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Finite Groups With Certain Permutability Criteria

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Abstract. Let G be a finite group. A subgroup H of G is said to be S-permutable in G if it permutes with all Sylow subgroups of G. In this note we prove that if P, the Sylow p-subgroup of G $(p > 2)$, has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-permutable in G, then G' is p-nilpotent.

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1. Introduction

Throughout this note, G denotes a finite group. The relationship between the properties of the Sylow subgroups of a group G and its structure has been investigated by many authors. Starting from Gaschűtz and Itő ([10], Satz 5.7, p.436) who proved that a group G is solvable if all its minimal subgroups are normal. In 1970, Buckely [4] proved that a group of odd order is supersolvable if all its minimal subgroups are normal (a subgroup of prime order is called a minimal subgroup). Recall that a subgroup is said to be S-permutable in G if it permutes with all Sylow subgroup of G. This concept, as a generalization of normality, was introduced by Kegel [11] in 1962 and has been studied extensively in many notes. For example, Srinivasan [15] in 1980 obtained the supersolvability of G under the assumption that the maximal subgroups of all Sylow subgroups are S-permutable in G. In 2000, Ballester-Bolinches et al. [3] introduced the c-supplementation concept of a finite group: A subgroup H of a group G is said to be c-supplemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$, where $H_G = Core_G(H)$ is the largest normal subgroup of G contained in H. By using this concept they were able to prove that a group G is solvable if and only if every Sylow subgroup of G is c-supplemented in G. Moreover, as an application, they got the supersolvability of a group G if all its minimal subgroups and the cyclic subgroups of order 4 are c-supplemented in G.

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In 2014, Heliel [8] proved that G is solvable if each subgroup of prime odd order of G is c-supplemented in G. Also he proved that G is solvable if and only if every Sylow subgroup of odd order of G is c-supplemented in G. This improved and generalized the results of Hall [6, 7], Ballester-Bolinches and Guo [2], and Ballester-Bolinches et al. [3]. Heliel also posted the following conjecture:

Let G be a finite group such that every non-cyclic Sylow subgroup P of odd order of G has a subgroup D such that $1 < |D| \leq |P|$ and all subgroups H of P with $|H| = |D|$ are c-supplemented in G. Is G solvable?

In the same year, Li et al. [12] presented a counterexample to show that the answer of this conjecture is negative in general and then gave a generalization of Heliel's theorems.

Example 1. *Let $G = A_5 \times H$, where A_5 is the alternating group of degree 5 and H is an elementary group of order p^n with $p > 5$ and $n \geq 2$. Then G satisfies the condition of the preceding conjecture, but G is not solvable.*

In 2015, Hijazi [9] continued the above mentioned investigations and proved the following: Suppose that each Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-permutable in G. Then G is solvable.

The main goal of this note is to prove the following main theorem:

Main Theorem 1. *Let P be a Sylow p-subgroup of G ($p > 2$). Suppose that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-permutable in G. Then G' is p-nilpotent.*

As immediate consequences of the main theorem we have:

Corollary 1. *Let P be a Sylow p-subgroup of G ($p > 2$). Suppose that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are permutable in G. Then G' is p-nilpotent.*

Corollary 2 ([9], Theorem 3.1)**.** *Suppose that each Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are S-permutable in G. Then G is solvable.*

Corollary 3 (Gaschűtz and Itő [10], Satz 5.7, p.436)**.** *A group G is solvable if all its minimal subgroups are normal.*

2. Proofs

We first prove the following theorems:

Theorem 2. *Let P be a Sylow p-subgroup of a group G, where p is an odd prime. If each subgroup of P of order p is S-permutable in G, then $G'$is p-nilpotent.*

Proof. We prove the theorem by induction on $|G|$. Hence if each subgroup of P of order p is normal in G, then each subgroup of G' of order p is normal in G'. Let L be a subgroup of G' such that $|L| = p$. Then $G/C_G(L) \subseteq Aut(L)$ and, since $Aut(L)$ is cyclic of order $p-1$, we have $G/C_G(L)$ is abelian. Thus $G' \leq C_G(L)$ and so $L \leq Z(G')$. By ([10], Satz 5.5(a), p. 435), $G'$is p-nilpotent. Thus we may assume that there exists a subgroup H of P of order p such that H is not normal in G. By the hypothesis, H is S-permutable in G and hence by ([13], Lemma A), $O^p(G) \leq N_G(H) < G$. Let M be a maximal subgroup of G such that $N_G(H) \leq M < G$. Then $M \lhd G$ and $|\ G/M\ | = p$. By induction on $|G|$, M' is p-nilpotent. Hence if $O_{p'}(G) \neq 1$, $G/\ O_{p'}(G)$ satisfies the hypothesis of the theorem and so $(G/\ O_{p'}(G))' = G'O_{p'}(G)/O_{p'}(G) \cong G'/(G' \cap O_{p'}(G))$ is p-nilpotent which implies that $G'$is p-nilpotent. Thus assume that $O_{p'}(G) = 1$. Since M' *char* M and $M \lhd G$, we have $M' \lhd G$. As M' is p-nilpotent and $O_{p'}(G) = 1$, we have M' is a p-group. Then $P_1 \lhd M$ where P_1 is a Sylow p-subgroup of M. By Schur-Zassenhaus Theorem [5, Theorem 6.2.1, p. 221], $M = P_1K$, where K is a p'-Hall subgroup of M. Hence if $C_G(P_1) \leq P_1$, K is a p'-group of automorphisms of P_1, and since K leaves each subgroup of P_1 invariant because every subgroup of P of prime order is S-permutable, then by ([14], Lemma 2.20), K is cyclic. Let Q be a Sylow q-subgroup of K, where q is a prime divisor of the order of K. Hence if $p < q$, then $P_1Q = P_1 \times Q$ and this means that $Q \leq C_G(P_1)$, a contradiction. Thus p is the largest prime dividing $|G|$ and since K is cyclic, it follows, by Burnside′s p-Nilpotent Theorem ([10], Satz 2.8, p.420), that $P \lhd G$. But $G/P \cong K$, therefore G/P is cyclic and so abelian, then $G' \leq P$. This completes the proof of the theorem.

As a corollary of Theorem 2.1:

Corollary 4. *If each subgroup of prime order of G is S-permutable in G, then G is solvable, $S \lhd G'$and G'/S is nilpotent,where S is a Sylow 2-subgroup of G'.*

Proof. By Theorem 2.1, G' is p-nilpotent for each odd prime p dividing $|G|$. So $G'\ /S$ is nilpotent, S is a Sylow 2-subgroup of $G'$and hence G is solvable.

Theorem 3. *Let p be an odd prime and let P be a Sylow p-subgroup of G. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ are normal in G. Then $G'$is p-nilpotent.*

Proof. We prove the theorem by induction on $|G|$. Clearly, $P \cap G'$ is a Sylow p-subgroup of G'. Set $P_1 = P \cap G'$. We deal with the following two cases:

Case 1. $|P_1| \leq |D|$.

Hence if $|D| = p$, $|P_1| = p$, and $P_1 \lhd G$. Then $G' \leq C_G(P_1)$ and so $P_1 \leq Z(G')$. Hence, by Schur-Zassenhaus Theorem, $G' = P_1 \times K$, where K is a p'-Hall subgroup of G'.

In particular, G' is p-nilpotent.

Thus we may assume that $|D| = p^n$ $(n \geq 2)$. Let H be a subgroup of P with $|H| = |D|$ such that $P_1 \leq H < P$. By the hypothesis, $H \lhd G$. Assume that $\Phi(H) \neq 1$ and consider the factor group $G/\Phi(H)$. Obviously, $G/\Phi(H)$ satisfies the theorem hypothesis and so $(G/\Phi(H))' = G'\Phi(H)/\Phi(H)$ is p-nilpotent by the induction on $|G|$. But $G'\Phi(H)/\Phi(H) \cong G'/G' \cap \Phi(H)$ and $\Phi(H) \leq \Phi(G)$, then we have $G' \cap \Phi(H) \leq G' \cap \Phi(G)$ and therefore

$G'/G' \cap \Phi(G)$ is p-nilpotent. Now $G'\Phi(G)/\Phi(G) \cong G'/G' \cap \Phi(G)$ is p-nilpotent implies that $G'\Phi(G)$ is p-nilpotent and consequently G' is p-nilpotent.

Thus we may assume that $\Phi(H) = 1$ and so H is elementary abelian p-group of order p^n $(n \geq 2)$. Let L be a subgroup of P contains H such that H is maximal in L. Clearly, L is not cyclic because H is elementary abelian group of order p^n $(n \geq 2)$. Then L contains a subgroup H_1 such that $|H_1| = |D|$ and $H_1 \neq H$. By the hypothesis, $H_1 \lhd G$ and since $H \lhd G$, we have $L = H_1 H \lhd G$ and so $\Phi(L) \leq \Phi(G)$. Hence if $\Phi(L) \neq 1$, $\Phi(L) \leq H_1 < L \leq P$. Since L is not cyclic, we have $\Phi(L)$ is contained properly in H_1. Now it is easy to notice that the factor group $G/\Phi(L)$ satisfies the hypothesis of the theorem, so by induction on $|G|$, G' is p-nilpotent. Thus we may assume that $\Phi(L) = 1$ and so P_1 is elementary abelian p-group. Since $P_1 \leq H < L \leq P$ and H is maximal in L, it follows that $|L| = p^{n+1}$. Let $L_1 =< x_1 >$ be a subgroup of P_1 of order p. Then $L =< x_1 > \times < x_2 > \times \ldots \times < x_{n+1} >$. By the hypothesis, each maximal subgroup of L is normal in G. Applying ([1], Lemma 2.9) implies that each subgroup of L of order p is normal in G; in particular each subgroup L_1 of P_1 of order p is normal in G. So, $G^{\backprime} \leq C_G(L_1)$ and consequently $P_1 \leq Z(G')$. By Schur-Zassenhaus Theorem, $G' = P_1 \times K_1$, where K_1 is a p'-Hall subgroup of G; in particular G' is p-nilpotent.

Case 2. $|P_1| > |D|$.

Hence if $|D| = p$, then every subgroup of P_1 of order p is normal in G, so $\Omega_1(P_1) \leq Z(G')$ which implies that G' is p-nilpotent by ([10], Satz 5.5(a), p 435). Thus assume that $|D| = p^n$ $(n \geq 2)$. Hence if $\Phi(D) \neq 1$, $G/\Phi(D)$ satisfies the hypothesis of the theorem and so $(G/\Phi(D))\prime = G'\Phi(D)/\Phi(D)$ is p-nilpotent by induction on $|G|$ which implies that $G'/G' \cap \Phi(G)$ is p-nilpotent; in particular G' is p-nilpotent. Thus we may assume that $\Phi(D) = 1$. Let $L \leq P_1$ such that D is maximal in L. Then $|L| = p^{n+1}(n \geqslant 2)$. Clearly L is not cyclic. Then there exists a maximal subgroup $L_1 \neq D$ in L. By the hypothesis $L_1 \lhd G$ and $D \lhd G$ which implies that $L = L_1 D \lhd G$. Hence if $\Phi(L) \neq 1$, $\Phi(L) \leq D < L \leq P_1$ and since L is not cyclic, it follows that $\Phi(L) < D$. By induction on $|G|$, $G'\Phi(L)/\Phi(L) \cong G'/G' \cap \Phi(L)$ is p-nilpotent. In particular, $G'\Phi(G)/\Phi(G)$ is p-nilpotent and it follows easily that G' is p-nilpotent. So we may assume that $\Phi(L) = 1$ and so L is elementary abelian. Let $L_1 < P$ such that $|L_1| = p$. Then $L_1 < L \leq P_1$ and so $L_1 \lhd G$ by ([1], Lemma 2.9). In particular, $\Omega_1(P_1) \leq Z(G')$. Again by ([10], Satz 5.5(a), p 435), G' is p-nilpotent. This completes the proof of the theorem.

Now we can move forward to prove our main theorem:

Proof. We prove the theorem by induction on $|G|$. Hence if $O_{p'}(G) \neq 1$, $G/O_{p'}(G)$ satisfies the hypothesis of the theorem and so $(G/O_{p'}(G))\prime$ is p-nilpotent by induction on $|G|$; in particular, G' is p-nilpotent. Thus we may assume that $O_{p'}(G) = 1$. If each subgroup H of P with $|H| = |D|$ is normal in G, then G' is p-nilpotent by Theorem 2.2. So we may assume that there exists a subgroup H of P with $|H| = |D|$ and H is not normal in G. By hypothesis, H is S-permutable in G. Since $H \ntrianglelefteq G$ and H is S-permutable in G, we have by ([13], Lemma A) that $O^p(G) \leq N_G(H) < G$. Let M be a maximal subgroup of G contains $N_G(H)$ properly. Then $M \lhd G$ and $|G/M| = p$. Let $P_1 = P \cap M$ be a Sylow p-subgroup of M. By the hypothesis, $|D| \leq |P_1|$. If $|D| = |P_1|$, then $|H| = |P_1|$ and so

$P \leq N_G(H)$, and since $O^p(G) \leq N_G(H)$, we have $PO^p(G) = G \leq N_G(H) < M$ which is impossible. Thus we may assume that $|D| < |P_1|$. Now M' is p-nilpotent, by the inductive hypothesis, implies that M' is a p-group because $O_{p'}(G) = 1$. Then P_1 is characteristic in M and since $M \lhd G$, we have $P_1 \lhd G$. If $P \lhd G$, then G/P is abelian and since all subgroups H of P with $|H| = |D|$ are S-permutable in G, we have that G is supersolvable by ([14], Theorem 1.3) and so G' is nilpotent; in particular G' is p-nilpotent. Thus we may assume that $P \ntriangleleft G$ and $P_1 = F(G)$ the Fitting subgroup of G (recall that $O_{p'}(G) = 1$ and that $F(G) =< O_p(G)$ for all p divides $|G| >$). Consider the subgroup $\Phi(P_1)$ and assume that $\Phi(P_1) \neq 1$. Hence if $|\Phi(P_1)| < |D|$, then $(G/\Phi(P_1))'$ is p-nilpotent by induction on $|G|$; in particular G' is p-nilpotent. So assume that $|\Phi(P_1)| \geq |D|$. If $|\Phi(P_1)| = |D|$, then $P/\Phi(P_1)$ is not cyclic. Let $L/\Phi(P_1)$ be a proper subgroup of $P/\Phi(P_1)$ such that $|L/\Phi(P_1)| = p$ (L is not cyclic; otherwise $\Phi(P_1)$ is cyclic and this implies that there exists $L_1 \leq \Phi(P_1)$ such that $L_1 \lhd G$; in particular $G/C_G(L_1)$ is isomorphic to a subgroup of $Aut(L_1)$ and so $G' \leq C_G(L_1)$ and we conclude then that G' is p-nilpotent). As $|L/\Phi(P_1)| = p$, then there exists a maximal subgroup L_1 of L such that $|L_1| = |\Phi(P_1)| = |D|$ and $L_1 \neq \Phi(P_1)$. But $L_1\Phi(P_1)$ is S-permutable in G, then $L_1\Phi(P_1)/\Phi(P_1) = L/\Phi(P_1)$ is S-permutable in $G/\Phi(P_1)$. By Theorem 2.1, $(G/\Phi(P_1))' = G'\Phi(P_1)/\Phi(P_1)$ is p-nilpotent and so G' is p-nilpotent. Thus we may assume that $\Phi(P_1) = 1$ and P_1 is elementary abelian. Since all subgroups H of P_1 with $|H| = |D|$ are normal in M, we have by ([1], Lemma 2.9) that all subgroups of P_1 of order p are normal in M. So $P_1 \cap Z(P) \neq 1$. Let $L \leq P_1 \cap Z(P)$ such that $|L| = p$. Then $L \lhd G$ and since $G/C_G(L)$ is isomorphic to a subgroup of $Aut(L)$, we have that $G' \leq C_G(L)$, in particular $G'L/L$ is p-nilpotent and so G' is p-nilpotent. This completes the proof of the theorem.

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