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On a nonsingular equation of length 9 over torsion free groups

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Abstract. In [11], Levin conjectured that every equation is solvable over a torsion free group. In this paper we consider a nonsingular equation $g_1tg_2tg_3tg_4tg_5tg_6t^{-1}g_7tg_8tg_9t^{-1} = 1$ of length 9 and show that it is solvable over torsion free groups modulo some exceptional cases.

2010 Mathematics Subject Classifications: 20F05, 57M05

Key Words and Phrases: Asphericity; relative group presentations; torsion-free groups; group equations.

1. Introduction

Let G be a non-trivial group, t be an unknown and let F be a free group generated by t. An equation in t over G is an expression of the form

$$s(t) = g_1t^{\epsilon_1}g_2t^{\epsilon_2}\cdots g_nt^{\epsilon_n} = 1 \ (g_i \in G,\ \epsilon_i = \pm 1)$$

in which it is assumed that $\epsilon_i + \epsilon_{i+1} = 0$ implies $g_{i+1} \neq 1$ in G. We call the integer n the length of the equation. A solution of $s(t) = 1$ over G is an embedding ϕ of G into a group H and an element $h \in H$ such that $\phi(g_1)h^{\epsilon_1}\phi(g_2)h^{\epsilon_2}\cdots\phi(g_n)h^{\epsilon_1} = 1$ in H. Equivalently $s(t) = 1$ is solvable over G if and only if the natural map from G to $\langle G*F|s(t)\rangle$ is injective, where $G * F$ is the free product of G and F. If G is a torsion free group then by Levin's conjecture every equation is solvable [11]. The conjecture is known to be true for $n \leq 7$ [5, 7, 9, 12]. The authors have done significant work in [1, 2, 4] to establish the conjecture for $n = 8$.

The equation of length 9 have been consider in [3]. It has been proved that there are only three equations of length 9 which are still open. In this paper we consider a nonsingular equation of length 9 (one of three) and show that the equation has a solution

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over G modulo some exceptional cases. This paper is the first step in proving Levin's conjecture for equations of length 9.

We first give some basic definitions. A relative group presentation is a presentation of the form $\mathcal{P} = \langle G,\ x \mid r\rangle$ where r is a set of cyclically reduced words in $G*\langle x\rangle$. If the relative presentation is orientable and aspherical then the natural map from G to $\langle G,\ x \mid r\rangle$ is injective. In our case x and r consist of the single element t and $s(t)$ respectively, therefore $\mathcal{P}$ is orientable and so asphericity implies $s(t) = 1$ is solvable. In this paper we use the weight test and the curvature distribution method to show that $\mathcal{P}$ is aspherical [6].

The star graph Γ of $\mathcal{P}$ has vertex set $x \cup x^{-1}$ and edge set r^*, where r^* is the set of all cyclic permutations of the elements of $r \cup r^{-1}$ which begin with an element of $x \cup x^{-1}$. For $R \in r^*$ write $R = Sg$ where $g \in G$ and S begins and ends with x symbols. Then $\mathfrak{i}(R)$ is the inverse of the last symbol of S, $\tau(R)$ the first symbol of S and $\lambda(R) = g$. A weight function θ on Γ is a real valued function on the set of edges of Γ which satisfies $\theta(Sh) = \theta(S^{-1}h^{-1})$. A weight function θ is called aspherical if the following three conditions are satisfied

(W1) Let $R \in r^*$ with $R = x_1^{\epsilon_1} g_1 \ \cdots \ x_n^{\epsilon_n} g_n$. Then

$$\sum_{i=1}^{n} (1 - \theta(x_i^{\epsilon_i} g_i \ \cdots \ x_n^{\epsilon_n} g_n x_1^{\epsilon_1} g_1 \ \cdots \ x_{i-1}^{\epsilon_{i-1}} g_{i-1})) \geq 2.$$

(W2) Each admissible cycle in Γ has weight at least 2 (where admissible means having a label trivial in G).

(W3) Each edge of Γ has a non-negative weight.

If Γ admits an aspherical weight function then $\mathcal{P}$ is aspherical [6]. The following lemma [10] tells us that we can apply asphericity test in $k-$steps.

Lemma 1. *Let the relative presentation $P = \langle H, x : r\rangle$ define a group G and let $Q = \langle G, t : s\rangle$ be another relative presentation. If Q and P are both aspherical, then the relative presentation $R = \langle H, x \cup t : r \cup \tilde{s}\rangle$ is aspherical, where $\tilde{s}$ is an element of $H * F(x) * F(t)$ obtained from s by lifting.*

For a detailed account on the curvature distribution method see [3]. It is clear from our definition of a group equation that if g_i is a coefficient between a negative and a positive power of t than g_i is not trivial in G. This fact will be used in all subsequent proofs without reference.

2. Main Results

We now turn our attention to length 9 equations. A list of these equations is given in [3]. Consider the nonsingular equation of length 9 given by $atbtctdtetft^{-1}gthtit^{-1} = 1$. We write this as $\mathcal{P} = \langle G, t|s(t)\rangle$, where $s(t) = atbtctdtetft^{-1}gthtit^{-1}$.

Here $b, c, d, e, h \in G$ and $a, f, g, i \in G\backslash\{1\}$. The star graph Γ for $\mathcal{P}$ is given in Figure 2. We apply the transformation $x = tb$ to get that $b = 1$ in G. By using the methods

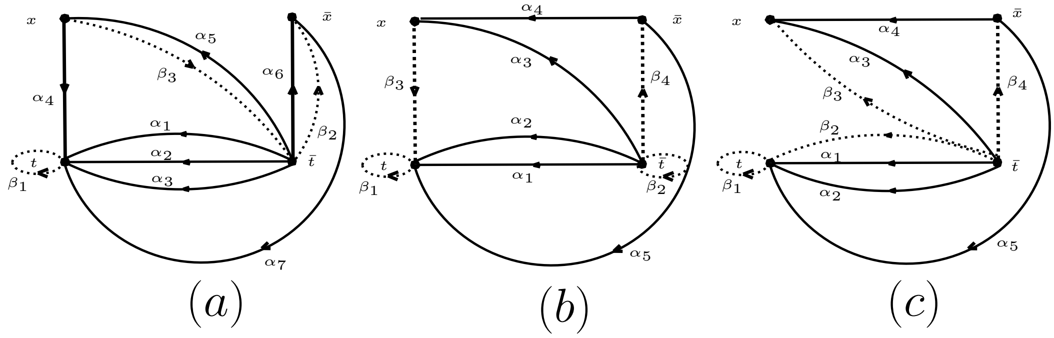

Figure 1: Star graph Γ

given in [5, 8] we conclude that possible vertices of degree 2 (in the diagram associated to $\mathcal{P}$) are (upto cyclic permutation and inversion)

$$S = \{ag,\, ag^{-1},\, fi,\, fi^{-1},\, bc^{-1},\, bd^{-1}\, be^{-1},\, bh^{-1},\, cd^{-1},\, ce^{-1},\, ch^{-1},\, de^{-1},\, dh^{-1},\, eh^{-1}\}.$$

Since G is torsion free therefore it is clear that ag and ag^{-1} can not both hold at the same time. Similarly fi and fi^{-1} can not both hold at the same time. The following lemma gives some general results that will greatly simplify the proofs. This is an application of the results given in [1, 2].

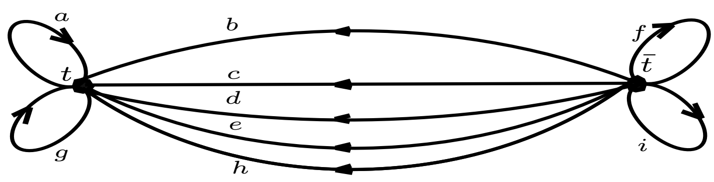

Figure 2: Star graph Γ

Lemma 2. *The presentation $\mathcal{P} = \langle G, t|s(t)\rangle$, where $s(t) = atbtctdtetft^{-1}gthtit^{-1}$ is aspherical if any one of the following holds:*

(i) $a = g^{-1}$

(ii) $a = g,\ f = i$

(iii) $a = g,\ b = h$

Proof. A new generator x will be introduced to obtain the presentation $\mathcal{Q} = \langle G, t, x | r_1, r_2 \rangle$.

(i) Let $a = g^{-1}$. The relator $s(t)$ is given by $s(t) = atbtctdtetft^{-1}a^{-1}thtit^{-1}$. We substitute $x = t^{-1}a^{-1}t$ to get $r_1 = x^{-1}btctdtetfxhti$ and $r_2 = t^{-1}a^{-1}tx^{-1}$. The star graph Γ for $\mathcal{Q}$ is given by Figure 1 (a) in which (using r_1) $\alpha_1 = e$, $\alpha_2 = d$, $\alpha_3 = c$, $\alpha_4 = b$, $\alpha_5 = f$, $\alpha_6 = i$, $\alpha_7 = h$; and (using r_2) $\beta_1 = a^{-1}$, $\beta_2 = 1$, $\beta_3 = 1$. We assign a weight function θ such that $\theta(\alpha_4) = \theta(\alpha_6) = \theta(\beta_1) = \theta(\beta_2) = 0$. All other edges are assigned a weight 1. Then $\Sigma(1 - \theta(\alpha_i)) = \Sigma(1 - \theta(\beta_j)) = 2$ shows that (W1) is satisfied. Also each cycle in Γ of weight less than 2 has label a^m or i^m, $(m \neq 0)$ and $(a, i \neq 1)$ and since G is torsion free (W2) is satisfied. Moreover (W3) clearly holds.

(ii) We have

$$s(t) = atbtctdtetit^{-1}athtit^{-1},\ r_1 = xbtctdtexh,\ r_2 = tit^{-1}atx^{-1}.$$

The star graph Γ is given by Figure 1 (b) in which $\alpha_1 = c$, $\alpha_2 = d$, $\alpha_3 = e$, $\alpha_4 = h$, $\alpha_5 = b$; and $\beta_1 = a$, $\beta_2 = i$, $\beta_3 = 1$, $\beta_4 = 1$. We assign a weight function θ such that $\theta(\alpha_3) = \theta(\alpha_5) = \theta(\beta_1) = \theta(\beta_2) = 0$. All other edges are assigned a weight 1. Then $\Sigma(1 - \theta(\alpha_i)) = \Sigma(1 - \theta(\beta_j)) = 2$ shows that (W1) is satisfied. Also each cycle in Γ of weight less than 2 has label a^m or i^m, $(m \neq 0)$ and $(a, i \neq 1)$ and since G is torsion free (W2) is satisfied. Moreover (W3) clearly holds.

(iii) We have

$$s(t) = atbtctdtetft^{-1}atbtit^{-1},\ r_1 = xctdtetfxi,\ r_2 = t^{-1}atbtx^{-1}.$$

The star graph Γ is given by Figure 1 (c) in which $\alpha_1 = e$, $\alpha_2 = d$, $\alpha_3 = f$, $\alpha_4 = i$, $\alpha_5 = c$; and $\beta_1 = a$, $\beta_2 = b$, $\beta_3 = 1, \beta_4 = 1$. We assign a weight function θ such that $\theta(\alpha_3) = \theta(\alpha_5) = \theta(\beta_1) = \theta(\beta_3)$. All other edges are assigned a weight 1.

We have the desired result.

Corollary 1. *The presentation $\mathcal{P} = \langle G, t | s(t) \rangle$, is aspherical if any one of the following holds:*

(i) $a = g^{-1}$ *and* $R \in \{fi,\ fi^{-1},\ bc^{-1},\ bd^{-1}\, be^{-1},\ bh^{-1},\ cd^{-1},\ ce^{-1},\ ch^{-1},\ de^{-1},\ dh^{-1},\ eh^{-1}\}$

(ii) $a = g,\ f = i$ *and* $R \in \{bc^{-1},\ bd^{-1}\, be^{-1},\ bh^{-1},\ cd^{-1},\ ce^{-1},\ ch^{-1},\ de^{-1},\ dh^{-1},\ eh^{-1}\}$

(iii) $a = g,\ b = h$ *and* $R \in \{cd^{-1},\ ce^{-1},\ ch^{-1},\ de^{-1},\ dh^{-1},\ eh^{-1}\}$

Proof. The result is clear from lemma 3 by taking the weight function as given in lemma 3, part 1, 2 and 3 respectively.

Lemma 3. *The presentation $\mathcal{P} = \langle G, t | s(t) \rangle$, where $s(t) = atbtctdtetft^{-1}gthtit^{-1}$ is aspherical if any one of the following holds:*

(i) $a = g,\ b = c$

(ii) $a = g,\ b = d$

(iii) $a = g,\ b = e$

(iv) $a = g,\ c = d$

(v) $a = g,\ c = e$

(vi) $a = g,\ c = h$

(vii) $a = g,\ d = e$

(viii) $a = g,\ d = h$

(ix) $a = g,\ e = h$

(x) $b = c,\ d = h$

(xi) $b = c,\ e = h$

(xii) $b = d,\ c = e$

(xiii) $b = d,\ c = h$

(xiv) $b = d,\ e = h$

(xv) $b = e,\ c = h$

(xvi) $b = h,\ c = d$

(xvii) $b = c,\ d = e$

Proof.

(i) In this case Δ is shown in Figure 3. Since $d_\Delta(v_a) = d_\Delta(v_b) = 2$ or $d_\Delta(v_b) = d_\Delta(v_c) = 2$ can not occur therefore it can be assumed that $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_g) = 2$ as shown in Figure 3. In this case $c(\Delta) \leq 0$.

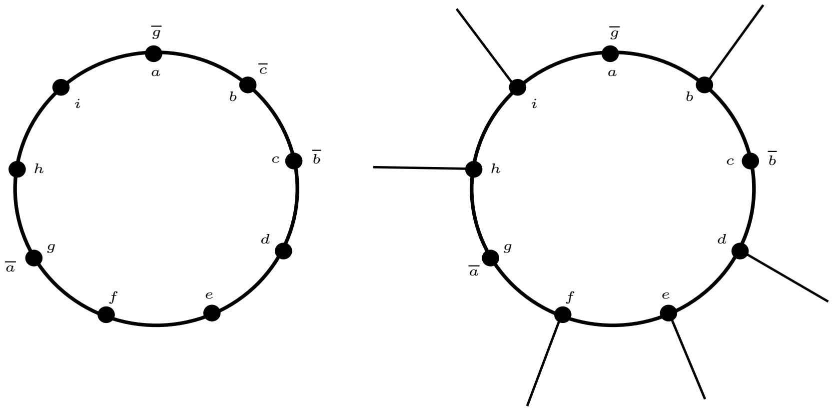

Figure 3: Region Δ

(ii) In this case Δ is shown in Figure 4. Since degree of vertices v_a and v_b can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_a) = d_\Delta(v_d) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_b) = d_\Delta(v_d) = d_\Delta(v_g) = 2$.

as shown in Figure 4. In both of these cases $c(\Delta) \leq 0$.

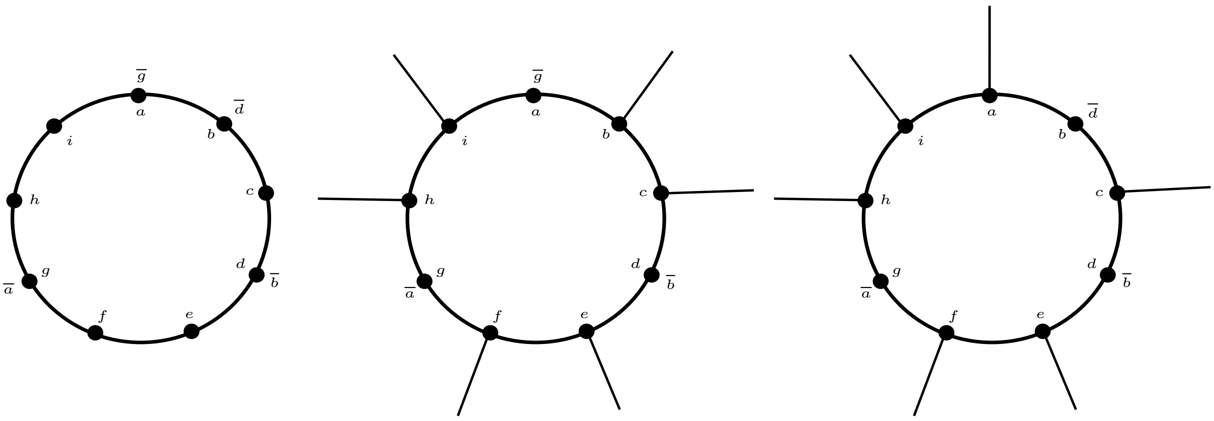

Figure 4: Region Δ

(iii) In this case Δ is shown in Figure 5. Since degree of vertices v_a and v_b can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_a) = d_\Delta(v_e) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_b) = d_\Delta(v_e) = d_\Delta(v_g) = 2$.

as shown in Figure 5. In both of these cases $c(\Delta) \leq 0$.

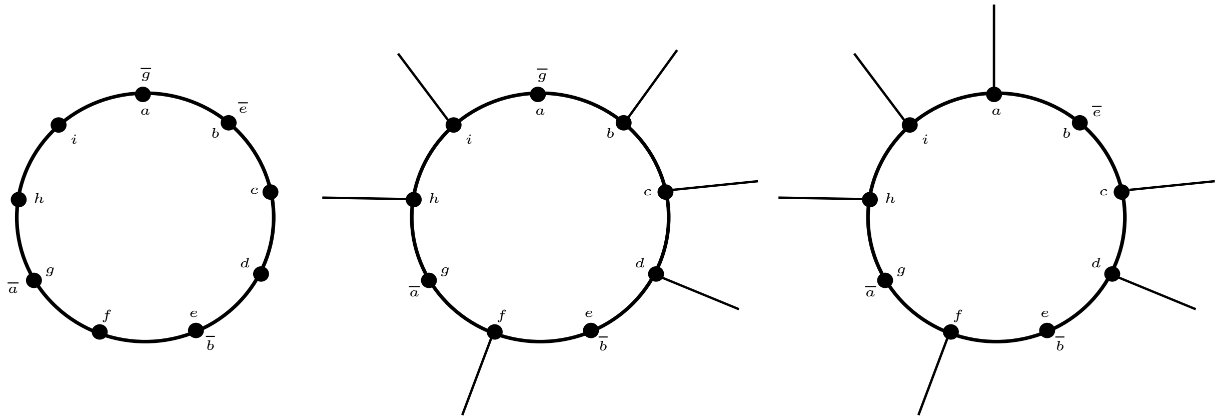

Figure 5: Region Δ

(iv) In this case Δ is shown in Figure 6. Since degree of vertices v_a and v_b can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_a) = d_\Delta(v_d) = d_\Delta(v_g) = 2$.

as shown in Figure 6. In both of these cases $c(\Delta) \leq 0$.

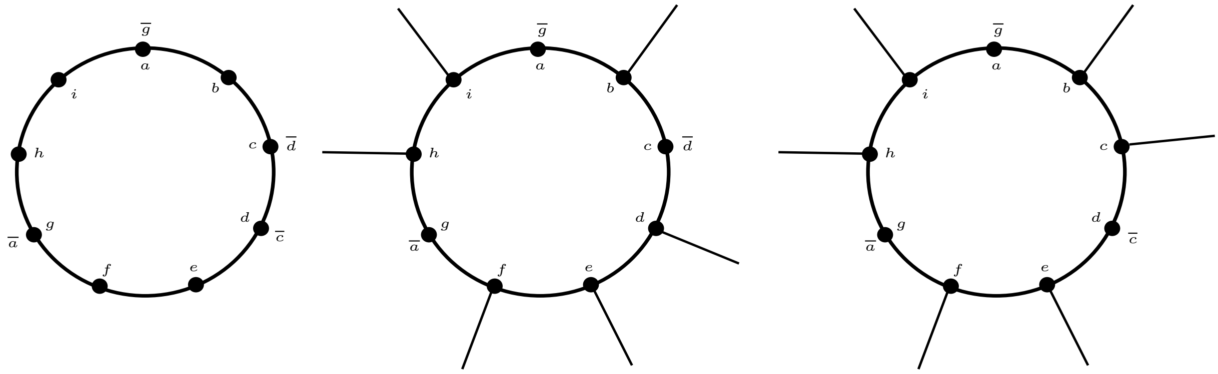

Figure 6: Region Δ

(v) In this case Δ is shown in Figure 7. Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$ which implies $l_\Delta(v_a) = ag^{-1},\ l_\Delta(v_c) = ce^{-1},\ l_\Delta(v_e) = ec^{-1}$ ans $l_\Delta(v_g) = ga^{-1}$ as shown in Figure 7. In order to have positive curvature the remaining vertices must be of degree 3. Observe that $l_\Delta(v_a) = ag^{-1}$ and $l_\Delta(v_c) = ce^{-1}$ implies that $l_\Delta(v_b) = h^{-1}bd^{-1}w$ where $w \in \{b, c, d, e, h\}$ which implies $d_\Delta(v_b) > 3$. Notice that $l_\Delta(v_e) = ec^{-1}$ and $l_\Delta(v_g) = ga^{-1}$ implies that $l_\Delta(v_f) = d^{-1}fi^{-1}w$ where $w \in \{b, c, d, e, h\}$ which implies $d_\Delta(v_f) > 3$. Since $d_\Delta(v_b) > 3$ and $d_\Delta(v_f) > 3$ so $c(\Delta) \leq 0$.

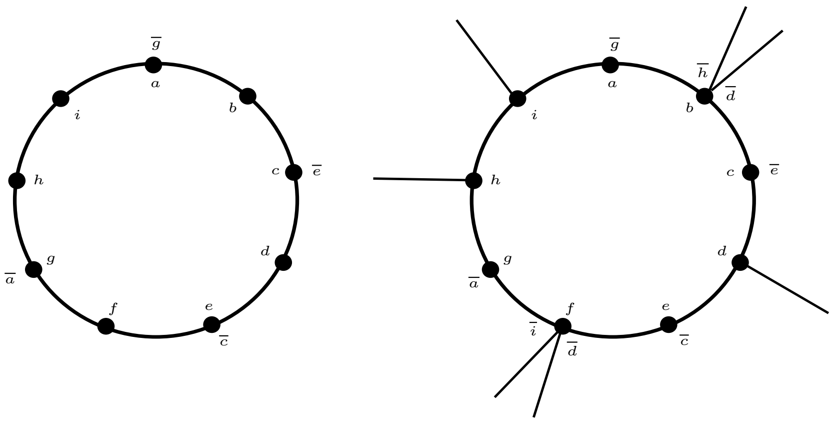

Figure 7: Region Δ

(vi) In this case Δ is shown in Figure 8. Since degree of vertices v_g and v_h can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_h) = 2$.

as shown in Figure 8. In both of these cases $c(\Delta) \leq 0$.

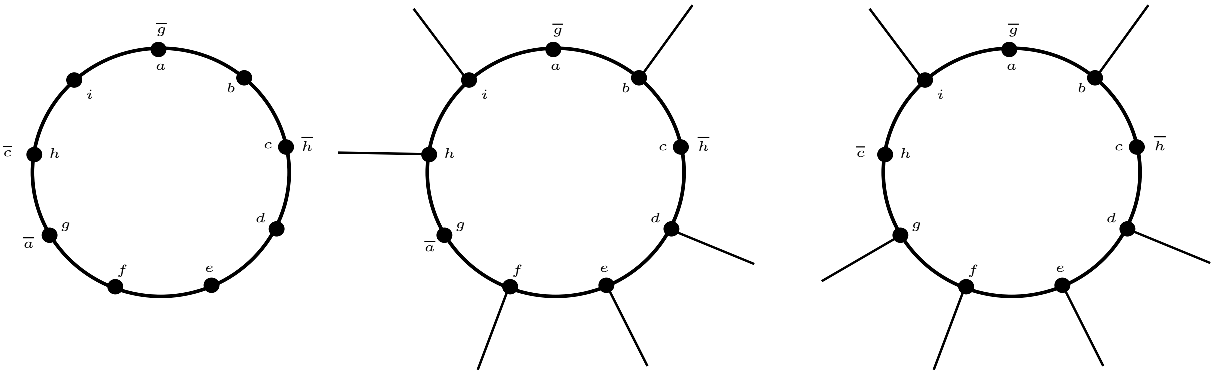

Figure 8: Region Δ

(vii) In this case Δ is shown in Figure 9. Since degree of vertices v_d and v_e can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_a) = d_\Delta(v_d) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_a) = d_\Delta(v_e) = d_\Delta(v_g) = 2$.

as shown in Figure 9. In both of these cases $c(\Delta) \leq 0$.

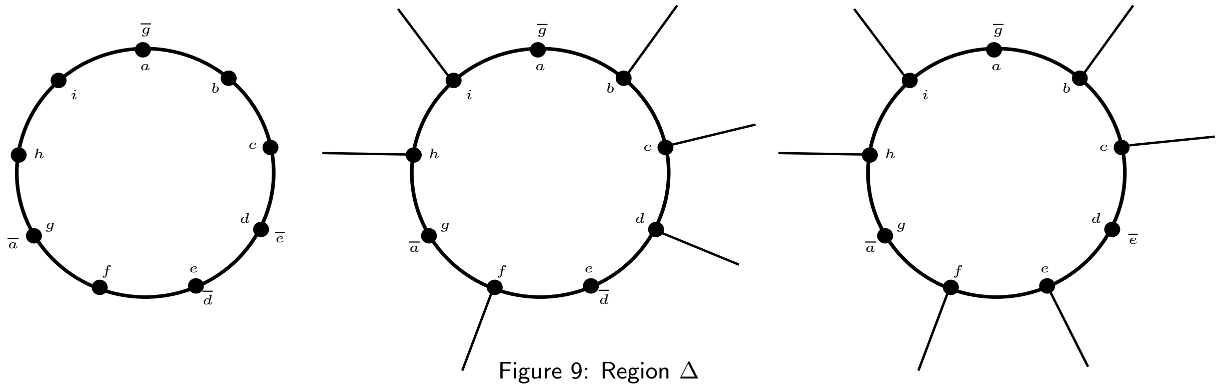

Figure 9: Region Δ

(viii) In this case Δ is shown in Figure 10. Since degree of vertices v_g and v_h can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_a) = d_\Delta(v_d) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_a) = d_\Delta(v_d) = d_\Delta(v_h) = 2$.

as shown in Figure 10. In both of these cases $c(\Delta) \leq 0$.

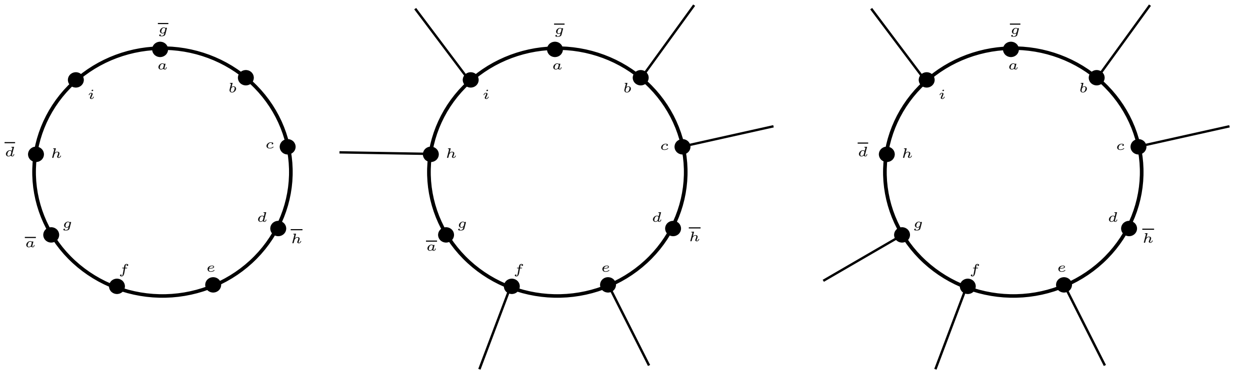

Figure 10: Region Δ

(ix) In this case Δ is shown in Figure 11. Since degree of vertices v_g and v_h can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_a) = d_\Delta(v_e) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_a) = d_\Delta(v_e) = d_\Delta(v_h) = 2$.

as shown in Figure 11. In both of these cases $c(\Delta) \leq 0$.

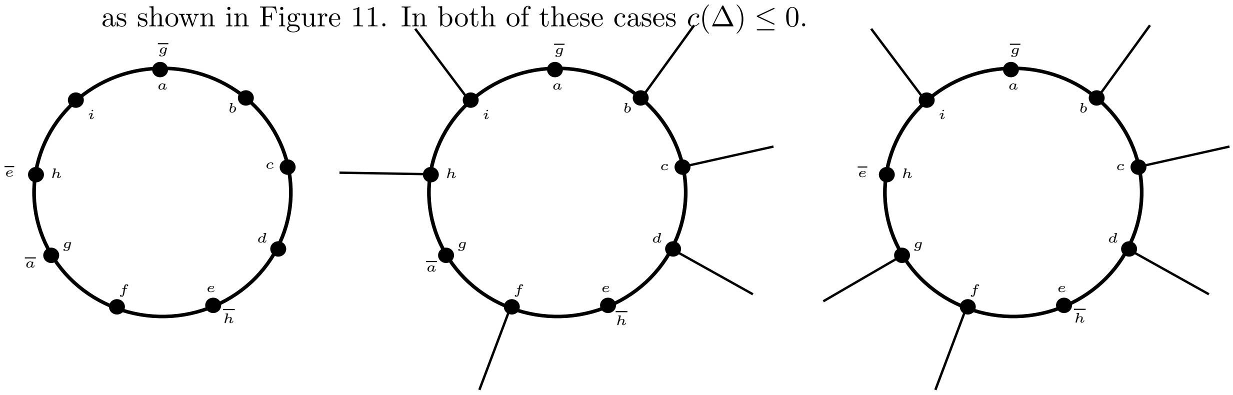

Figure 11: Region Δ

(x) In this case Δ is shown in Figure 12. Since $d_\Delta(v_b) = d_\Delta(v_c) = 2$ or $d_\Delta(v_c) = d_\Delta(v_d) = 2$ can not occur therefore it can be assumed that $d_\Delta(v_b) = d_\Delta(v_d) = d_\Delta(v_h) = 2$ as shown in Figure 12. In this case $c(\Delta) \leq 0$.

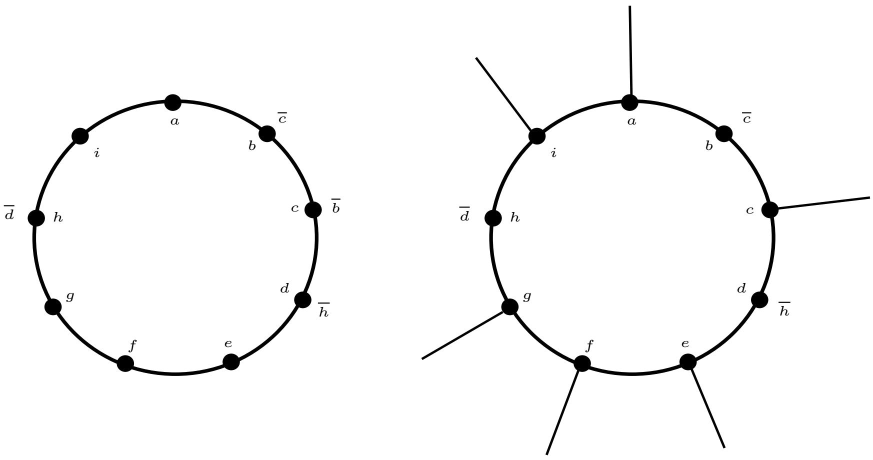

Figure 12: Region Δ

(xi) In this case Δ is shown in Figure 13. Since degree of vertices v_b and v_c can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_b) = d_\Delta(v_e) = d_\Delta(v_h) = 2$;

(b) $d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_h) = 2$.

as shown in Figure 13. In both of these cases $c(\Delta) \leq 0$.

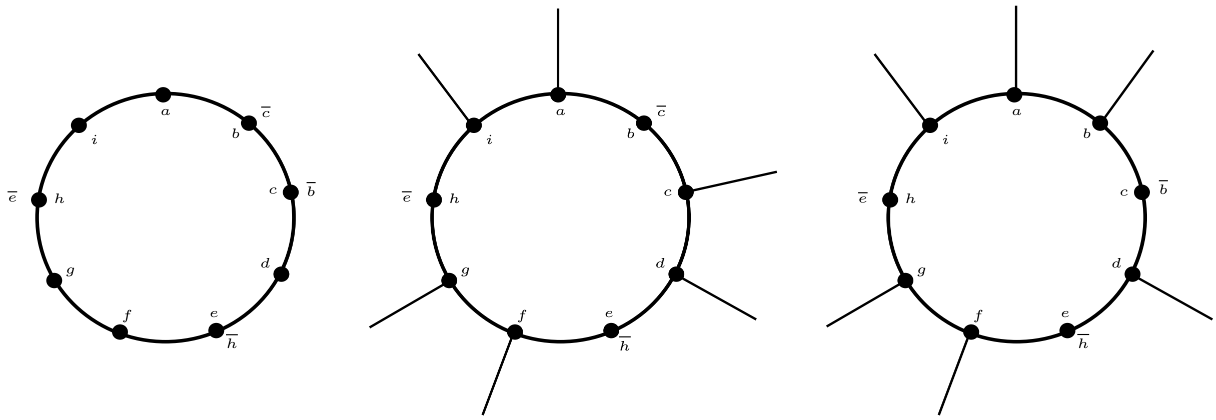

Figure 13: Region Δ

(xii) In this case Δ is shown in Figure 14. Since $d_\Delta(v_b) = d_\Delta(v_c) = 2$ or $d_\Delta(v_c) = d_\Delta(v_d) = 2$ or $d_\Delta(v_d) = d_\Delta(v_e) = 2$ can not occur therefore $c(\Delta) \leq 0$.

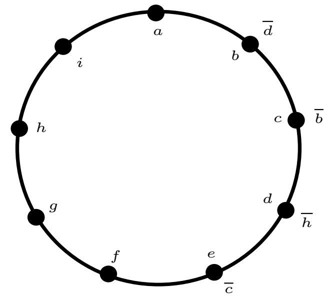

Figure 14: Region Δ

(xiii) In this case Δ is shown in Figure 15. Since $d_\Delta(v_b) = d_\Delta(v_c) = 2$ or $d_\Delta(v_c) = d_\Delta(v_d) = 2$ can not occur therefore it can be assumed that $d_\Delta(v_b) = d_\Delta(v_d) = d_\Delta(v_h) = 2$ as shown in Figure 15. In this case $c(\Delta) \leq 0$.

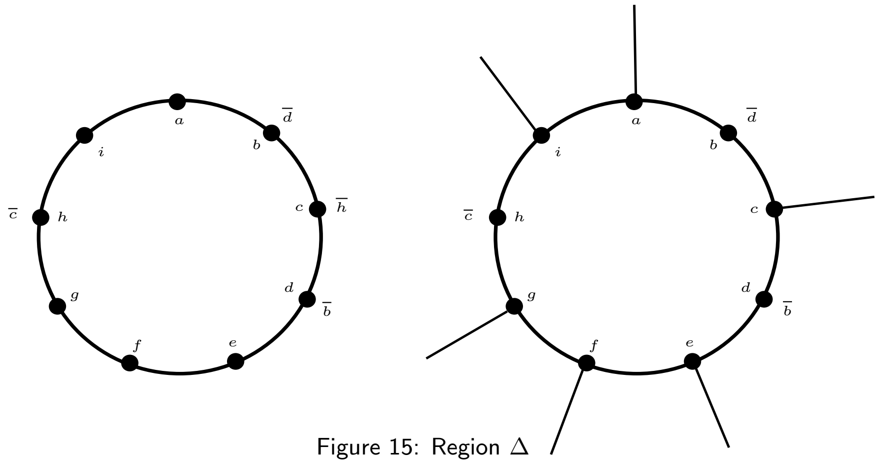

Figure 15: Region Δ

(xiv) In this case Δ is shown in Figure 16. Since degree of vertices v_d and v_e can not be 2 together so there are the following two cases to consider:

(a) $d_\Delta(v_b) = d_\Delta(v_d) = d_\Delta(v_h) = 2$;

(b) $d_\Delta(v_b) = d_\Delta(v_e) = d_\Delta(v_h) = 2$.

as shown in Figure 16. In both of these cases $c(\Delta) \leq 0$.

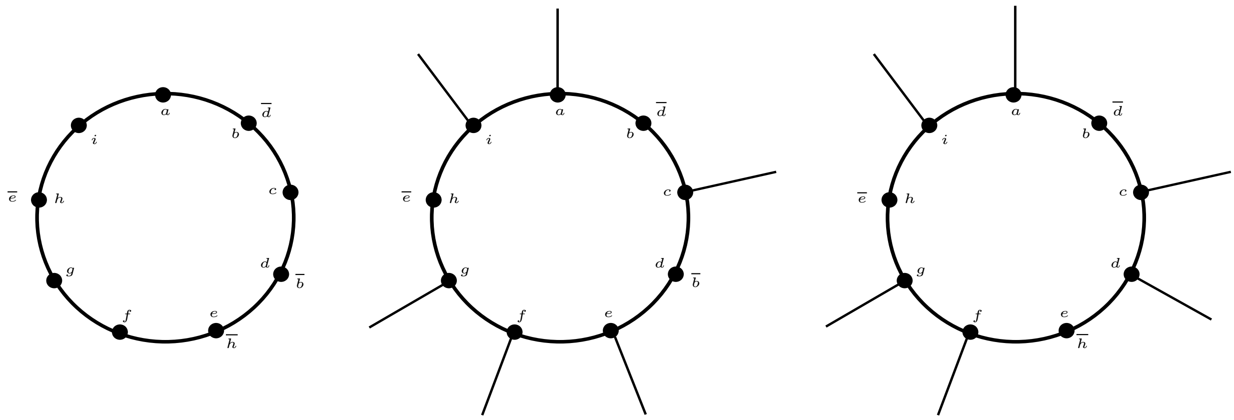

Figure 16: Region Δ

(xv) In this case Δ is shown in Figure 17. Since degree of vertices v_b and v_c can not be 2 together so there are the following two cases to consider:

(a) $d_{\Delta}(v_b) = d_{\Delta}(v_e) = d_{\Delta}(v_h) = 2$;

(b) $d_{\Delta}(v_c) = d_{\Delta}(v_e) = d_{\Delta}(v_h) = 2$.

as shown in Figure 17. In both of these cases $c(\Delta) \leq 0$.

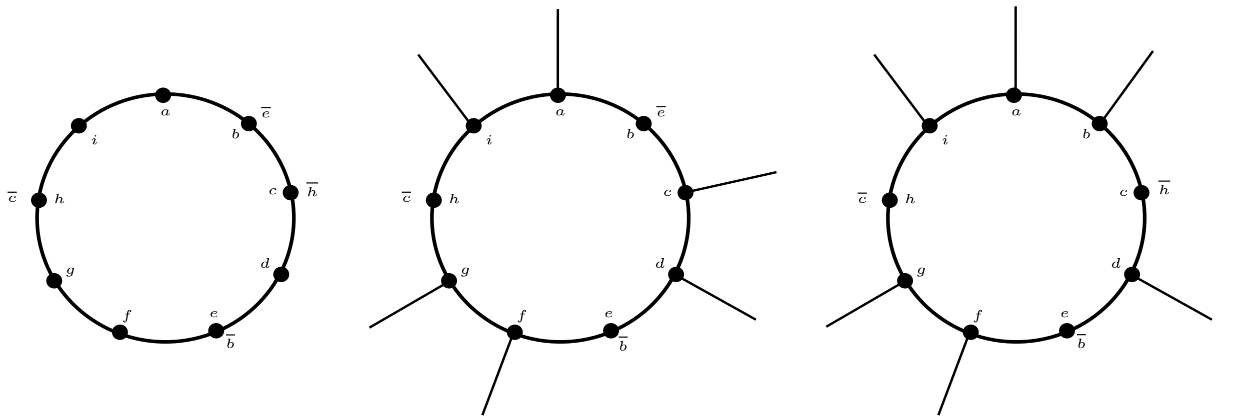

Figure 17: Region Δ

(xvi) In this case Δ is shown in Figure 18. Since $d_{\Delta}(v_b) = d_{\Delta}(v_c) = 2$ or $d_{\Delta}(v_c) = d_{\Delta}(v_d) = 2$ can not occur therefore it can be assumed that $d_{\Delta}(v_b) = d_{\Delta}(v_d) = d_{\Delta}(v_h) = 2$ as shown in Figure 18. In this case $c(\Delta) \leq 0$.

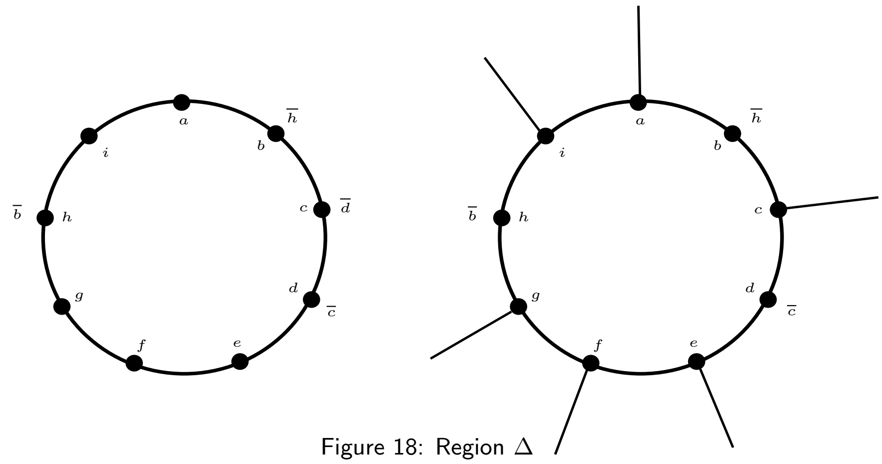

Figure 18: Region Δ

(xvii) In this case Δ is shown in Figure 19. Since $d_\Delta(v_b) = d_\Delta(v_c) = 2$ or $d_\Delta(v_c) = d_\Delta(v_d) = 2$ or $d_\Delta(v_d) = d_\Delta(v_e) = 2$ can not occur therefore $c(\Delta) \leq 0$.

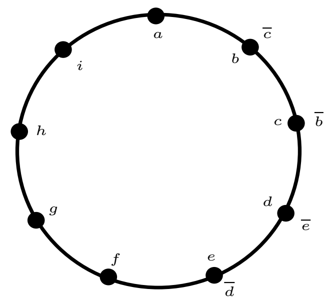

Figure 19: Region Δ

Remark 1. *It is worth mentioning here that a few of the cases still remain open for this equation. These cases are extremely technical in detail and will be considered in a different article.*

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