
\Rightarrow \|x\|_p &\leq \frac{\|x\|_p}{\|r\|_p}.
\end{aligned}$$

Hence,

$$\frac{1}{2\pi}\int_0^{\pi}\|x+re^{i\theta}y\|d\theta \leq \frac{1}{2\pi}\int_0^{\pi}\frac{\|x\|}{\|re^{i\theta}y\|}d\theta.$$

The converse inequality of theorem (6) is obvious. Thus, define

Definition 8.

$$H_p^E(\epsilon) = \inf \Big\{ \sup \Big(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\|^p d\theta \Big)^{\frac{1}{p}} - 1 : \|y\| = 1,\ \|x\| = \epsilon \Big\},\ \ 0 < p < \infty,\ \ \textit{and}\ \ \epsilon \leq 1,$$

and provide the converse result in corollary 3 below.

Corollary 3. *Suppose that $(X, \|\cdot\|_q)$ is a continuously quasi-normed space. Then there exists $0 < p < \infty$, such that whenever x and y are in X with $\delta = \delta(x, y) > 0$, then*

$$\|x\| \leq \Big(\frac{1}{2\pi} \int_0^{2\pi} \Big\| \frac{x}{re^{i\theta} y} \Big\|^p \Big)^{\frac{1}{p}}, \qquad 0 < r \leq \delta \leq 1.$$

Proof: We see that:

$$\Big(\frac{1}{2\pi} \int_0^{2\pi} \Big\| \frac{x}{re^{i\theta} y} \Big\|^p d\theta \Big)^{\frac{1}{p}} = \Big(\frac{1}{2\pi} \int_0^{2\pi} \frac{\|x\|^p}{\|re^{i\theta} y\|^p} d\theta \Big)^{\frac{1}{p}}.$$

Using the first quotient inequality on the right hand side of the above equation, we obtain

$$\begin{aligned}
\Rightarrow \Big(\frac{1}{2\pi} \int_0^{2\pi} \Big\| \frac{x}{re^{i\theta} y} \Big\|^p d\theta \Big)^{\frac{1}{p}} &\geq \Big(\int_0^{2\pi} \frac{\|x\|^p}{\|r\|^p} \Big)^{\frac{1}{p}} \geq \Big(\int_0^{2\pi} \|x\|^p \|r\|^p \Big)^{\frac{1}{p}} \\
\Rightarrow \Big(\int_0^{2\pi} \|x\|^p \|r\|^p \Big)^{\frac{1}{p}} &\leq \Big(\int_0^{2\pi} \frac{\|x\|^p}{\|r\|^p} \Big)^{\frac{1}{p}} \\
\Rightarrow \Big(\int_0^{2\pi} \|x\|^p \Big)^{\frac{1}{p}} &\leq \Big(\int_0^{2\pi} \frac{\|x\|^p}{\|r\|^p} \Big)^{\frac{1}{p}} \\
\Rightarrow \|x\|_p &\leq \frac{\|x\|_p}{\|r\|_p}.
\end{aligned}$$

It follows that,

$$\|x\|_p \leq \Big(\int_0^{2\pi} \frac{\|x\|^p}{\|r\|^p} \Big)^{\frac{1}{p}}.$$

Hence,

$$\|x\| \leq \Big(\frac{1}{2\pi} \int_0^{2\pi} \Big\| \frac{x}{re^{i\theta} y} \Big\|^p \Big)^{\frac{1}{p}}, \qquad 0 < r \leq \delta \leq 1.$$

Proposition 2. *Suppose that $(X, \|\cdot\|_q)$ is a continuously quasi-normed space. The following statements are equivalent:*
(i) $(X, \|\cdot\|)$ is locally PL-convex;
(ii) there exists $0 < p < \infty$, such that whenever x and y are in X with $\delta = \delta(x, y) > 0$ such that

$$\|x\| \leq \Big(\frac{1}{2\pi} \int_0^{2\pi} \Big\| \frac{x}{re^{i\theta} y} \Big\|^p \Big)^{\frac{1}{p}}, \qquad 0 < r \leq \delta \leq 1;$$

(iii) $\ln \|x\|$ is a pluri-subharmonic function on X.

5. Quotient Inequalities involving Powers

In this section, the inequalities involving the relationships among index product of numbers and its quotient of numbers as bases are established for given regions of validity. Thus, we introduce in this paper the index power quotient inequalities; the first and second index power quotient inequalities. The main result is expressed in theorem (7), which is the fundamental tool for proving other results in this paper. involving propositions and corollaries.

Theorem 7 (First index power quotient inequality). *Let $T : X \to Y$ be a Banach space. Then*

$$\|x\|^{pq} \leq \|x\|^{\frac{p}{q}}, \ \forall \ x \in X, \ 0 \leq p < \infty, \ and\ 0 < q \leq 1,$$

where equality occurs at either $p = 0$ or $q = 1$.

Proof: Setting $0 \leq p < \infty$, and $0 < q \leq 1$. We can see that:

$$\begin{aligned} f(x) &= -(pqx + \frac{p}{q}x) \leq 0 \\ \Rightarrow f(x) &= -(pqx + \frac{p}{q}x) = -(x^{pq} + x^{\frac{p}{q}}) \leq 0 \\ \Rightarrow -(x^{pq} + x^{\frac{p}{q}}) &\leq 0 \\ \Rightarrow -x^{pq} &\leq -x^{\frac{p}{q}} \\ \Rightarrow \|x\|^{pq} &\leq \|x\|^{\frac{p}{q}}, \quad \forall\, x \in X. \end{aligned}$$

This completes the proof.

The converse of theorem 7 is stated in theorem 8 below.

Theorem 8 (Second index power quotient inequality). *Let $T : X \to Y$ be a Banach space. Then*

$$\|x\|^{\frac{p}{q}} \leq \|x\|^{pq}, \ \forall \ x \in X, \ 1 \leq p < \infty, \ and\ 1 \leq q < \infty.$$

Proof: The proof of theorem 8 is similar to theorem 7.

Theorem 9. *Let $T : X \to Y$ be a Banach space. Then*

$$\|a\|\|b\| \leq pq^2\|a\|^p + p^2q\|b\|^q, \forall \quad a, b \in X \quad p, q \leq 1.$$

Proof: Using the Young's inequality, we have

$$\begin{aligned} \|ab\| &\leq \|\frac{qa^p + pb^q}{pq}\| \\ \|a\|\|b\| &\leq \frac{\|qa^p + pb^q\|}{\|pq\|}. \end{aligned}$$

Applying the second quotient inequality on the right hand side of the above inequality yields

$$\begin{aligned} \|a\|\|b\| &\leq \Big(\|qa^p + pb^q\|\Big)\|pq\| \\ \|a\|\|b\| &\leq pq^2\|a\|^p + p^2q\|b\|^q, \ \ p,q \geq 1. \end{aligned}$$

Theorem 10. *Let $T : X \to Y$ is a Banach space. Then*

$$pq^2\|a\|^p + p^2q\|b\|^q \leq \|a\|\|b\|, \forall \ \ a,b \in X \ \ p,q \in [0,1).$$

Proof: We see that:

$$\begin{aligned} \|ab\| &\leq \|\frac{qa^p + pb^q}{pq}\| \\ \Rightarrow \|a\|\|b\| &\leq \frac{\|qa^p + pb^q\|}{\|pq\|}. \end{aligned}$$

Applying the first quotient inequality on the right hand side of the above inequality yields

$$\begin{aligned} \Big(\|qa^p + pb^q\|\Big)\|pq\| &\leq \|a\|\|b\| \\ \Rightarrow pq^2\|a\|^p + p^2q\|b\|^q &\leq \|a\|\|b\|, \ \ p,q \in [0,1). \end{aligned}$$

6. The Applications of the Index Power Quotient Inequalities to H$\ddot{o}$lder Spaces

In this section, the quotient inequalities involving index are introduced.

Theorem 11. *Let $0 < p < +\infty$ and $0 < q < 1$ such that $|\frac{p}{q}| < 1$. Then a mapping $T : X \to Y$ is Hölder-type continuous of exponent $\frac{p}{q}$ at x_o, if*

$$\|T(x) - T(x_o)\| \leq L\|x - x_o\|^{\frac{p}{q}}, \ \ \forall \ x, \ x_o \in X,$$

where L is the $\frac{p}{q}-$th Hölder coefficient of T.

Proof: By the Hölder continuity of T of exponent γ, we have

$$\|T(x) - T(x_o)\| \leq L\|x - x_o\|^{\gamma}, \ \ \forall \ x, \ x_o \in X \ and \ \gamma \in (0,1].$$

Setting $\gamma = |p||q|$ and applying the power quotient inequality, we obtain

$$\begin{aligned} \|T(x) - T(x_o)\| &\leq L\|x - x_o\|^{pq} \leq L\|x - x_o\|^{\frac{p}{q}} \\ \|T(x) - T(x_o)\| &\leq L\|x - x_o\|^{\frac{p}{q}} \ \ \forall \ x, \ x_o \in X, \ \ \left|\frac{p}{q}\right| \leq 1. \end{aligned}$$

Theorem 12. *Let A and B be selfadjoint operators with $S_p(A),\ S_p(B) \subseteq [m, M]$ for some real numbers $m < M$. If $f : [m, M] \to \mathbf{R}$ is of $\frac{p}{q} - L-$Hölder type. Thus, for a given $|p||q| \leq 1$ and $L > 0$, we have:*

$$\Big|f(s) - f(t)\Big| \leq L\Big|s - t\Big|^{\frac{p}{q}}, \quad \forall\ \ s, t \in [m, M].$$

Then the Ostrowski type inequality for selfadjoint operators becomes:

$$\Big|f(s) - \langle f(A)x, x\rangle\Big| \leq \Big[\frac{1}{2}(M-m) + |s - \frac{m+M}{2}|\Big]^{\frac{p}{q}}, \quad \forall\ \ s \in [m, M] \text{ and } x \in H \text{ with } \|x\| = 1.$$

Moreover, we have:

$$\begin{aligned}\Big|\langle f(B)y, y\rangle - \langle f(A)x, x\rangle\Big| &\leq \Big\langle\Big|f(B) - \langle f(A)x, x\rangle \cdot 1_H\Big|_y, y\Big\rangle. \\ &\leq L\Big[\frac{1}{2}(M-m) + \langle|B - \frac{m+M}{2} \cdot 1_H|y, y\rangle\Big]^{\frac{p}{q}}, \\ &\quad \forall\ \ x, y \in H \ \ with\ \ \|x\| = \|y\| = 1,\end{aligned}$$

and H denotes Hilbert space.

Proof: We can see that, using the Ostrowski-type inequality for the Riemann-Stieltjes integral, we have:

$$\Big|f(s)[u(b) - u(a)] - \int_a^b f(t)du(t)\Big| \leq L\Big[\frac{1}{2}(b-a) + |s - \frac{a+b}{2}|\Big]^r \bigvee_a^b(u).$$

Setting $r = |p||q| \leq 1$ and applying the first index power quotient inequality, we obtain

$$\begin{aligned}\Big|f(s)[u(b) - u(a)] - \int_a^b f(t)du(t)\Big| &\leq L\Big[\frac{1}{2}(b-a) + |s - \frac{a+b}{2}|\Big]^{pq} \bigvee_a^b(u) \\ &\leq L\Big[\frac{1}{2}(b-a) + |s - \frac{a+b}{2}\Big]^{\frac{p}{q}} \bigvee_a^b(u) \\ \Big|f(s)[u(b) - u(a)] - \int_a^b f(t)du(t)\Big| &\leq L\Big[\frac{1}{2}(b-a) + |s - \frac{a+b}{2}|\Big]^{\frac{p}{q}} \bigvee_a^b(u), \quad s \in [a, b],\end{aligned}$$

u is a bounded variation on $[a, b]$ and $\bigvee_a^b$ is the total variation of u on $[a, b]$. Then the functional $f(t)$ is of $\frac{p}{q} - L-$Hölder-type on $[a, b]$.

Also, setting $u(\lambda) = g_x(\lambda) = \langle E_{\lambda x}, x\rangle$, where $x \in H$ with $\|x\| = 1$, then

$$\Big|f(s) - \int_m^M f(\lambda)d(\langle E_{\lambda x}, x\rangle)\Big| \quad \leq \quad L\Big[\frac{1}{2}(M-m) + |s - \frac{m+M}{2}|\Big]^{pq} \bigvee_m^M(g(x))$$

$$
\begin{aligned}
& \leq \; L\Big[\frac{1}{2}(M-m)+|s-\frac{m+M}{2}|\Big]^{\frac{p}{q}}\bigvee_{m}^{M}(g(x)) \\
\Big|f(s)-\int_{m}^{M} f(\lambda)d(\langle E_{\lambda x},x\rangle)\Big| & \leq \; L\Big[\frac{1}{2}(M-m)+|s-\frac{m+M}{2}|\Big]^{\frac{p}{q}}\bigvee_{m}^{M}(g(x)).
\end{aligned}
$$

Again, we see that:

$$
\begin{aligned}
\Big\langle |f(B)-\langle f(A)x,x\rangle\cdot 1_H|_y,y\Big\rangle & \leq \; \Big\langle\Big[\frac{1}{2}(M-m)+|B-\frac{m+M}{2}\cdot 1_H|\Big]^{pq}y,y\Big\rangle \\
& \leq \; \Big\langle\Big[\frac{1}{2}(M-m)+|B-\frac{m+M}{2}\cdot 1_H|\Big]^{\frac{p}{q}}y,y\Big\rangle, \\
& \quad \forall \;\; x,y\in H, \;\; \|x\|=\|y\|=1.
\end{aligned}
$$

This completes the proof.

7. Conclusion

In a nutshell, the first and second quotient inequalities are introduced which extend some findings in functional spaces such as L^p space, Hilbert space, unitary space, Hölder spaces and Sobolev spaces. The relationship between the inequality involving norm of quotient of the vectors or functions and its product counterpart has not been observed over the years and vice versa. These inequalities do not only establish the relationship between two mathematical structures but also, useful for establishing many functional properties such as continuity, boundedness, expansive of the operators in functional spaces.

References

[1] W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann, The complex of quasi-normed linear spaces *Journal of Functional Analysis,* **55** , (1984). pp. 110-150.

[2] M. Agueh, N. Ghoussoub and X. Kang, Geometric inequalities via a general comparison principle for interacting gases, *Geom. Funct. Anal.,* **14,**(2004). pp. 215-244.

[3] B. Barnes, C. Sebil and E. Harris, A note on the quasi-normed linear space, *Advances in theoretical and applied mathematics,***13, No. 2**(2018). pp. 107-114.

[4] B. Barnes, E. D. J. Owusu-Ansah, S. K. Amponsah and C. Sebil, The proofs of product inequalities in a generalized vector spaces, *European journal of pure and applied mathematics,***11, No. 2**(2018). pp. 375-389.

[5] B. Barnes, I. A. Adjei and C. Sebil and E. Harris, product-normed linear spaces, *European journal of pure and applied mathematics,***11, No. 3**(2018). pp. 740-750.

[6] H. Royden and P. Fitxpatrick, Real Analysis. Pearson Education, Inc, 4^{th} ed., Upper saddle River; 2010.

[7] J. T. Oden, Applied Functional Analysis: A first course for students of mechanics and engineering science, Prentice-Hall, Inc, New Jersey; 1979.

[8] J. Horvath, An introduction to distributions, *American mathematical monthly,* **77**(1970).pp. 227-240.

[9] H. Kozono and H. Wadade, Remarks on Gagliardo-Nirenberg type inequality with critical Sobolev space and BMO, *Mathematische Zeitschrift,***259**(2008).pp. 935-950.

[10] S. S. Dragomir, Operators inequalities of the Jensen, *Čhebyšev* and Grüss type, Springer New York Dordrecht Heidelberg, London; 2012.

[11] S. S. Dragomir, A generalization of Grüss's inequality in inner product spaces and applications *Journal of Mathematical Analysis and Applications* **237**, (2009).pp. 74-82.