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Cubic Transmuted Uniform Distribution: An Alternative to Beta and Kumaraswamy Distributions

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Abstract. In this article, a new cubic transmuted (CT) family of distributions has been proposed by adding one more parameter. We have introduced cubic transmuted uniform (CTU) distribution by using the proposed class. We have also provided a detail description of the statistical properties of the proposed CTU distribution along with its estimation and real-life application.

2010 Mathematics Subject Classifications: 33B20, 46N30, 60B15

Key Words and Phrases: Transmuted Distributions, Uniform Distribution, Maximum Likelihood Estimation

1. Introduction

In statistical data analysis, quality of the procedures mainly depends upon the assumed probability model of the phenomenon. Large numbers of probability models are, therefore, being developed by the researcher. Enormous practical problems are on hand where the standard and extended probability distributions does not work.

The generalization of standard probability distributions has been an area of interest by several authors. Azzalini [4] proposed a new method of generalizing the probability distributions by adding skewness parameter to symmetric distributions and this class is referred to as the skew-symmetric distributions. Eugene et al. [9] developed the $Beta-G$ family of distributions by using the logit of beta distribution. The $Beta-G$ family of distributions extends the distribution of order statistics. Cordeiro and de Castro [8] proposed an alternative to $Beta-G$ family of distributions by using Kumaraswamy distribution in place of beta distribution and named the family as $Kum-G$ family. Alzaatreh et al. [3] have developed a more general method of extending probability distributions for any

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baseline distribution and named the family as $T-X$ family of distributions. Shaw and Buckley [16] introduced transmuted family of distributions for any baseline distribution and has *cdf*

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x),\ \lambda \in [-1,1] \tag{1}$$

The transmutation approach (1) captures the quadratic behavior in the data. Rahman et al. [13, 14] have extended the transmuted family of distributions, given in (1), by introducing two cubic transmuted families of distributions for any base distribution.

In this article, we have proposed a new cubic transmuted family of distributions and have linked it both with the distribution of order statistics and with the $T-X$ family of distributions.

The rest of the paper is organized as follows: The new family of cubic transmuted distributions is proposed in Section 2. In Section 3, cubic transmuted uniform distributions are proposed by using existing families, as given by Rahman et al. [13, 14], and by using a new family. In Section 4, statistical properties of the cubic transmuted uniform distribution, obtained by using the new family, are given. Section 5 provides distribution of the order statistics and the maximum likelihood estimation is given in Section 6. Simulation study and real-life application are presented in Section 7. The paper ends with some concluding remarks.

2. New Cubic Transmuted Family of Distributions

In this section, we will propose a new cubic transmuted family of distributions. The *cdf* of new family of distributions is given as

$$F(x) = (1-\lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x),\ x \in \mathbb{R}, \tag{2}$$

with corresponding *pdf*

$$f(x) = (1-\lambda)g(x) + 6\lambda G(x)g(x) - 6\lambda G^2(x)g(x),\ x \in \mathbb{R}, \tag{3}$$

where $\lambda \in [-1,1]$. The proposed family of distributions provides the base distribution for $\lambda = 0$.

The proposed cubic transmuted family given in (2) can be obtain as the distribution of order statistics and as a member of $T-X$ family of distributions. We will obtain the propose family as distribution of order statistics and as a member of $T-X$ family in the following theorems.

Theorem 1. *Suppose X_1, X_2 and X_3 be the iid random variables each with cdf $G(x)$, then the proposed cubic transmuted family of distributions given in (2) can be obtained as a weighted sum of three order statistics.*

Proof. Consider following three order statistics

$$X_{(1)} = min(X_1, X_2, X_3),\ X_{(2)} \text{ and } X_{(3)} = max(X_1, X_2, X_3).$$

Now consider the random variable Z as

$$Z \overset{d}{=} X_{(3)}, \quad \text{with probability } p_1,$$

$$Z \overset{d}{=} X_{(2)}, \quad \text{with probability } p_2,$$

$$Z \overset{d}{=} X_{(1)}, \quad \text{with probability } p_3,$$

where $p_1 + p_2 + p_3 = 1$. Now, $F_Z(x)$ is given as

$$\begin{aligned} F_Z(x) &= p_1 F_{(3)}(x) + p_2 F_{(2)}(x) + p_3 F_{(1)}(x) \\ &= (3 - 3p_1 - 3p_2)G(x) - (3 - 3p_1 - 6p_2)G^2(x) + (1 - 3p_2)G^3(x). \end{aligned} \tag{4}$$

Now setting $3p_1 + 3p_2 - 2 = \lambda$ and $3p_2 - 1 = 2\lambda$ the distribution function given in (4) corresponds to the cubic transmuted family given in (2).

Theorem 2. *Let a random variable X has cdf $G(x)$ and $p(t)$ be the pdf of a bounded random variable T with support on [0,1] such that $p(t)$ can be written as weighted sum of three bounded densities $p_1(t)$, $p_2(t)$ and $p_3(t)$ each with support [0,1]. The cubic transmuted family given in (2) can be obtained by using $T - X$ family, introduced by Alzaatreh et al. [3], for suitable choice of $p_1(t)$, $p_2(t)$ and $p_3(t)$.*

Proof. The *cdf* of $T - X$ family of distributions, proposed by Alzaatreh et al. [3], is

$$F(x) = \int_0^{G(x)} p(t)\ \mathrm{d}t, \ \ x \in \mathbb{R}, \tag{5}$$

where $p(t)$ denotes a *pdf* with support on $[0, 1]$. As noticed by Alizadeh et al. [2], the *cdf* given in (1), corresponds to the *cdf* given by (5) for the *pdf* $p(t) = 1 + \lambda - 2\lambda t$. Now, considering

$$p(t) = (1 - \lambda)p_1(t) + 3\lambda p_2(t) - 2\lambda p_3(t), \tag{6}$$

where $p_1(t) = 1,\ p_2(t) = 2t$ and $p_3(t) = 3t^2$, each having support of $[0, 1]$ and using (6) in (5) we have

$$F(x) = \int_0^{G(x)} [(1 - \lambda)p_1(t) + 3\lambda p_2(t) - 2\lambda p_3(t)]\ \mathrm{d}t. \tag{7}$$

On simplification, the *cdf* given in (7), turned out to be the *cdf* of cubic transmuted family of distributions given in (2).

Some cubic transmuted distributions are given in the Table 1 by using different base distributions in the proposed cubic transmuted family of distributions given in (2).

We will now propose some *CTU* distributions by using the cubic transmuted families proposed by Rahman et al. [13, 14] and by using (2). These distributions are obtained in the following section.

Table 1: Special Cases of Proposed Cubic Transmuted Family of Distributions.

Distribution	Cumulative Distribution Function
CT-Normal	$(1-\lambda)\Phi(x)+3\lambda\Phi^2(x)-2\lambda\Phi^3(x),\ x\in\mathbb{R}$
CT-Exponential	$e^{-\frac{3x}{\theta}}\left[2\lambda+(\lambda-1)e^{\frac{2x}{\theta}}-3\lambda e^{x/\theta}\right]+1,\ x\in[0,\infty)$
CT-Rayleigh	$e^{-\frac{3x^2}{2\sigma^2}}\left[(\lambda-1)e^{\frac{x^2}{\sigma^2}}-3\lambda e^{\frac{x^2}{2\sigma^2}}+2\lambda\right]+1,\ x\in[0,\infty)$
CT-Weibull	$\left[(\lambda-1)e^{2\left(\frac{x}{\lambda}\right)^k}-3\lambda e^{\left(\frac{x}{\lambda}\right)^k}+2\lambda\right]e^{-3\left(\frac{x}{\lambda}\right)^k}+1,\ x\in[0,\infty)$
CT-Gompertz	$(\lambda-1)e^{\eta-\eta e^{bx}}+\lambda e^{-3\eta\left(e^{bx}-1\right)}\left[2-3e^{\eta\left(e^{bx}-1\right)}\right]+1,\ x\in[0,\infty)$
CT-Kumaraswamy	$\left[(1-x^a)^b-1\right]\left[\lambda(1-x^a)^b\left\{2(1-x^a)^b-1\right\}-1\right],\ x\in[0,1]$
CT-Log-logistic	$\frac{x^\beta}{\left(\alpha^\beta+x^\beta\right)^3}\left[(1-\lambda)\alpha^{2\beta}+x^{2\beta}+(\lambda+2)\alpha^\beta x^\beta\right],\ x\in[0,\infty)$
CT-Pareto	$\left[\left(\frac{k}{x}\right)^\theta-1\right]\left[\lambda\left\{2\left(\frac{k}{x}\right)^\theta-1\right\}\left(\frac{k}{x}\right)^\theta-1\right],\ x\in[k,\infty)$
CT-Dagum	$\frac{\left[3\lambda\left\{\left(\frac{x}{b}\right)^{-a}+1\right\}^p-(\lambda-1)\left\{\left(\frac{x}{b}\right)^{-a}+1\right\}^{2p}-2\lambda\right]}{\left[\left(\frac{x}{b}\right)^{-a}+1\right]^{3p}},\ x\in\mathbb{R}^+$
CT-Burr XII	$\frac{\left[(x^c+1)^k-1\right]\left[\lambda\left\{(x^c+1)^k-2\right\}+(x^c+1)^{2k}\right]}{(x^c+1)^{3k}},\ x\in\mathbb{R}^+$

3. Cubic Transmuted Uniform Distributions

The uniform distribution is a useful distribution and plays important role in sampling from any distribution. Several researchers have extended the uniform distribution, see for example, Shaw and Buckley [16], Nadarajah and Aryal, [12] among others.

In this section, we will propose different cubic transmuted uniform distributions. We will first propose cubic transmuted uniform distributions by using cubic transmuted families of distributions given by Rahman et al. [13, 14]. We will also propose a cubic transmuted uniform distribution by using the new proposed family of distributions, given in (2). The cubic transmuted uniform distributions, obtained by using Rahman et al. [13, 14], will be named as cubic transmuted uniform distribution-I (for short, CTU_I) and cubic transmuted uniform distribution-II (for short, CTU_{II}). These distributions are given in the following subsection.

3.1. Cubic Transmuted Uniform Distributions Using Existing Families

The density and distribution function of uniform distribution over the interval $[0,1]$ are, respectively, $g(x)=1$ and $G(x)=x$. Using *pdf* and *cdf* of standard uniform distribution in equation (3) of Rahman et al. [13], we have following *cdf* of CTU_I distribution

$$F_I(x)=(1+\lambda_1)\,x+(\lambda_2-\lambda_1)x^2-\lambda_2x^3,\ x\in[0,1],$$

with corresponding *pdf*

$$f_I(x)=(1+\lambda_1)+2(\lambda_2-\lambda_1)x-3\lambda_2x^2,\ x\in[0,1],$$

where $\lambda_1, \lambda_2 \in [-1, 1]$ and $-2 \leq \lambda_1 + \lambda_2 \leq 1$. Again, using *pdf* and *cdf* of standard

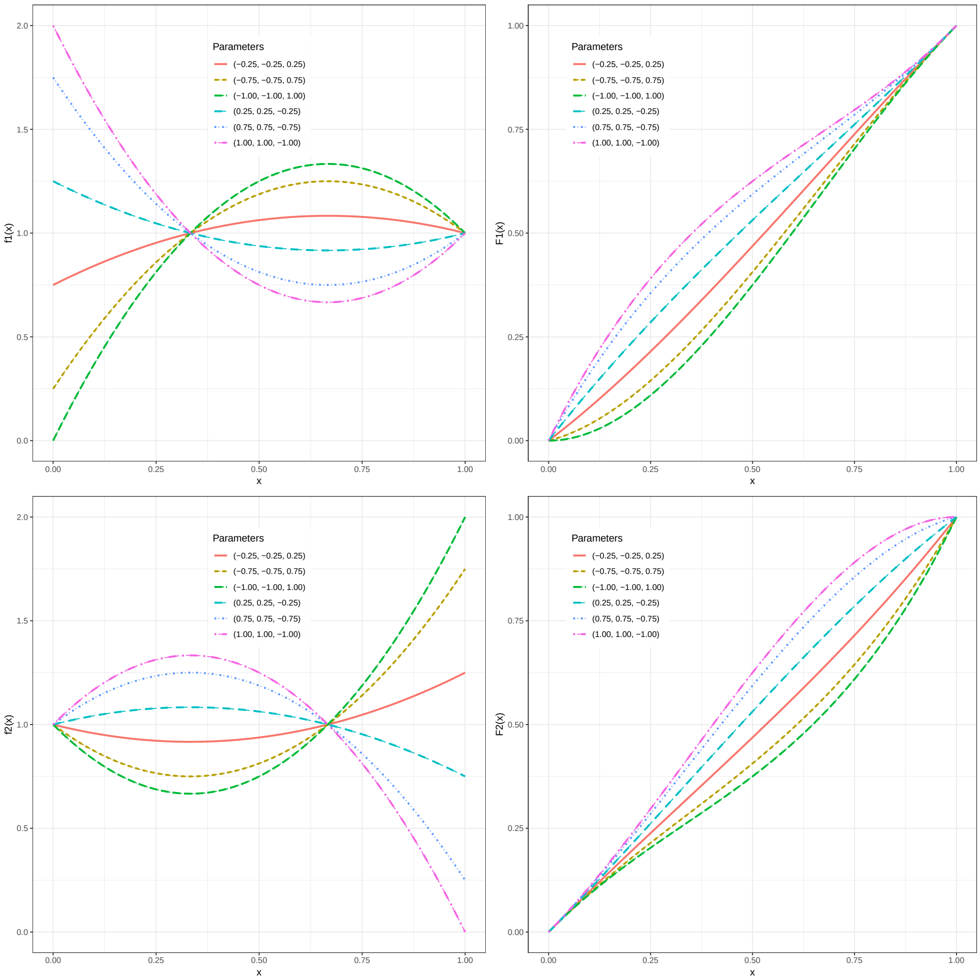

Figure 1: Density and Distribution Functions Plots for CTU_I (Top) and CTU_{II} (Bottom) Distributions.

uniform distribution in (7) of Rahman et al. [14], we have the *cdf* of CTU_{II} distribution as

$$F_{II}(x) = (1 + \lambda_1 + \lambda_2)\, x - (\lambda_1 + 2\lambda_2)x^2 + \lambda_2 x^3,\ x \in [0, 1],$$

with corresponding *pdf* as

$$f_{II}(x) = (1 + \lambda_1 + \lambda_2) - 2(\lambda_1 + 2\lambda_2)x + 3\lambda_2 x^2,\ x \in [0, 1],$$

where $\lambda_1 \in [-1, 1]$ and $\lambda_2 \in [0, 1]$.

Figure 1 presents the density and distribution functions of CTU_I and CTU_{II} distributions. It can be seen that CTU_I distribution is skewed to the left and CTU_{II} distribution is skewed to the right.

3.2. Cubic Transmuted Uniform Distribution Using Proposed Family

We will now propose a cubic transmuted uniform distribution (CTU for short) by using the family of distributions given in (2). Using *pdf* and *cdf* of standard uniform distribution in (2), we obtain the *cdf* of CTU distribution as

$$F(x) = (1-\lambda)x + 3\lambda x^2 - 2\lambda x^3, \ x \in [0,1], \tag{8}$$

with corresponding *pdf*

$$f(x) = (1-\lambda) + 6\lambda x - 6\lambda x^2, \ x \in [0,1], \tag{9}$$

where $\lambda \in [-1, 1]$.

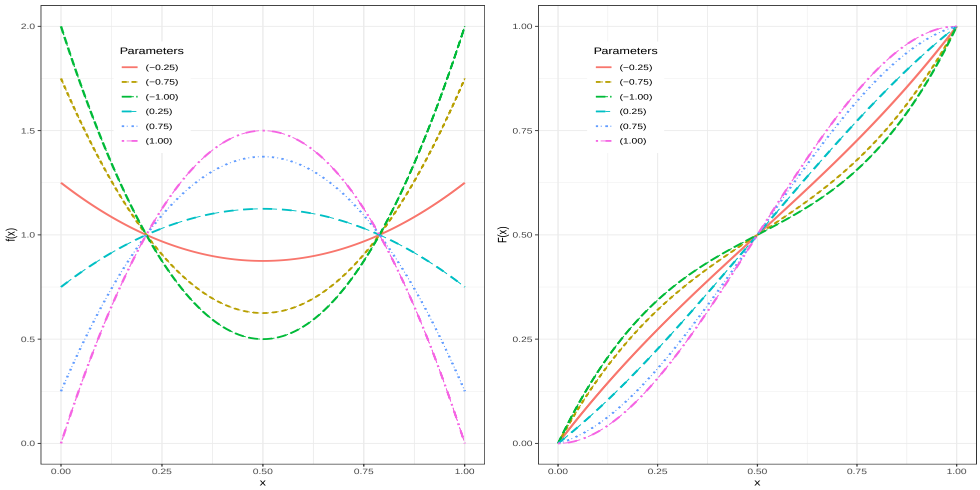

Figure 2: Density and Distribution Functions Plots for Proposed CTU Distribution.

Figure 2 present the plots of density and distribution functions for the proposed CTU distribution. It can be seen that the proposed CTU distribution capture the complexity alone that were captured by CTU_I and CTU_{II} distributions together.

We will now present the distributional properties, alongside real data application, of the proposed CTU distribution in the following sections.

4. Statistical Properties

In this section we will discuss some distributional properties of the proposed CTU distribution. These properties are discussed in the following subsections.

4.1. Moments

Moments play an important role in studying certain properties of the distribution. We will give the rth moment of proposed CTU distribution in the following theorem.

Theorem 3. *The rth raw moment of CTU distribution, with density (9), is*

$$\mu_r' = \frac{r(\lambda - \lambda r + r + 5) + 6}{(r+1)(r+2)(r+3)}.$$

The mean, variance, skewness and kurtosis are, respectively, $\frac{1}{2}$, $\frac{1}{60}(5-2\lambda)$, *0 and* $-\frac{6[2\lambda(7\lambda-20)+35]}{7(5-2\lambda)^2}$ *respectively.*

Proof. The rth raw moment is defined as

$$\begin{aligned}
\mu_r' = E(X^r) &= \int_0^1 x^r f(x)\mathrm{d}x \\
&= \int_0^1 x^r \left[(1-\lambda) + 6\lambda x - 6\lambda x^2\right] \mathrm{d}x \\
&= (1-\lambda)\int_0^1 x^r \mathrm{d}x + 6\lambda \int_0^1 x^{r+1}\mathrm{d}x - 6\lambda \int_0^1 x^{r+2}\mathrm{d}x \\
&= \frac{(1-\lambda)}{(r+1)} + \frac{6\lambda}{(r+2)} - \frac{6\lambda}{(r+3)} \\
&= \frac{r(\lambda - \lambda r + r + 5) + 6}{(r+1)(r+2)(r+3)}. \qquad (10)
\end{aligned}$$

Mean of the distribution is obtained by using $r = 1$ in (10) and is

$$\mu = \mu_1' = \frac{1}{2}.$$

We can see that mean of CTU distribution is same as the uniform distribution over $[0, 1]$.

The variance of proposed CTU distribution obtained as

$$\begin{aligned}
\sigma^2 = \mu_2 &= \mu_2' - \left(\mu_1'\right)^2 \\
&= \frac{1}{60}(2(7-\lambda) + 6) - \frac{1}{4} = \frac{1}{60}(5 - 2\lambda).
\end{aligned}$$

The coefficient of skewness for the proposed CTU distribution is

$$\gamma_1 = \sqrt{\beta_1} = E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right] = 0,$$

whereas the coefficient of kurtosis is

$$\begin{aligned}
\gamma_2 = \beta_2 - 3 &= E\left[\left(\frac{x-\mu}{\sigma}\right)^4\right] - 3 \\
&= \frac{45(7-4\lambda)}{7(5-2\lambda)^2} - 3 \\
&= -\frac{6\left[2\lambda(7\lambda - 20) + 35\right]}{7(5-2\lambda)^2}.
\end{aligned}$$

The proof is completed.

4.2. Moments Generating Function

The moment generating function (MGF) is used to obtain the moments of a distribution. The MGF of CTU distribution is given in the following theorem.

Theorem 4. *Let X follows the CTU distribution, then the moment generating function, $M_X(t)$ is given as*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{r(\lambda - \lambda r + r + 5) + 6}{(r+1)(r+2)(r+3)}, \tag{11}$$

where $t \in \mathbb{R}$.

Proof. The moment generating function of CTU distribution is obtained by using

$$M_X(t) = \int_0^1 e^{tx} f(x) \mathrm{d}x$$

where $f(x)$ is given in (9). Using the series representation of e^{tx} given in Gradshteyn and Ryzhik [10], we have

$$\begin{aligned} M_x(t) &= \int_0^1 \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x) \mathrm{d}t \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r). \end{aligned} \tag{12}$$

Using $E(X^r)$ from (10) in (12), we have (11).

4.3. Characteristic Function

Characteristic function defines its probability distribution completely. The characteristic function of CTU distribution is stated by the following theorem.

Theorem 5. *Let X have the CTU distribution, then characteristic function, $\phi_X(t)$, of X is given as*

$$\phi_X(t) = \sum_{r=0}^{1} \frac{(it)^r}{r!} \frac{r(\lambda - \lambda r + r + 5) + 6}{(r+1)(r+2)(r+3)},$$

where $i = \sqrt{-1}$ is the imaginary unit and $t \in \mathbb{R}$.

Proof. The proof is simple as MGF.

4.4. Mean Absolute Deviation

Let a random variable X has the CTU distribution. The mean absolute deviation (MAD) about mean can be obtained for CTU distribution as

$$\eta = E|x - E(x)| = \frac{4-\lambda}{16}.$$

This MAD is easier to understand and easier to compute.

4.5. Quantile Function and Median

The quantile function for CTU distribution is obtained by solving (8) for x and is obtain as

$$x_q = \frac{1}{2} - \frac{\sqrt[3]{\eta_1 + \sqrt{4\eta_2^3 + \eta_1^2}}}{6\sqrt[3]{2}\lambda} + \frac{\eta_2}{3\ 2^{2/3}\lambda\sqrt[3]{\eta_1 + \sqrt{4\eta_2^3 + \eta_1^2}}}, \tag{13}$$

where $\eta_1 = -54\lambda^2 + 108\lambda^2 q$ and $\eta_2 = -3\lambda^2 - 6\lambda$.

The first quartile, median and third quartile can be obtained by setting $q = 0.25$, 0.50 and 0.75 in (13) respectively.

4.6. Simulating Random Sample

The random numbers can be drawn from CTU distribution by solving

$$(1-\lambda)x + 3\lambda x^2 - 2\lambda x^3 = u,$$

for x, where $u \sim U(0,1)$. The equation for generating random sample from CTU distribution can be further expressed as

$$X = \frac{1}{2} - \frac{\sqrt[3]{\eta_1 + \sqrt{4\eta_2^3 + \eta_1^2}}}{6\sqrt[3]{2}\lambda} + \frac{\eta_2}{3\ 2^{2/3}\lambda\sqrt[3]{\eta_1 + \sqrt{4\eta_2^3 + \eta_1^2}}}, \tag{14}$$

where $\eta_1 = -54\lambda^2 + 108\lambda^2 u$ and $\eta_2 = -3\lambda^2 - 6\lambda$.

One can generate random sample from CTU distribution using (14) for various values of model parameter λ.

4.7. Reliability Analysis

The reliability function is defined as $R(t) = 1 - F(t)$ and, for CTU distribution, it is

$$R(t) = 1 - (1-\lambda)t - 3\lambda t^2 + 2\lambda t^3,\ t \in [0,1].$$

The hazard function is the ratio of the probability distribution function to the reliability function and is given as

$$h(t) = \frac{(1-\lambda) + 6\lambda t - 6\lambda t^2}{1 - (1-\lambda)t - 3\lambda t^2 + 2\lambda t^3},\ t \in [0,1].$$

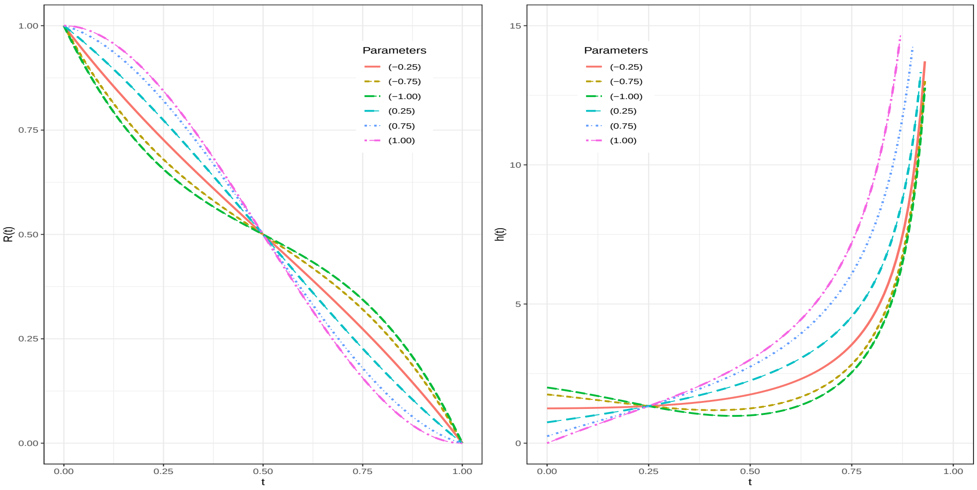

Figure 3: Reliability and Hazard Rate Functions Plots for Proposed CTU Distribution.

Figure 3 describes some possible shapes for the reliability and hazard functions of CTU distribution for different values of model parameter λ.

4.8. Shannon Entropy

The uncertainty of a random variable X can be easily measured by using entropy. The entropy H of a random variable X is defined by Shannon [15], and, for the CTU distribution, it is obtained as

$$\begin{aligned} H &= -E[log\{f(x)\}] \\ &= -E\left[log\left\{(1-\lambda)+6\lambda x-6\lambda x^2\right\}\right] \\ &= -\int_0^1 log\left\{(1-\lambda)+6\lambda x-6\lambda x^2\right\}\left\{(1-\lambda)+6\lambda x-6\lambda x^2\right\}\,dx \\ &= -\frac{3\sqrt{\lambda}\left\{\lambda-3\log(1-\lambda)+4\right\}+2\sqrt{3}(-\lambda-2)^{3/2}\tan^{-1}\left(\frac{\sqrt{3}\sqrt{\lambda}}{\sqrt{-\lambda-2}}\right)}{9\sqrt{\lambda}} \end{aligned}$$

and can be computed numerically.

5. Order Statistics

The density function of rth order statistics for CTU distribution is given as

$$\begin{aligned} f_{X_{(r)}}(x) &= \frac{n!}{(r-1)!(n-r)!}\left\{(1-\lambda)+6\lambda x-6\lambda x^2\right\}\left[(1-\lambda)x+3\lambda x^2-2\lambda x^3\right]^{r-1} \\ &\quad \times\left[1-(1-\lambda)x-3\lambda x^2+2\lambda x^3\right]^{n-r}, \end{aligned} \tag{15}$$

where $r = 1, 2, \cdots, n$. Therefor, for $r = 1$, we have the *pdf* of the smallest order statistics $X_{(1)}$, and is given as

$$f_{X_{(1)}}(x) = n\left\{(1-\lambda) + 6\lambda x - 6\lambda x^2\right\}\left[1 - (1-\lambda)x - 3\lambda x^2 + 2\lambda x^3\right]^{n-1},$$

and for $r = n$, the *pdf* of the largest order statistics $X_{(n)}$, is

$$f_{X_{(n)}}(x) = n\left\{(1-\lambda) + 6\lambda x - 6\lambda x^2\right\}\left[(1-\lambda)x + 3\lambda x^2 - 2\lambda x^3\right]^{n-1}.$$

Note that $\lambda = 0$, we have *pdf* of the rth order statistics for uniform distribution The kth moment of rth order statistics for CTU distribution is obtained as

$$E(X_{(r)}^k) = \int_0^1 x_{(r)}^k \cdot f_{X_{(r)}}(x) \cdot \mathrm{d}x,$$

where $f_{X_{(r)}}(x)$ is given in (15). Setting $r = 1$ and $r = n$ one can obtain the kth moment of smallest and largest order statistics for the CTU distribution.

6. Parameter Estimation

This section is dedicated to maximum likelihood estimation of the parameter of CTU distribution. For this, suppose a random sample of size n is available from the CTU distribution, then the likelihood function is

$$L = \prod_{i=1}^{n}\left[(1-\lambda) + 6\lambda x_i - 6\lambda x_i^2\right].$$

The log-likelihood function is

$$l = \sum_{i=1}^{n} ln\left[(1-\lambda) + 6\lambda x_i - 6\lambda x_i^2\right]. \tag{16}$$

The derivatives of (16) with respect to λ is

$$\frac{\delta l}{\delta\lambda} = \sum_{i=1}^{n}\frac{6x_i - 6x_i^2 - 1}{(1-\lambda) + 6\lambda x_i - 6\lambda x_i^2},$$

Now setting $\frac{\delta l}{\delta\lambda} = 0$ one can get numerical estimate of λ. For the numerical estimate of λ we apply R-package "bbmle", for details see [6].

7. Numerical Studies

In this section, simulation study has been conducted to assess the performance of estimation method. Also, the CTU distribution has been applied on real-life data set to observed the practicality. These are given by the following two subsection.

Table 2: Average Estimate of Parameter and MSE for CTU Distribution.

Sample Size	Estimate of λ	MSE of λ
50	-0.526	0.106
100	-0.546	0.053
200	-0.540	0.026
500	-0.550	0.011
1000	-0.550	0.005

7.1. Simulation Study

In order to conduct this simulation study, we have drawn random samples of sizes 50, 100, 200, 500 and 1000 from CTU distribution by setting $\lambda = -0.55$. Maximum likelihood estimate of λ is obtained by using each sample. The procedure is repeated 10000 times and we have then taken the average value of the estimate alongside the mean square error. The results are described in Table 2 which shows that the estimated value is very close to the pre-selected value of λ. We can also see that the mean square error reduces with increase in the sample size. This shows the adequacy of the estimation method.

7.2. Application

In this section we have given application of the proposed CTU distribution by using a real data set. The data set is about lifetimes (in days) of 30 electronic devices and is given in Table 3.

Table 3: Lifetimes of 30 Electronic Devices.

0.020, 0.029, 0.034, 0.044, 0.057, 0.096, 0.106, 0.139, 0.156, 0.164, 0.167, 0.177, 0.250, 0.326, 0.406, 0.607, 0.650, 0.672, 0.676, 0.736, 0.817, 0.838, 0.910, 0.931, 0.946, 0.953, 0.961, 0.981, 0.982 and 0.990.

Table 4: Summary Statistics for Lifetimes of 30 Electronic Devices.

Data Set	Min.	Q_1	Median	Mean	Q_3	Max.	Skew.
Lifetimes	0.020	0.143	0.506	0.494	0.892	0.990	0.062

The summary statistics for the data is given in Table 4. It has been observed from table 4 that data follows slightly positively skewed distribution.

The probability-probability (for short, $P-P$) and quantile-quantile (for short, $Q-Q$) plots are presented in Figure 4. We can see, from the Figure 4, that CTU distribution closely agree with the actual data.

We have fitted various distributions; namely skew uniform, Beta and Kumaraswamy distribution; alongside the CTU distribution on this data. The estimates of the model parameters along with standard errors are obtained and presented in Table 5.

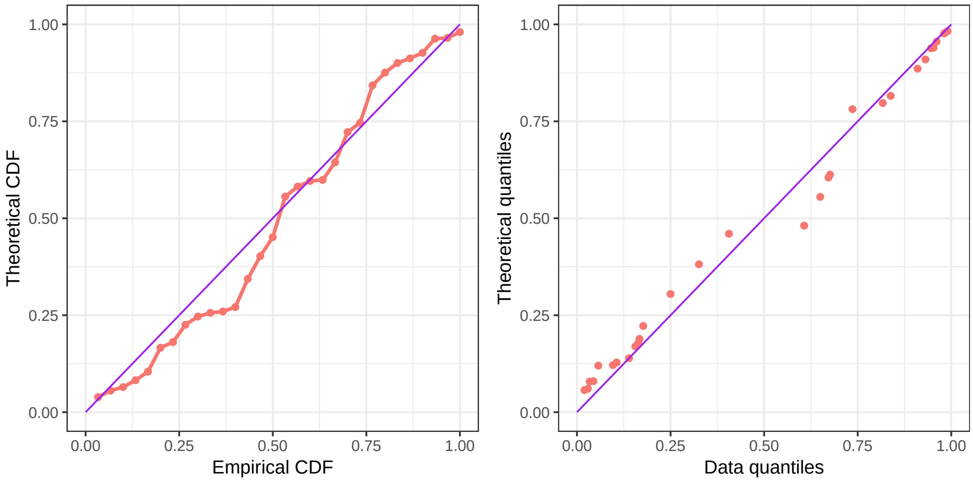

Figure 4: $P-P$ and $Q-Q$ Plots of CTU Distribution for Lifetimes of 30 Electronic Devices.

Table 5: $MLE's$ of the Parameters and Respective $SE's$ for Selected Models.

Distribution	Parameter	Estimate	SE
CTU	λ	-1	0.444
Skew Uniform	λ	0.022	0.246
Beta	α, β	0.607, 0.591	0.142, 0.138
Kumaraswamy	α, β	0.588, 0.612	0.161, 0.134

Estimated density and reliability curves for the selected models have been plotted over empirical histogram and empirical reliability curves in the upper left and upper right corner of the Figure 5. It has been observed that CTU distribution fitted very well as compare with other competitive models used in this study.

In order to verify the shape of hazard rate function, the total time on test (TTT) plot is used. For more details of TTT plot see [1, 5, 11]. TTT plot of CTU distribution are presented in lower left of Figure 5. It has been observed a series of convex and concave TTT plot with upward trend. Estimated hazard curve has been plotted in lower right of Figure 5 and observed slight decreasing failure rate then constant failure rate and dramatic increasing failure rate at the end.

Table 6: Selection Criteria Estimated for Selected Models.

Distribution	logLike	AIC	$AICc$	BIC
CTU	6.196	-10.393	-10.250	-8.991
Skew Uniform	0.004	1.992	2.135	3.393
Beta	3.625	-3.250	-2.805	-0.447
Kumaraswamy	3.503	-3.005	-2.561	-0.203

We have considered various selection criteria; like log-likelihood, Akaike's information criterion (AIC), corrected Akaike's information criterion ($AICc$) and Bayesian informa-

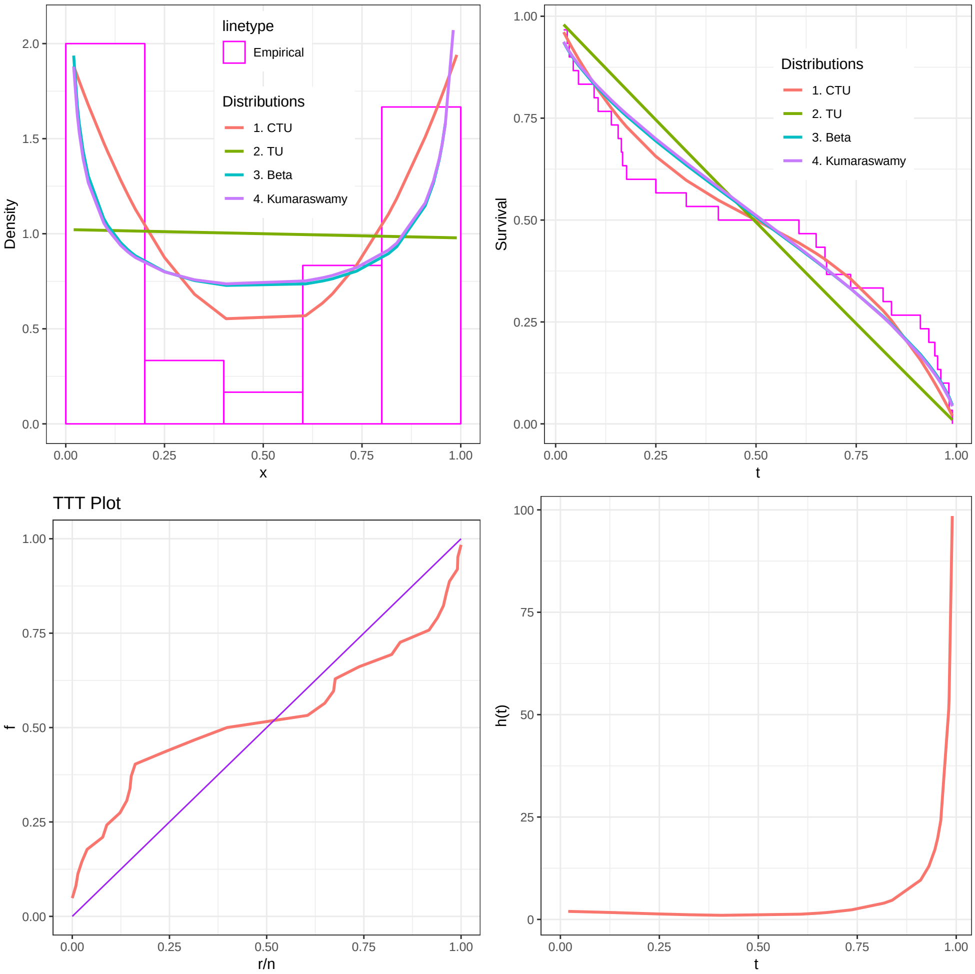

Figure 5: Estimated Density Curves (Top Left), Estimated Reliability Curves (Top Right), TTT Plot (Bottom Left) and Hazard Estimate Curve (Bottom Right).

tion criterion (BIC); to see the goodness of fit of various models. The computed values of these selection criteria are given in Table 6 and we have observed the positive log-likelihood along with negative AIC, $AICc$ and BIC values (for example, see [7]) for CTU, beta and Kumaraswamy distributions. The computed selection criteria values are positive for skew uniform distribution. According to the values of highest log-likelihood and lowest AIC, $AICc$ and BIC, the proposed CTU distribution is the most competitive model for this data.

8. Concluding Remarks

We have proposed cubic transmuted family of distributions and have introduced CTU distribution. Various statistical properties; including moments, quantile and generating functions, random number generation, reliability function and Shannon entropy; of the proposed *CTU* distribution have been studied along with the distribution of order statistics. The maximum likelihood estimate of parameter has been discussed and a simulation study has been conducted to see the performance of estimation procedure. We have also fitted the proposed *CTU* distribution on a real data set and have found that our proposed distribution is most adequate fit to the data as compared with other competing models used in the study.

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