EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 2, 2019, 279-293
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Hankel Transform of (q, r)-Dowling Numbers

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Abstract. In this paper, the authors establish certain combinatorial interpretation for q-analogue of r-Whitney numbers of the second kind defined by Corcino and Cañete in the context of A-tableaux. They derive convolution-type identities by making use of the combinatorics of A-tableaux. Finally, they define a q-analogue of r-Dowling numbers and obtain some necessary properties including its Hankel transform.

Key Words and Phrases: Whitney numbers, Dowling numbers, generating function, q-analogue, q-exponential function, A-tableau, convolution formula, Hankel transform, Hankel matrix, binomial transform.

1. Introduction

The binomial transform B of a sequence $A = \{a_n\}$ is the sequence $\{b_n\}$ defined by

$$b_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k.$$

That is, $B(A) = b_n$. It is one of the common and useful transforms that frequently appeared in the literature of integer sequences (see [16]). The inverse binomial transform (or inverse transform) C of a sequence A is the sequence $\{c_n\}$ defined by

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$

That is, $C(A) = c_n$.

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DOI: https://doi.org/10.29020/nybg.ejpam.v12i2.3406

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The *Hankel matrix* H_n of order n of a sequence $A = \{a_0, a_1, \ldots, a_n\}$ is given by $H_n = (a_{i+j})_{0\leq i,j\leq n}$. The *Hankel determinant* h_n of order of n of A is the determinant of the corresponding Hankel matrix of order n. That is, $h_n = det(H_n)$. The *Hankel transform* of the sequence A, denoted by $H(A)$, is the sequence $\{h_n\}$ of Hankel determinants of A. For instance, the Hankel transform of the sequence of Catalan numbers $C = \{\frac{1}{n+1}\binom{2n}{n}\}_{n=1}^{\infty}$, is given by

$$H(C) = \{1, 1, 1, \ldots, \}$$

and the sequence of the sum of two consecutive Catalan numbers, $a_n = c_n + c_{n+1}$, with c_n the nth Catalan numbers, has the Hankel transform

$$H(a_n) = \{F_{2n+1}\}_{n=0}^{\infty}$$

where F_n is the nth Fibonacci numbers [12].

One remarkable property of Hankel transform is established by Layman [12], which states that the Hankel transform of an integer sequence is invariant under binomial and inverse transforms. That is, if A is an integer sequence, B is binomial transform of A and C is the inverse transform of A, then

$$H(B(A)) = H(A) \text{ and } H(C(A)) = H(A).$$

This property played an important role in proving that the Hankel transform of the sequence of Bell number $\{B_n)\}$ [1] and that of r-Bell numbers $\{B_{n,r}\}$ [14] are equal. Recently, in the paper by R. Corcino and C. Corcino [7], this property has also been used in proving that the Hankel transform of the sequence of generalized Bell numbers $\{G_{n,r,\beta}\}$ is given by

$$H(G_{n,r,\beta}) = \prod_{j=0}^{n} \beta^j j!$$

where $G_{n,r,\beta}$ is the sum of (r,β)-Stirling numbers $\left\{ {n \atop k} \right\}_{r,\beta}$

$$G_{n,r,\beta} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{r,\beta}$$

(see [5, 8]), which are also known as (r,β)-Bell numbers. In the same paper, the authors have made an attempt to establish the Hankel transform for the q-analogue of (r,β)-Bell numbers. However, they are not successful with their attempt and have conjectured that the Hankel transform for the q-analogue of (r,β)-Bell numbers when $r = 0$ is equal to

$$H\left(\mathcal{G}^q_{n,\beta,0}\right) = \prod_{k=0}^{n} q^{f(n,k)}[\beta]_q^k [k]_{q^\beta}! \tag{1}$$

for some number $f(n,k)$, which is a function of n and k. With this, the present authors have decided to use other method. Recently, R. Corcino et al.[9] have successfully established the Hankel transform for the q-analogue of noncentral Bell numbers. This motivates the present authors to use this method to establish the Hankel transform for the q-analogue of (r, β)-Bell numbers $G_{n,r,\beta}$. It is important to note that the numbers $G_{n,r,\beta}$ are equivalent to the r-Dowling numbers $D_{m,r}(n)$, which are defined as the sum of r-Whitney numbers of the second kind, denoted by $W_{m,r}(n, k)$. That is,

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n, k).$$

The term "r-Dowling numbers" was introduced by Cheon and Jung [3].

2. A q-Analogue of $W_{m,r}(n, k)$: Second Form

A q-analogue of both kinds of Stirling numbers was first defined by Carlitz in [2]. The second kind of which, known as q-Stirling numbers of the second kind, is defined in terms of the following recurrence relation

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q S_q[n-1, k] \tag{2}$$

in connection with a problem in abelian groups, such that when $q \to 1$, this gives the triangular recurrence relation for the classical Stirling numbers of the second kind $S(n, k)$

$$S(n, k) = S(n-1, k-1) + kS(n-1, k). \tag{3}$$

A different way of defining q-analogue of Stirling numbers of the second kind has been adapted in the paper by [10] which is given as follows

$$S_q[n, k] = q^{k-1} S_q[n-1, k-1] + [k]_q S_q[n-1, k]. \tag{4}$$

This type of q-analogue gives the Hankel transform of q-exponential polynomials and numbers which are certain q-analogue of Bell polynomials and numbers. Recently, a q-analogue of r-Whitney numbers of the second kind was defined by Corcino and Cañete [6] parallel to the definition for q-analogue of noncentral Stirling numbers of the second kind as follows:

Definition 1. *For non-negative integers n and k, and real number a, a q-analogue $W_{m,r}[n, k]_q$ of $W_{m,r}(n, k)$ is defined by*

$$W_{m,r}[n, k]_q = q^{m(k-1)+r} W_{m,r}[n-1, k-1]_q + [mk + r]_q W_{m,r}[n-1, k]_q. \tag{5}$$

where $W_{m,r}[0, 0]_q = 1$, $W_{m,r}[n, k]_q = 0$ for $n < k$ or $n, k < 0$ and $[t-k]_q = \frac{1}{q^k}([t]_q - [k]_q)$.

Remark 1. *When $m = 1$ and $r = 0$, the relation* (5) *reduces to* (4)*. This implies that*

$$W_{1,0}[n, k]_q = S_q[n, k]. \tag{6}$$

The q-analogue $W_{m,r}[n,k]_q$ satisfies the following properties:

Vertical and Horizontal Recurrence Relations

$$W_{m,r}[n+1,k+1]_q = q^{mk+r}\sum_{j=k}^{n}[m(k+1)+r]_q^{n-j}W_{m,r}[j,k]_q; \tag{7}$$

$$W_{m,r}[n,k]_q = \sum_{j=0}^{n-k}(-1)^j q^{-r-m(k+j)}\frac{r_{k+j+1,q}}{r_{k+1,q}}W_{m,r}[n+1,k+j+1]_q; \tag{8}$$

Horizontal Generating Function

$$\sum_{k=0}^{n} W_{m,r}[n,k]_q[t-r|m]_{k,q} = [t]_q^n. \tag{9}$$

Explicit Formula

$$W_{m,r}[n,k]_q = \frac{1}{[k]_{q^m}![m]_q^k}\sum_{j=0}^{k}(-1)^{k-j}q^{m\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{q^m}[jm+r]_q^n \tag{10}$$

$$= \frac{1}{[k]_{q^m}![m]_q^k}\left[\Delta^k_{q^m,m}[x+r]_q^n\right]_{x=0} \tag{11}$$

Exponential Generating Function

$$\sum_{n\geq 0} W_{m,r}[n,k]_q\frac{[t]_q^n}{[n]_q!} = \frac{1}{[k]_q m![m]_q^k}\left[\Delta_{q^m,m^k}e_q\big([x+jm+r]_q[t]_q\big)\right]_{x=0}. \tag{12}$$

Rational Generating Function

$$\Psi_k(t) = \sum_{n\geq k} W_{m,r}[n,k]_q[t]_q^n = \frac{q^{m\binom{k}{2}+kr}[t]_q^k}{\prod_{j=0}^{k}(1-[mj+r]_q[t]_q)}.$$

Explicit Formula in Symmetric Function Form

$$W_{m,r}[n,k]_q = q^{m\binom{k}{2}+kr}\sum_{S_1+S_2+\cdots S_k=n-k}\prod_{j=0}^{k}[mj+r]_q^{S_j}$$

$$= \sum_{0\leq j_1\leq j_2\leq\cdots j_{n-k}\leq k} q^{m\binom{k}{2}+kr}\prod_{i=1}^{n-k}[mj+r]_q.$$

We now define another form of q-analogue of r-Whitney numbers of the second, denoted by $W^*_{m,r}[n,k]_q$, as follows

$$W^*_{m,r}[n,k]_q := q^{-kr-m\binom{k}{2}} W_{m,r}[n,k]_q.$$

Hence,

$$W^*_{m,r}[n,k] = \sum_{0\le j_1\le j_2\le \ldots \le j_{n-k}\le k} \prod_{i=1}^{n-k} [mj_i + r]_q. \tag{13}$$

All other properties parallel to those of $W_{m,r}[n,k]_q$ can easily be established by imbedding the factor $q^{-kr-m\binom{k}{2}}$ in the derivations or multiply directly to the resulting identities/formula.

Definition 2. [13] An A-*tableau* is a list ϕ of column c of a Ferrer's diagram of a partition λ(by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A = (r_i)_{i\ge 0}$, a strictly increasing sequence of nonnegative integers.

Let ω be a function from the set of nonnegative integers N to a ring K. Suppose Φ is an A-tableau with l columns of lengths $|c| \le h$. We use $T^A_r(h,l)$ to denote the set of such A-tableaux. Then, we set

$$\omega_A(\Phi) = \prod_{c\in\Phi} \omega(|c|).$$

Note that Φ might contain a finite number of columns whose lengths are zero since $0 \in A = \{0, 1, 2, \ldots, k\}$ and if $\omega(0) \neq 0$.

From this point onward, whenever an A-tableau is mentioned, it is always associated with the sequence $A = \{0, 1, 2, \ldots, k\}$.

We are now ready to mention the following theorem.

Theorem 1. *Let $\omega : N \to K$ denote a function from N to a ring K (column weights according to length) which is defined by $\omega(|c|) = [m|c| + r]_q$ where r is a complex number, and $|c|$ is the length of column l of an A-tableau in $T^A_r(k, n-k)$. Then*

$$W^*_{m,r}[n,k] = \sum_{\phi\in T^A_r(k,n-k)} \prod_{c\in\phi} \omega(|c|).$$

Proof. Let $\Phi \in T^A_r(k, n-k)$. This means that Φ has exactly $n-k$ columns say $c_1, c_2, \cdots, c_{n-k}$ whose lengths are $j_1, j_2, \cdots, j_{n-k}$, respectively. Now, for each column $c_i \in \Phi, i = 1, 2, 3, \cdots, n-k$, we have $|c_i| = j_i$ and

$$\omega(|c_i|) = [m|j_i| + r]_q.$$

Then

$$\prod_{c\in\Phi}\omega(|c|)=\prod_{i=1}^{n-k}\omega(|c_i|)=\prod_{i=1}^{n-k}[m|j_i|+r]_q.$$

Since $\Phi\in T_r^A(k,n-k)$, then

$$\begin{aligned}\sum_{\Phi\in T_r^A(k,n-k)}\prod_{c\in\Phi}\omega(|c|) &= \sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}\le k}\prod_{c\in\Phi}\omega(|c|)\\ &= \sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}\le k}\prod_{i=1}^{n-k}[m|j_i|+r]_q\\ &= W^*_{m,r}[n,k].\end{aligned}$$

□

Suppose that for some numbers r_1 and r_2, we have $r=r_1+r_2$. Then, equation (13) yields

$$W^*_{m,r}[n,k]_q=\sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}\le k}\prod_{i=1}^{n-k}[(mj_i+r_1)+r_2]_q.$$

That is, for any $\phi\in T_r^A(k,n-k)$,

$$\omega_A(\phi)=\prod_{c\in\phi}[(mj_i+r_1)+r_2]_q,$$

where $|c|\in\{0,1,2,\ldots,k\}$. Note that the weight of each column of ϕ can be considered as a finite sum with additive constant r_2, that is, for each $c\in\phi$, we can write

$$\omega(|c|)=\frac{1}{q^{r_2}}(\omega^*(|c|)+[r_2]_q), \tag{14}$$

where $\omega^*(|c|)=[m|c|+r_1]_q$. The following theorem determines how an additive constant affects the recurrence formula for $W_{m,r}[n,k]_q$. From Theorem 1,

$$W^*_{m,r}[n,k]_q=\sum_{\phi\in T_r^A(k,n-k)}\omega_A(\phi)=\sum_{\phi\in T_r^A(k,n-k)}\prod_{c\in\phi}\omega(|c|)$$

where

$$\begin{aligned}\omega_A(\phi) &= \prod_{c\in\phi}[m|c|+r]_q,\ \ \text{where}\ \ |c|\in\{0,1,\ldots,k\}\\ &= \prod_{i=1}^{n-k}[mj_i+r]_q,\ \ \text{where}\ \ j_i\in\{0,1,\ldots,k\}.\end{aligned}$$

If $r = r_1 + r_2$ for some r_1 and r_2, then by (14),

$$\begin{aligned}\omega_A(\phi) &= \prod_{i=1}^{n-k} \frac{1}{q^{r_2}} \left(\omega^*(j_i) + [r_2]_q\right), \quad \text{where} \ \ \omega^*(j_i) = [mj_i + r_1]_q \\ &= q^{-(n-k)r_2} \left(\omega^*(j_1) + [r_2]_q\right) \left(\omega^*(j_2) + [r_2]_q\right) \cdots \left(\omega^*(j_{n-k}) + [r_2]_q\right)) \\ &= q^{-(n-k)r_2} \sum_{l=0}^{n-k} ([r_2]_q)^{n-k-l} \sum_{j_1 \leq j_1 \leq j_2 \leq \ldots \leq j_l \leq j_{n-k}} \prod_{i=1}^{l} \omega^*(j_i).\end{aligned}$$

Suppose B_ϕ is the set of all A-tableaux corresponding to ϕ such that for each $\psi \in B_\phi$, either

$$\begin{array}{ll} \psi & \text{has no column whose weight is } [r_2]_q, \text{ or} \\ \psi & \text{has one column whose weight is } [r_2]_q, \text{ or} \\ \psi & \text{has two columns whose weights are } [r_2]_q, \text{ or} \\ & \vdots \\ \psi & \text{has } (n-k) \text{ columns whose weights are } [r_2]_q. \end{array}$$

Then, we may write

$$\omega_A(\phi) = \sum_{\psi \in B_\phi} \omega_A(\psi).$$

Now, if l columns in ψ have weights other than $[r_2]_q$, then

$$\omega_A(\psi) = \prod_{c \in \psi} \omega^*(|c|) = q^{-(n-k)r_2} ([r_2]_q)^{n-k-r} \prod_{i=1}^{r} \omega^*(q_i)$$

where $q_1, q_2, \ldots, q_r \in \{j_1, j_2, \ldots, j_{n-k}\}$. Note that for each l, there corresponds

$$\binom{n-k}{l}$$

tableaux with l columns having weights $\omega^*(j_i) = [mj_i + r_1]_q$. It can be easily verified that,

$$|T_r^A(k, n-k)| = \binom{(n-k)+k}{n-k} = \binom{n}{n-k} = \binom{n}{k}.$$

Thus, $\forall \phi \in T_r^A(k, n-k)$, B_ϕ contains a total of

$$\binom{n}{k}\binom{n-k}{l}$$

tableaux with l columns of weights $\omega^*(j_i)$. However, only $\binom{l+k}{l}$ tableaux with l columns in B_ϕ are distinct. Hence, every distinct tableaux ψ with l columns of weights other than $[r_2]_q$ appears

$$\frac{\binom{n}{k}\binom{n-k}{l}}{\binom{l+k}{l}} = \binom{n}{l+k}$$

times in the collection. Thus,

$$\sum_{\phi\in T_r^A(k,n-k)} \omega_A(\phi) = \sum_{l=0}^{n-k} \binom{n}{l+k} q^{-(n-k)r_2}([r_2]_q)^{n-k-l} \sum_{\varphi\in \bar{B}_l} \prod_{c\in\varphi} \omega^*(|c|)$$

where $\bar{B}_l$ denotes the set of all tableaux φ having l columns of weights $\omega^*(j_i) = [mj_i + r_1]_q$. Reindexing the double sum, we get

$$\sum_{\phi\in T_r^A(k,n-k)} \omega_A(\phi) = \sum_{j=k}^{n} \binom{n}{j} q^{-nr_2}([r_2]_q)^{n-j} \sum_{\varphi\in \bar{B}_{j-k}} \prod_{c\in\varphi} \omega^*(|c|)$$

where $\bar{B}_{j-k}$ is the set of all tableaux φ with $j-k$ columns of weights $\omega^*(j_i) = [mj_i + r_1]_q$ for each $i = 1, 2, \ldots, j-k$. Clearly $\bar{B}_{j-k} = T_{r_1}^A(k, j-k)$. Hence,

$$\sum_{\phi\in T_r^A(k,n-k)} \omega_A(\phi) = \sum_{j=k}^{n} \binom{n}{j} q^{-nr_2}([r_2]_q)^{n-j} \sum_{\varphi\in T_{r_1}^A(k,j-k)} \omega_A(\varphi).$$

Applying Theorem 1, we obtain the following theorem.

Theorem 2. *The q-analogue $W^*_{m,r}[n,k]_q$ satisfies the following identity*

$$W^*_{m,r}[n,k]_q = \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} q^{-nr_2} [r_2]_q^{n-j} W^*_{m,r_1}[j,k]_q$$

where $r = r_1 + r_2$ for some numbers r_1 and r_2.

Suppose

ϕ_1 is a tableau with $k-s$ columns whose lengths are in the set $\{0, 1, \ldots, s\}$, and

ϕ_2 be a tableau with $n-k-j$ columns whose lengths are in the set $\{s+1, s+2, \ldots, s+j+1\}$

Then

$$\phi_1 \in T^{A_1}(s, k-s) \text{ and } \phi_2 \in T^{A_2}(j, n-k-j)$$

where $A_1 = \{0, 1, \ldots, s\}$ and $A_2 = \{s+1, s+2, \ldots, s+j+1\}$. Notice that by joining the columns of ϕ_1 and ϕ_2, we obtain an A-tableau ϕ with $n-s-j$ columns whose lengths are in the set $A = A_1 \cup A_2 = \{0, 1, \ldots, s+j+1\}$. That is, $\phi \in T^A(s+j+1, n-s-j)$. Then,

$$\sum_{\phi \in T^A(s+j+1,n-s-j)} \omega_A(\phi) = \sum_{k=s}^{n-j} \left\{ \sum_{\phi_1 \in T^{A_1}(s,\, k-s)} \omega_{A_1}(\phi_1) \right\} \left\{ \sum_{\phi_2 \in T^{A_2}(j,\, n-k-j)} \omega_{A_2}(\phi_2) \right\}.$$

Note that

$$\begin{aligned} \sum_{\phi_2 \in T^{A_2}(j,\, n-k-j)} \omega_{A_2}(\phi_2) &= \sum_{\phi_2 \in T^{A_2}(j,\, n-k-j)} \prod_{c \in \phi_2} [m|c| + r]_q \\ &= \sum_{s+1 \le g_1 \le \ldots \le g_{\,n-k-j} \le\, s+j+1} \prod_{i=1}^{n-k-j} [mg_i + r]_q \\ &= \sum_{0 \le g_1 \le \ldots \le g_{\,n-k-j} \le j} \prod_{i=1}^{n-k-j} [mg_i + m(s+1) + r]_q. \end{aligned}$$

Thus,

$$\sum_{0 \le g_1 \le \ldots \le g_{n-s-j} \le s+j+1} \prod_{i=1}^{n-s-j} [mg_i + r]_q$$

$$= \sum_{k=s}^{n-j} \left\{ \sum_{0 \le g_1 \le \ldots \le g_{k-s} \le s} \prod_{i=1}^{k-s} [mg_i + r]_q \right\} \left\{ \sum_{0 \le g_1 \le \ldots \le g_{n-k-j} \le j} \prod_{i=1}^{n-k-j} [mg_i + m(s+1) + r]_q \right\}.$$

By (13), we obtain the following theorem.

Theorem 3. *The q-analogue $W^*_{m,r}[n,k]$ satisfies the following convolution-type identity*

$$W^*_{m,r}[n+1, s+j+1]_q = \sum_{k=0}^{n} W^*_{m,r}[k,s]_q W^*_{m,r+m(s+1)}[n-k, j]_q.$$

The next theorem provides another form of convolution-type identity.

Theorem 4. *The q-analogue $W^*_{m,r}[n,k]_q$ satisfies the following second form of convolution formula*

$$W^*_{m,r}[s+j, n]_q = \sum_{k=s}^{n-j} W^*_{m,r}[s,k]_q W^*_{m,r+mk}[j, n-k]_q.$$

Proof. Let

ϕ_1 be a tableau with $s-k$ columns whose lengths are in $A_1 = \{0, 1, \ldots, k\}$, and

ϕ_2 be a tableau with $j-n+k$ columns whose lengths are in $A_2 = \{k, k+1, \ldots, n\}$.

Then $\phi_1 \in T^{A_1}(k, s-k)$ and $\phi_2 \in T^{A_2}(n-k, j-n+k)$. Using the same argument above, we can easily obtain the convolution formula. □

3. $(q,r)-$Dowling Number and Its Hankel Transform

In this section, we define a q-analogue of the r-Dowling numbers and obtain some combinatorial properties that will be used to establish its Hankel transform.

A q-analogue of the r-Dowling numbers, denoted by $\widetilde{D}_{m,r}[n]_q$, is defined by

$$\widetilde{D}_{m,r}[n]_q = \sum_{k=0}^{n} \widetilde{W}_{m,r}[n,k]_q$$

where

$$\widetilde{W}_{m,r}[n,k]_q = q^{kr} W^*_{m,r}[n,k]_q = q^{-m\binom{k}{2}} W_{m,r}[n,k].$$

For brevity, we use the term (q,r)-*Dowling numbers* for $\widetilde{D}_{m,r}[n]_q$.

Remark 2. *When $m = 1$ and $r = 0$, (6) yields*

$$\widetilde{W}_{1,0}[n,k]_q = q^{-\binom{k}{2}} W_{1,0}[n,k] = q^{-\binom{k}{2}} S_q[n,k] = \widetilde{S}_q[n,k]. \tag{15}$$

It follows that the (q,r)-Dowling numbers reduces to

$$\widetilde{D}_{1,0}[n]_q = \widetilde{e}_{q,n}[1] \tag{16}$$

where $\widetilde{e}_{q,n}[z]$ is the q-exponential polynomial in [11] defined by

$$\widetilde{e}_{q,n}[z] = \sum_{k=0}^{n} \widetilde{S}_q[n,k] z^k. \tag{17}$$

Remark 3. *We recall that the Hankel transform of the q-exponential polynomial $\widetilde{e}_{q,n}[z]$ is given by*

$$H\left(\widetilde{e}_{q,n}(z)\right) = q^{\binom{n+1}{3}} [0]![1]! \ldots [n]!(z)^{\binom{n+1}{2}}.$$

It can easily be verified that the Hankel transform of

$$\bar{e}_{q,n}[z] = \sum_{k=0}^{n} \widetilde{S}_q[n,k] z^{n-k} \tag{18}$$

is equal to that of $\widetilde{e}_{q,n}[z]$.

Remark 4. *Since*

$$W_{m,0}[n,k]_q = [m]_q^{n-k}\left\{\frac{1}{[k]_{q^m}!}\sum_{j=0}^{k}(-1)^{k-j}q^{m\binom{k-j}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{q^m}[j]_{q^m}^n\right\}$$
$$= [m]_q^{n-k}S_{q^m}[n,k],$$

we have

$$\widetilde{W}_{m,0}[n,k]_q = q^{-m\binom{k}{2}}W_{m,0}[n,k] = [m]_q^{n-k}\left(q^m\right)^{-\binom{k}{2}}S_{q^m}[n,k] = [m]_q^{n-k}\widetilde{S}_{q^m}[n,k].$$

This implies that

$$\widetilde{D}_{m,0}[n]_q = \sum_{k=0}^{n}\widetilde{W}_{m,0}[n,k]_q = \sum_{k=0}^{n}\widetilde{S}_{q^m}[n,k][m]_q^{n-k}. \tag{19}$$

Thus, using Remark 3, the Hankel transform of $\widetilde{D}_{m,0}[n]_q$ is given by

$$H\left(\widetilde{D}_{m,0}[n]_q\right) = H\left(\bar{e}_{q^m,n}([m]_q)\right) = q^{m\binom{n+1}{3}}[0]_{q^m}![1]_{q^m}!\dots[n]_{q^m}![m]_q^{\binom{n+1}{2}} \tag{20}$$

Clearly, when $q \to 1$, $\widetilde{D}_{m,r}[n]_q \to \widetilde{D}_{m,r}(n)$, the r-Dowling numbers. By making use of Theorem 2, with $r_1 = r-1$ and $r_2 = 1$ and multiplying both sides by q^{-kr},we have

$$\widetilde{W}_{m,r}[n,k]_q = \sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}q^{-n}\widetilde{W}_{m,r-1}[j,k]_q. \tag{21}$$

Summing up both sides of (21), we have

$$\begin{aligned}\widetilde{D}_{m,r}[n]_q &= \sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^{n-j}\binom{n}{j}q^{-n}\widetilde{W}_{m,r-1}[j,k]_q\\ &= \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{n-j}\binom{n}{j}q^{-n}\widetilde{W}_{m,r-1}[j,k]_q\\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}q^{-n}\sum_{k=0}^{j}\widetilde{W}_{m,r-1}[j,k]_q\\ &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}q^{-n}\widetilde{D}_{m,r-1}[j]_q.\end{aligned}$$

The following theorem states formally the above recurrence relation for $\widetilde{D}_{m,r}[n]_q$.

Theorem 5. *The (q,r)-Dowling numbers satisfy the following relation*

$$q^n\widetilde{D}_{m,r}[n]_q = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\widetilde{D}_{m,r-1}[j]_q. \tag{22}$$

The following corollary is a direct consequence of Theorem 5 which can be proved using the inversion formula by Riordan [4, 15].

Corollary 1. *The (q,r)-Dowling numbers satisfy the following relations*

$$\widetilde{D}_{m,r-1}[n]_q = \sum_{j=0}^{n} \binom{n}{j} q^j \widetilde{D}_{m,r}[j]_q. \tag{23}$$

To establish the Hankel transform of $\widetilde{D}_{m,r}[n]_q$, we need the concept of rising k-binomial transform by Spivey and Steil [17] as well as its property in relation to Hankel transform.

Definition 3. (Spivey-Steil [17]) The *rising k-binomial transform R* of a sequence $A = \{a_n\}$ is the sequence $R(A;k) = \{r_n\}$, where r_n is given by

$$r_n = \sum_{j=0}^{n} \binom{n}{j} k^j a_j, \quad k \neq 0. \tag{24}$$

We use $R(A,k)$ to denote the set of rising k-binomial transform of A. That is, $R(A,k) = \{r_n\}$. Then we have the following theorem by Spivey and Steil.

Theorem 6. (Spivey-Steil [17]) *Given a sequence $A = \{a_0, a_1, \ldots,\}$. Let $H(A) = \{h_n\}$. Then*

$$H(R(A,k)) = \{a_0, 0, 0, \ldots,\}.$$

If $k \neq 0$,

$$H(R(A,k)) = \{k^{n(n+1)} h_n\}.$$

Now, we are ready to state the main result of the paper.

Theorem 7. *The Hankel transform of the sequence of (q,r)-Dowling numbers $\{\widetilde{D}_{m,r}[n]_q\}$ is given by*

$$H(\widetilde{D}_{m,r}[n]_q) = q^{m\binom{n+1}{3} - rn(n+1)} [0]_{q^m}! [1]_{q^m}! \ldots [n]_{q^m}! [m]_q^{\binom{n+1}{2}}. \tag{25}$$

Proof. Using equation (18) in Remark 4, we have

$$H(\widetilde{D}_{m,0}[n]_q) = q^{m\binom{n+1}{3}} [0]_{q^m}! [1]_{q^m}! \ldots [n]_{q^m}! [m]_q^{\binom{n+1}{2}}. \tag{26}$$

From Corollary 1, we say that $\widetilde{D}_{m,r-1}[n]_q$ is the binomial transform of $q^n \widetilde{D}_{m,r}[n]_q$. This means that

$$B(q^n \widetilde{D}_{m,r}[n]_q) = \widetilde{D}_{m,r-1}[n]_q.$$

Hence, by Layman's Theorem [12],

$$H(B(q^n \widetilde{D}_{m,r}[n]_q)) = H(q^n \widetilde{D}_{m,r}[n]_q).$$

That is,

$$H(\widetilde{D}_{m,r-1}[n]_q) = H(q^n \widetilde{D}_{m,r}[n]_q).$$

Now, Corollary 1 can also be stated as $\widetilde{D}_{m,r-1}[n]_q$ is the rising q-binomial transform of $\widetilde{D}_{m,r}[n]_q$. Using Spivey-Steil Theorem, with $A = \{\widetilde{D}_{m,r}[n]_q\}$, $h_n = H(\widetilde{D}_{m,r}[n]_q)$ and $r_n = \widetilde{D}_{m,r-1}[n]_q$, we have

$$H(\widetilde{D}_{m,r-1}[n]_q) = q^{n(n+1)} H(\widetilde{D}_{m,r}[n]_q).$$

We observe that, when $r = 1$ and using (26), we have

$$\begin{aligned} H(\widetilde{D}_{m,1}[n]_q) &= q^{-n(n+1)} H(\widetilde{D}_{m,0}[n]_q) \\ &= q^{-n(n+1)} q^{m\binom{n+1}{3}} [0]_{q^m}![1]_{q^m}! \dots [n]_{q^m}![m]_q^{\binom{n+1}{2}} \\ &= q^{m\binom{n+1}{3} - n(n+1)} [0]_{q^m}![1]_{q^m}! \dots [n]_{q^m}![m]_q^{\binom{n+1}{2}} \end{aligned}$$

Also, when $r = 2$,

$$H(\widetilde{D}_{m,2}[n]_q) = q^{m\binom{n+1}{3} - 2n(n+1)} [0]_{q^m}![1]_{q^m}! \dots [n]_{q^m}![m]_q^{\binom{n+1}{2}}.$$

Continuing this argument, we obtain

$$H(\widetilde{D}_{m,r}[n]_q) = q^{m\binom{n+1}{3} - rn(n+1)} [0]_{q^m}![1]_{q^m}! \dots [n]_{q^m}![m]_q^{\binom{n+1}{2}}$$

□

Remark 5. *When $m = 1$, the Hankel transform in* (25) *reduces to*

$$H(\widetilde{D}_{1,r}[n]_q) = q^{\binom{n+1}{3} - rn(n+1)} [0]![1]! \dots [n]!,$$

which is exactly the Hankel transform for the q-noncentral Bell numbers in [9].

Remark 6. *When $q \to 1$, the Hankel transform in* (25) *yields*

$$H(\widetilde{D}_{m,r}[n]_q) = [0]![1]! \dots [n]! m^{\binom{n+1}{2}},$$

which is exactly the Hankel transform for the q-analogue of (r, β)-Bell numbers in [9].

Remark 7. *The Hankel transform in* (25) *can also be written as*

$$H(\widetilde{D}_{m,r}[n]_q) = q^{m\binom{n+1}{3}} \prod_{k=0}^{n} q^{-2rk} [m]_q^k [k]_{q^m}!$$

such that, when $r = 0$, we have

$$H(\widetilde{D}_{m,0}[n]_q) = q^{m\binom{n+1}{3}} \prod_{k=0}^{n} [m]_q^k [k]_{q^m}!,$$

which is exactly the conjectured Hankel transform in (1) *with $m = \beta$ and*

$$\prod_{k=0}^{n} f(n,k) = q^{m\binom{n+1}{3}}.$$

Acknowledgements

The first author would like to thank Cebu Normal University (CNU) and the Commission on Higher Education-Grants-in-Aid for Research (CHED-GIA) for the financial support extended to this project. The second author would also like to thank the Commission on Higher Education-Faculty Development Program (CHED-FDP) for providing her financial support in pursuing Ph.D. Math Program at the Mathematics Department of MSU - Iligan Institute of Technology. Lastly, the authors would like to thank the two anonymous referees for spending their free time in reviewing our paper. Their comments and suggestions have helped a lot in improving the paper.

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