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Application of the SBA method to solve the nonlinear biological population models

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Abstract. This paper presents numerical solution of some nonlinear degenerate parabolic equations arising in the spatial diffusion of biological populations. The SBA method based on combination of Adomian decomposition method, principle of Picard and successive approximations is used for solving these equations. The analytical obtained solutions show that the SBA method leads to more accurate results.
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1. Introduction

The spatial diffusion of some biological species is described by nonlinear partial differential equations. In the last years, various numerical powerful methods have been applied to get the solutions of general degenerate parabolic equations, such as collocation methods with mesh-free technique [2], variational iteration method [10], Adomian decomposition method (ADM) [1, 13], homotopy perturbation method [8, 9], homotopy analysis Sumudu transform method [12], etc.

The objective of this paper is to apply Some Blaise Abbo(SBA) method, which is an elegant combination of ADM [3], and Picard principle and successive approximations [5] to find the analytical solution of some degenerate parabolic equations arising in the spatial diffusion of time fractional biological populations. A particular form of these equations is given by :

$$\frac{\partial u(x,y,t)}{\partial t} = \Delta u^2(x,y,t) + g(u),\ (x,y) \in \mathbb{R}^2,\ t > 0 \qquad (1)$$

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with given initial conditions $u(x,y,0) = f(x,y)$

The function $u(x,y,t)$ denotes the population density (number of minimal species per-unit volume at position (x,y) and time t), $g(u)$ the population supply due to birth and death of species. In this study, a form of $g(u)$ is :

$$g(u) = hu^{\alpha}(x,y,t)\left(1 - ru^{\beta}(x,y,t)\right) \tag{2}$$

where h, α, β, r are real numbers.

The SBA method overcomes the difficulty arising in calculating the Adomian's polynomials which is an important advantage over the ADM and other numerical methods [6, 11]

2. The numerical SBA method

Let us consider the following functional equation:

$$Au = f \tag{3}$$

Where $A : H \to H$, is an operator not necessarily linear and H is a Hilbert space adequately chosen given the operator A, f is given function and u the unknown function.

Let :

$$A = L + R + N \tag{4}$$

Where L is an invertible operator in the Adomian "sense", R the linear remainder and N a nonlinear operator. The equation (3) therefore becomes :

$$Lu + Ru + Nu = f$$

$\Leftrightarrow$

$$u = \theta + L^{-1}(f) - L^{-1}(Ru) - L^{-1}(Nu) \tag{5}$$

Where θ is such that $L(\theta) = 0$. The equation (5) is the Adomian canonical form , using the successive approximations [10]we get :

$$u^k = \theta + L^{-1}(f) - L^{-1}\left(Ru^k\right) - L^{-1}\left(Nu^{k-1}\right); k \geq 1 \tag{6}$$

This yields the following Adomian algorithm [4, 7, 14, 15]

$$\begin{cases} u_0^k = \theta + L^{-1}(f) - L^{-1}\left(Nu^{k-1}\right); k \geq 1 \\ u_{n+1}^k = -L^{-1}\left(Ru_n^k\right); n \geq 0 \end{cases} \tag{7}$$

The Picard principle is then applied to the equation (7) let u^0 be such that $N\left(u^0\right) = 0$, for $k = 1$, we get :

$$\begin{cases} u_0^1 = \theta + L^{-1}(f) + L^{-1}\left(Nu^0\right) \\ u_{n+1}^1 = -L^{-1}\left(Ru_n^1\right); n \geq 0 \end{cases} \tag{8}$$

If the series $\left(\sum_{n=0}^{+\infty} u_n^1\right)$ converges, then $u^1 = \sum_{n=0}^{+\infty} u_n^1$. For $k = 2$, we get:

$$\begin{cases} u_0^2 = \theta + L^{-1}(f) + L^{-1}\left(Nu^1\right) \\ u_{n+1}^2 = L^{-1}\left(Ru_n^2\right); n \geq 0 \end{cases} \tag{9}$$

If the series $\left(\sum_{n=0}^{+\infty} u_n^2\right)$ converges, then $u^2 = \sum_{n=0}^{+\infty} u_n^2$. This process is repeated to k.

If the series $\left(\sum_{n=0}^{+\infty} u_n^k\right)$ converges, then $u^k = \sum_{n=0}^{+\infty} u_n^k$, therefore $u^k = \lim_{k\to+\infty} u^k$ is the solution of the problem.

With the following hypothesize : At the step k, $N(u^k) = 0, \forall k \geq 1$.

2.1. Test examples

In this section, we present some examples with analytical solution to show the efficiency of method described in previous section for solving equation (1)

2.2. Example 1

Consider equation (1) with $\alpha = \beta = 1, r \neq 0, h \neq 0$ and $f(x, y) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}(x + y)\right)$

Putting these values in this equation, we write :

$$\begin{cases} \frac{\partial u(x,y,t)}{\partial t} = \Delta u^2(x, y, t) + hu(x, y, t)\left(1 - ru(x, y, t)\right), \ (x, y) \in \mathbb{R}^2, t > 0 \\ u(x, y, 0) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}(x + y)\right) \end{cases} \tag{10}$$

$$\frac{\partial u(x, y, t)}{\partial t} = hu(x, y, t) + \Delta u^2(x, y, t) - rhu^2(x, y, t) \tag{11}$$

Let

$$\begin{cases} L_t(u(x, y, t)) = \frac{\partial}{\partial t}(.) \\ L_t^{-1} = \int_0^t (.)\, ds \\ R(u(x, y, t)) = hu(x, y, t) \\ N(u(x, y, t)) = \dfrac{\partial^2 u^2(x, y, t)}{\partial x^2} + \dfrac{\partial^2 u^2(x, y, t)}{\partial y^2} - rhu^2(x, y, t) \end{cases} \tag{12}$$

We obtain :

$$L(u(x, y, t)) = R(u(x, y, t)) + N(u(x, y, t)) \tag{13}$$

Integrating the equation (13) with respect to t gives the following canonical form:

$$u(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) + \int_0^t R(u(x,y,s))ds + \int_0^t N(u(x,y,s)ds \tag{14}$$

In Applying the method of successive approximations to equation (14) we obtain:

$$u^k(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) + \int_0^t R(u^k(x,y,s))ds + \int_0^t N(u^{k-1}(x,y,s)ds \tag{15}$$

If now we Apply the Adomian algorithm to equation (15), we obtain:

$$\begin{cases} u_0^k(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) + \int_0^t N(u^{k-1}(x,y,s)ds \\ u_n^k(x,y,t) = \int \_0^t R(u_{n-1}^k(x,y,s))ds, n \geq 1 \end{cases} \tag{16}$$

The solution at each stage is :

$$u^k(x,y,t) = \sum_{n=0}^{\infty} u_n^k(x,y,t), k = 1,2,3,\cdots \tag{17}$$

First step $k = 1$

In Applying the Picard principle and if we take $u^0(x,y,s)$ such as $N(u^0(x,y,s)) = 0$, we obtain :

$$\begin{cases} u_0^1(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \\ u_n^1(x,y,t) = \int_0^t R(u_{n-1}^1(x,y,s))ds, n \geq 1 \end{cases} \tag{18}$$

It follows that expressions of function $u_n^1(x,y,t)$ were :

$$\begin{cases} u_0^1(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \\ u_1^1(x,y,t) = (ht)\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \\ u_2^1(x,y,t) = \frac{1}{2!}(ht)^2\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \\ \vdots \\ u_n^1(x,y,t) = \frac{1}{n!}(ht)^n\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \end{cases} \tag{19}$$

In this representation,$u^1(x,y,t)$ form the finite sequence :

$$u^1(x,y,t)=\sum_{n=0}^{\infty}u_n^1(x,y,t)=\left(\sum_{n=0}^{\infty}\frac{1}{n!}(ht)^n\right)\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right) \tag{20}$$

and we can write :

$$u^1(x,y,t)=\exp(ht)\times\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right)=\exp\left(ht+\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right) \tag{21}$$

which is the solution in step 1

Second step $k=2$

$$\begin{cases} u_0^2(x,y,t)=\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right)+\int_0^t N(u^1(x,y,s)ds \\ u_n^2(x,y,t)=\int_0^t R(u_{n-1}^2(x,y,s))ds, n\geq 1 \end{cases} \tag{22}$$

gold

$$\begin{cases} N(u^1)=\frac{\partial^2\left(u^1\right)^2}{\partial x^2}+\frac{\partial^2\left(u^1\right)^2}{\partial y^2}-rh\left(u^1\right)^2 \\ =\frac{\partial^2\left(\exp\left(ht+\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right)\right)^2}{\partial x^2}+\frac{\partial^2\left(\exp\left(ht+\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right)\right)^2}{\partial y^2}- \\ rh\left(\exp\left(ht+\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right)\right)^2 \\ =\frac{\partial^2\left(\exp\left(\sqrt{\frac{hr}{2}}(x+y)+2ht\right)\right)}{\partial x^2}+\frac{\partial^2\left(\exp\left(\sqrt{\frac{hr}{2}}(x+y)+2ht\right)\right)}{\partial y^2}- \\ rh\exp\left(2ht+\sqrt{\frac{hr}{2}}(x+y)\right) \\ = (rh-\ rh)\exp\left(2ht+\sqrt{\frac{hr}{2}}(x+y)\right) \\ =0 \end{cases} \tag{23}$$

From where in step 2, we have the following SBA algorithm:

$$\begin{cases} u_0^2(x,y,t)=\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}(x+y)\right) \\ u_n^2(x,y,t)=\int_0^t R(u_{n-1}^2(x,y,s))ds, n\geq 1 \end{cases} \tag{24}$$

Again, we write :

$$\begin{cases} u_0^2(x,y,t) = \exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \\ u_1^2(x,y,t) = (ht)\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \\ u_2^2(x,y,t) = \frac{1}{2!}\,(ht)^2\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \\ \vdots \\ u_n^2(x,y,t) = \frac{1}{n!}\,(ht)^n\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \end{cases} \tag{25}$$

So the finite sequence form of $u^2(x,y,t)$ is:

$$u^2(x,y,t) = \sum_{n=0}^{\infty} u_n^2(x,y,t) = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\,(ht)^n\right)\exp\left(\frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \tag{26}$$

Thus,

$$u^2(x,y,t) = \exp\left(ht + \frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \tag{27}$$

is the solution in step 2

In a recurrent way, for the following steps ($k \geq 3$), we obtain:

$$u^k(x,y,t) = \sum_{n=0}^{\infty} u_n^k(x,y,t) = \exp\left(ht + \frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \tag{28}$$

Therefore, the exact solution of this equation is:

$$u(x,y,t) = \lim_{k\to+\infty} u^k(x,y,t) = \exp\left(ht + \frac{1}{2}\sqrt{\frac{hr}{2}}\,(x+y)\right) \tag{29}$$

2.3. Example 2

Consider equation (1) with $\alpha = \beta = 1, r = 0, h \neq 0$ and $f(x,y) = \sqrt{\cos x \cosh y}$

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} = hu(x,y,t) + N\left(u(x,y,t)\right),\ (x,y) \in \mathbb{R}^2,\ t > 0 \\ u(x,y,0) = \sqrt{\cos x \cosh y} \end{cases} \tag{30}$$

where

$$N\left(u(x,y,t)\right) = \Delta u^2(x,y,t) \tag{31}$$

From (30), we obtain the following canonical Adomian form:

$$u(x,y,t)=\sqrt{\cos x\cosh y}+h\int_0^t u(x,y,s)ds+\int_0^t N\left(u(x,y,s)\right)ds \tag{32}$$

From (32), the successive approximations give us

$$u^k(x,y,t)=\sqrt{\cos x\cosh y}+\widetilde{N}^{k-1}\left(u(x,y,t)\right)+h\int_0^t u^k(x,y,s)ds \tag{33}$$

Where

$$\widetilde{N}\left(u^{k-1}(x,y,t)\right)=\int_0^t N\left(u^{k-1}(x,y,s)\right)ds \tag{34}$$

From (33), we have the following algorithm of Adomian :

$$\begin{cases} u_0^k(x,y,t)=\sqrt{\cos x\cosh y}+\widetilde{N}^{k-1}\left(u(x,y,t)\right) \\ u_{n+1}^k(x,y,t)=h\displaystyle\int_0^t u_n^k(x,y,s)ds, n\geq 0 \end{cases} \tag{35}$$

Let us apply to (35), the principle of Picard. We remark that $u^0(x,y,t)=0$ is a root of the $\widetilde{N}\left(u^0(x,y,t)\right)=0$

For $k=1$, we obtain:

$$\begin{cases} u_0^1(x,y,t)=\sqrt{\cos x\cosh y} \\ u_1^1(x,y,t)=(th)\sqrt{\cos x\cosh y} \\ u_2^1(x,y,t)=\frac{1}{2!}(th)^2\sqrt{\cos x\cosh y} \\ u_3^1(x,y,t)=\frac{1}{3!}(th)^3\sqrt{\cos x\cosh y} \\ \vdots \\ u_n^1(x,y,t)=\frac{1}{n!}(th)^n\sqrt{\cos x\cosh y} \end{cases} \tag{36}$$

Let us put

$$\begin{aligned} u^1(x,y,t)&=u_0^1(x,y,t)+u_1^1(x,y,t)+u_2^1(x,y,t)+\cdots \\ &=\left(1+(th)+\frac{1}{2!}(th)^2+\frac{1}{3!}(th)^3+\cdots\right)\sqrt{\cos x\cosh y} \\ &=\sqrt{\cos x\cosh y}e^{th} \end{aligned} \tag{37}$$

Second step

For $k=2$, we have:

$$\begin{aligned}\widetilde{N}\left(u^1(x,y,t)\right) &= \int_0^t N\left(u^1(x,y,s)\right) ds \\ &= \int_0^t \left(\frac{\partial^2\left(\left(\sqrt{\cos x\cosh y}\right)e^{sh}\right)^2}{\partial x^2} + \frac{\partial^2\left(\left(\sqrt{\cos x\cosh y}\right)e^{sh}\right)^2}{\partial y^2}\right) ds \\ &= \int_0^t e^{2sh}\left(\frac{\partial^2\left(\cos x\cosh y\right)}{\partial x^2} + \frac{\partial^2\left(\cos x\cosh y\right)}{\partial y^2}\right) ds \\ &= \int_0^t e^{2sh}\left(-\cos x\cosh y + \cos x\cosh y\right) ds \\ &= 0\end{aligned} \tag{38}$$

From (35),we obtain:

$$\begin{cases} u_0^2(x,y,t) = \sqrt{\cos x\cosh y} \\ u_1^2(x,y,t) = (th)\sqrt{\cos x\cosh y} \\ u_2^2(x,y,t) = \frac{1}{2!}(th)^2\sqrt{\cos x\cosh y} \\ u_3^2(x,y,t) = \frac{1}{3!}(th)^3\sqrt{\cos x\cosh y} \\ \vdots \end{cases} \tag{39}$$

Therefore,

$$\begin{aligned}u^2(x,y,t) &= u_0^2(x,y,t) + u_1^2(x,y,t) + u_2^2(x,y,t) + \cdots \\ &= \left(1 + (th) + \tfrac{1}{2!}(th)^2 + \tfrac{1}{3!}(th)^3 + \cdots\right)\sqrt{\cos x\cosh y} \\ &= \left(\sqrt{\cos x\cosh y}\right)e^{th}\end{aligned} \tag{40}$$

Using the procedure for $k \geq 3$, the solution to the k step is

$$\begin{aligned}u^k(x,y,t) &= u_0^k(x,y,t) + u_1^k(x,y,t) + u_2^k(x,y,t) + \cdots \\ &= \left(1 + (th) + \tfrac{1}{2!}(th)^2 + \tfrac{1}{3!}(th)^3 + \cdots\right)\sqrt{\cos x\cosh y} \\ &= e^{th}\sqrt{\cos x\cosh y}\end{aligned} \tag{41}$$

So the exact solution of equation (30) with initial boundaries $u(x,y,0) = \sqrt{\cos x\cosh y}$ is

$$u(x,y,t) = \lim_{k\to+\infty} u^k(x,y,t) = e^{th}\sqrt{\cos x\cosh y} \tag{42}$$

3. Conclusion

The SBA numerical method permitted us to resolve a few nonlinear partial differential equations modelling diffusion, convection, reaction problems Cauchy type. The SBA method permitted us to resolve the problems proposed in this paper. It is then a very powerful numerical tool of analysis for the resolution of these kinds of problems.

References

[1] K. Abbaoui. *Les fondements de la méthode décompositionnelle dAdomian et application à la résolution de problèmes issus de la biologie et de la médécine.* PhD thesis, Université Paris VI, 1995.

[2] Abbasbandy and E.Shivanian. Numerical simulation based on meshless technique to study the biological population model. *Math sci*, 10(3):123–130, 2016.

[3] G. Adomian. *Solving Frontier Problems of Physics: The Decomposition Method.* 1994.

[4] G. Bissanga and Joseph Bonazebi Yindoula. A new approach of construction of adomian algorithm - application of the adomian decomposition method (adm) and the some blaise abbo (sba) method to solving the convection equation. *Advances in Theoretical and Applied Mathematics*, 13(2):127–138, 2015.

[5] Georges Bouligand. *Fonctions harmoniques. Principes de Picard et de Dirichlet.* 1926.

[6] Salih M.Elzaki. Exact solution of nonlinear time fractional biological population equation. *International Journal of Advanced Research in Science engineering and Technology*, 1(3):829–838, 2014.

[7] Pare.Youssouf, Yaro.Rasmane Elysée, and Gouba Some Blaise. Solving a system of nonlinear equations second kind of volterra by the sba method. *Far East.J.Appl.Math*, 71(1):43–54, 2012.

[8] P Roul. Application of homotopy perturbation method to biological polpulation model. *Application and Applied Mathematics And International Journal*, 5(2):272–281, 2010.

[9] P. Roul. Application of homotopy perturbation method to biological population model. *Applications and Applied Mathematics*, 10(3):1369–1378, 2010.

[10] F. Shakeri and M. Dehghan. Numerical solution of a biological population model using he's variational iteration method. *Computers and Mathematics with applications*, 54(2):1197–1209, 2007.

[11] V.K Srivastava, S.Kumar, M.K.Awasthi, and B.K Singh. Two dimensional time fractional biological population model and its analytical solution. *Egyptian J. Basic Appl.Sci*, 17(1):71–76, 2014.

[12] V.G.Gupta and P.Kumar. Approximate solutions of fractional biological population model by homotopy analysis sumudu transform method. *International Journal of Science and Research*, 10(5):14–23, 2015.

[13] E. Yee. Application of the decomposition method to the solution of the reaction convection diffusion equation. *Applied Mathematics and Computation*, 56(1):1–27, 1993.

[14] Joseph Bonazebi Yindoula, Gabriel Bissanga, Pare Youssouf, Francis Bassono, and Blaise Some. Application of the adomian decomposition method(adm) and the some blaise abbo(sba) method to solving the diffusion-reaction equations. *Advances in Theoretical and Applied Mathematics*, 9(2):97–104, 2014.

[15] Pare Youssouf. *Résolution de quelques équations fonctionnelles par la méthode SBA (Somé Blaise-Abbo)*. PhD thesis, Université de Ouagadougou, 2010.