



## Convergence of an exponential Runge–Kutta method for non-smooth initial data

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**Abstract.** The paper presents error bounds for the second order exponential Runge-Kutta method for parabolic abstract linear time-dependent differential equations incorporating non-smooth initial data. As an example for this particular type of problems, the paper presents a spatial discretization of a partial integro-differential equation arising in financial mathematics, where non-smooth initial conditions occur in option pricing models. For this example, numerical studies of the convergence rate are given.

**2010 Mathematics Subject Classifications:** 35K90

**Key Words and Phrases:** Exponential integrators, Runge–Kutta methods, Integro-differential equations

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### 1. Introduction

To give numerical solution of stiff differential equations, exponential integrators have been constructed. Through Exponential integrators, unlike standard numerical integrators, the exponential and related functions (often called  $\varphi$ -functions) of large matrices can be used explicitly. The exponential Runge-Kutta methods of collocation type have been constructed by Hochbruck & Ostermann [9] and their convergence properties were analyzed for linear and semi-linear parabolic problems. Hochbruck & Ostermann [8] also studied explicit exponential Runge-Kutta methods for the time integration of semi-linear parabolic problems. Gondal [4] considered exponential Rosenbrock integrators for option pricing. Different types of exponential integrators and their applications are discussed in details in [6, 10, 11, 18, 19].

Henry [7] and Pazy [16] studied semi-linear problems and contributed significantly. Le Roux [17] introduced for the first time non-smooth data error estimates for time discretizations of linear parabolic problems. The error bounds for time discretizations of

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DOI: <https://doi.org/10.29020/nybg.ejpam.v12i3.3423>

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semi-linear parabolic equations with non-smooth initial data have been inferred in [12]. Linearly implicit time discretization of semi-linear parabolic equations with non-smooth initial data was studied by the authors in [15]. In [5], author proved convergence result of an exponential Euler method using non-smooth initial data for option pricing. We mean to examine convergence properties of exponential Runge-Kutta method for linear parabolic problems that spring up in financial problems. To evaluate this, we cultivate in an abstract Banach space framework of sectorial operators and analytic semi-groups and prove convergence for Exponential Runge-Kutta method for non-smooth initial data.

The jump diffusion model, proposed in [14], is chosen as an application of our analysis. In particular, we discuss the partial integro-differential equations(PIDE) for Mertons model. Briani, La Chioma & Natalini [13] used an explicit method to solve Mertons model and constituted a convergence theory for explicit schemes for varied integro-differential Cauchy problems. Cont & Voltchkova [2] used implicit-explicit finite difference methods successfully for European and barrier options in jump diffusion and exponential Levy models. In a paper [3], one can find different option pricing problems solved numerically through Chebychev discretisation schemes and exponential integrators.

This article is attributed to a theoretical convergence analysis of exponential integrators which is transported out within the framework of evolution equations in Banach spaces. In financial applications, the initial information is generally non-smooth and lies of the payoff function of the option. Therefore, in case of non-smooth initial data, the matter of concern is to have practical error bounds. A bound of this nature is developed in [5] of order one for the exponential Euler method. Following, in Section 3, a error bound is established for the method called an exponential RungeKutta method of order two, and the result is given in Theorem 1.

Besides this preamble, the paper comprises of four sections. Section 2 describes the exponential Rung-Kutta and exponential Euler time integrators. Section 3 present the main results and originate new error bounds. Although in case of non-smooth initial data, error bounds derived in [12] and [15]. But the results for exponential integrators, however, have not been experienced. For the application of analysis, Section 4 offers an example from the Mertons models. The Conclusion includes few final remarks.

## 2. Numerical method

In this section, the abstract form of evaluation equation that results from partial integro-differential equations, that arise in financial mathematics, is considered as follows:

$$u'(t) = Au(t) + Bu(t) + g(t), \quad u(t_0) = u_0, \quad 0 < t \leq T, \quad (1)$$

The variation-of-constants formula with the exact solution representation of (1) is

$$u(t_{n+1}) = e^{hA}u(t_n) + \int_0^h e^{(h-\tau)A}B \cdot u(t_n + \tau)d\tau + \int_0^h e^{(h-\tau)A}g(t_n + \tau)d\tau.$$

The approximation obtained through the left rectangular rule is

$$u(t_{n+1}) \approx e^{hA}u(t_n) + \int_0^h e^{(h-\tau)A} B \cdot u(t_n) d\tau + \int_0^h e^{(h-\tau)A} g(t_n) d\tau$$

and

$$u_{n+1} = e^{hA}u_n + h\varphi_1(hA)(Bu_n + g(t_n)), \quad \varphi_1(hA) = \frac{1}{h} \int_0^h e^{(h-\tau)A} d\tau. \tag{2}$$

which is known as the exponential Euler method of order one for problem given in (1).

Now for (1), we assume the following exponential Runge–Kutta methods

$$u_{n+1} = e^{hA}u_n + h \sum_{i=1}^s b_i(hA)(Bu_{ni} + g(t_n + c_i h)), \tag{3}$$

$$u_{ni} = e^{c_i hA}u_n + h \sum_{j=1}^{i-1} a_{ij}(hA)(Bu_{nj} + g(t_n + c_j h)), \quad 1 \leq i \leq s.$$

The exponential Runge–Kutta method (3) for a second-order method with two stages can be written as

$$u_{n+1} = e^{hA}u_n + h \left( b_1(hA)(Bu_{n1} + g(t_n + c_1 h)) + b_2(hA)(Bu_{n2} + g(t_n + c_2 h)) \right), \tag{4}$$

$$u_{n1} = e^{c_1 hA}u_n,$$

$$u_{n2} = e^{c_2 hA}u_n + ha_{21}(hA)(Bu_{n1} + g(t_n + c_1 h)).$$

Further we know that for a second-order method with two stages it must satisfy the following three order conditions given in Hochbruck & Ostermann [9]

$$\begin{aligned} b_1(hA) + b_2(hA) &= \varphi_1(hA), \\ c_1 b_1(hA) + c_2 b_2(hA) &= \varphi_2(hA), \\ a_{21}(hA) &= c_2 \varphi_1(c_2 hA), \end{aligned} \tag{5}$$

where

$$\varphi_1(z) = \frac{e^z - 1}{z}, \quad \varphi_2(z) = \frac{\varphi_1(z) - 1}{z}.$$

By taking  $c_1 = 0$  and  $c_2 = 1$ , we find the values of  $b_1 = \varphi_1(hA) - \varphi_2(hA)$ ,  $b_2 = \varphi_2(hA)$  and  $a_{21} = \varphi_1(hA)$ . Hence we can write (4) as

$$u_{n+1} = e^{hA}u_n + h(\varphi_1(hA) - \varphi_2(hA))Bu_{n1} + h\varphi_2(hA)Bu_{n2} + h(\varphi_1(hA) - \varphi_2(hA))g(t_n) + h\varphi_2(hA)g(t_n + h), \tag{6}$$

$$\begin{aligned} u_{n1} &= u_n, \\ u_{n2} &= e^{hA}u_n + h\varphi_1(hA)(Bu_{n1} + g(t_n)). \end{aligned} \tag{7}$$

This is the exponential Runge–Kutta method of order two for the problem (1).

### 3. Convergence of an exponential Runge–Kutta method of order two for non-smooth initial data

In this section we study an exponential Runge–Kutta method of order two for discretizing an abstract problem (1) in time. On  $A$ ,  $B$  and  $g$ , our assumptions are the same as given in Gondal [5].

Now first we are going to prove vital properties of the exact solution and then we will move to the numerical solution.

**Lemma 1.** *Assume that problem (1) fulfill the hypotheses of Lemma 2 given in Gondal [5]. Then the bounds*

$$\|L^{-1}u''(t)\| \leq \frac{C}{t}, \quad \text{on } (0, T], \tag{8}$$

hold uniformly on  $0 \leq t \leq T$  for non-smooth initial data.

*Proof.* From Lemma 2 given in Gondal [5] we use equation (18) given in Gondal [5] in

$$u''(t) = Lu'(t) + g'(t), \tag{9}$$

and we get

$$u''(t) = L^2e^{tL}u_0 + Le^{tL}g(0) + e^{tL}g'(0) + t\varphi_1(tL)g''(0) + \dots \tag{10}$$

Premultiplying with  $L^{-1}$  on both sides of (10), we get

$$L^{-1}u''(t) = Le^{tL}u_0 + e^{tL}g(0) + \dots \tag{11}$$

Now multiplying with  $t$  on both sides of (11), we have

$$tL^{-1}u''(t) = tLe^{tL}u_0 + te^{tL}g(0) + \dots \tag{12}$$

Therefore

$$\|tL^{-1}u''(t)\| \leq C \quad \text{or} \quad \|L^{-1}u''(t)\| \leq \frac{C}{t}. \tag{13}$$

In this section, we will also derive error bounds for exponential Runge–Kutta discretizations of (1). The exponential Runge–Kutta method of order two for given problem is (6). To analyze (6), one can write the exact solution of (1) as

$$u(t_{n+1}) = e^{hA}u(t_n) + \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}Bu(\tau)d\tau + \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}g(\tau)d\tau. \tag{14}$$

To write (14) in the form of (6), below result will be helpful

$$\int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}d\tau = \int_0^h e^{(h-s)A}ds = h\varphi_1(hA) \tag{15}$$

$$\int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}(\tau - t_n)d\tau = \int_0^h e^{(h-s)A}sds = h^2\varphi_2(hA). \tag{16}$$

**Theorem 1.** Assume that problem (1) fulfill the hypotheses of Lemma 2 given in Gondal [5] and that  $AB = BA$ . For the numerical solution, we consider the exponential Runge-Kutta method (6). Also suppose that  $g', g''$  are bounded and  $g : [0, T] \rightarrow \mathcal{X}$  is differentiable. Then the following error bound

$$\|u(t_n) - u_n\| \leq \frac{Ch^2}{t_n}(|\log h| + 1) \tag{17}$$

holds uniformly in  $0 \leq t_n \leq T$  for non-smooth initial data.

*Proof.* We can write the difference between the  $g$  terms from (14) and (6) as

$$\begin{aligned} \epsilon_{n+1}(g) &= \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} g(\tau) d\tau - h(\varphi_1(hA) - \varphi_2(hA))g(t_n) \\ &\quad - h\varphi_2(hA)g(t_n + h). \end{aligned} \tag{18}$$

Using results (15) and (16) in (18) and simplifying, we get

$$\begin{aligned} \epsilon_{n+1}(g) &= \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} \left( g(\tau) - g(t_n) + \frac{\tau - t_n}{h} g(t_n) \right. \\ &\quad \left. - \frac{\tau - t_n}{h} g(t_n + h) \right) d\tau. \end{aligned} \tag{19}$$

By using Taylor series we can write

$$g(\tau) = g(t_n) + (\tau - t_n)g'(t_n) + \frac{1}{2}(\tau - t_n)^2g''(t_n) + \dots \tag{20}$$

$$g(t_n + h) = g(t_n) + hg'(t_n) + \frac{h^2}{2}g''(t_n) + \dots \tag{21}$$

Now substitute  $g(\tau)$  and  $g(t_n + h)$  from equations (20) and (21) in (19) and after simplification and neglecting higher order terms we can write (19) as

$$\begin{aligned} \epsilon_{n+1}(g) &= \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} \left( \frac{(\tau - t_n)^2}{2} - \frac{h(\tau - t_n)}{2} \right) g''(t_n) d\tau, \\ \|\epsilon_{n+1}(g)\| &\leq \int_{t_n}^{t_{n+1}} \|e^{(t_{n+1}-\tau)A}\| \left\| \frac{(\tau - t_n)^2}{2} - \frac{h(\tau - t_n)}{2} \right\| \|g''(t_n)\| d\tau, \\ &\leq C \int_{t_n}^{t_{n+1}} \left( \frac{(\tau - t_n)^2}{2} + \frac{h(\tau - t_n)}{2} \right) d\tau, \\ &\leq C \frac{h^3}{6} + C \frac{h^3}{2} = Ch^3. \end{aligned} \tag{22}$$

This is the one way to solve (19) in which we have to assume that all higher order derivatives of  $g$  are bounded. There is another good and tricky way to prove that  $\epsilon_{n+1}(g)$  is bounded. For this trick, we can write  $g(\tau)$  instead of Taylor series in the following form

$$g(\tau) = g(t_n) + \int_{t_n}^{\tau} g'(s) ds,$$

$$\begin{aligned}
 &= g(t_n) + \int_{t_n}^{\tau} 1 \cdot g'(s) ds, \\
 &= g(t_n) + \int_{t_n}^{\tau} (s - t_n)' g'(s) ds.
 \end{aligned}
 \tag{23}$$

Integration by part yields

$$g(\tau) = g(t_n) + (\tau - t_n)g'(t_n) + \int_{t_n}^{\tau} (\tau - s)g''(s) ds,
 \tag{24}$$

since we can write  $g(t_n + h) = g(t_{n+1})$ . Using (24), one can write

$$\begin{aligned}
 g(t_{n+1}) &= g(t_n) + (t_{n+1} - t_n)g'(t_n) + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)g''(s) ds, \\
 &= g(t_n) + hg'(t_n) + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)g''(s) ds.
 \end{aligned}
 \tag{25}$$

Now we can use expressions (24) and (25) for  $g(\tau)$  and  $g(t_n + h)$  instead of (20) and (21) in (19) to get (22). In this case we only assume that first and second order derivatives of  $g$  are bounded.

Now in the same way we can write the difference between the  $u$  terms from (14) and (6) as

$$\begin{aligned}
 \epsilon_{n+1}(u) &= \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} Bu(\tau) d\tau - h(\varphi_1(hA) - \varphi_2(hA))Bu_n \\
 &\quad - h\varphi_2(hA)Bu_{n2},
 \end{aligned}
 \tag{26}$$

since we can write

$$\begin{aligned}
 \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} Bu(\tau) d\tau &= \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} B \left( u(t_n) - \frac{\tau - t_n}{h} u(t_n) + \frac{\tau - t_n}{h} u(t_n + h) \right. \\
 &\quad \left. + u(\tau) - u(t_n) + \frac{\tau - t_n}{h} u(t_n) - \frac{\tau - t_n}{h} u(t_n + h) \right) d\tau \\
 &= h\varphi_1(hA)Bu(t_n) - h\varphi_2(hA)Bu(t_n) + h\varphi_2(hA)Bu(t_{n+1}) \\
 &\quad + \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} B \left( u(\tau) - u(t_n) + \frac{\tau - t_n}{h} u(t_n) \right. \\
 &\quad \left. - \frac{\tau - t_n}{h} u(t_n + h) \right) d\tau.
 \end{aligned}
 \tag{27}$$

Using (27) in (26) and simplifying, we have

$$\begin{aligned}
 \epsilon_{n+1}(u) &= h\varphi_1(hA)B(u(t_n) - u_n) - h\varphi_2(hA)B(u(t_n) - u_n) \\
 &\quad + h\varphi_2(hA)B(u(t_{n+1}) - u_{n2}) + R_{n+1}, \\
 \|\epsilon_{n+1}(u)\| &\leq Ch\|\epsilon_n\| + Ch\|\epsilon_n\| + Ch\|u(t_{n+1}) - u_{n2}\| + \|R_{n+1}\|,
 \end{aligned}
 \tag{28}$$

where

$$R_{n+1} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} B \left( u(\tau) - u(t_n) + \frac{\tau - t_n}{h} u(t_n) - \frac{\tau - t_n}{h} u(t_n + h) \right) d\tau. \tag{29}$$

Since one can write

$$\begin{aligned} u(t_{n+1}) - u_{n2} &= u(t_{n+1}) - u_{n+1} + u_{n+1} - u_{n2}, \\ &= \epsilon_{n+1} + u_{n+1} - u_{n2}, \\ \|u(t_{n+1}) - u_{n2}\| &\leq \|\epsilon_{n+1}\| + \|u_{n+1} - u_{n2}\|. \end{aligned} \tag{30}$$

Now from (6) and (7) we can write

$$\begin{aligned} u_{n+1} - u_{n2} &= h\varphi_2(hA)B(u_{n2} - u_n) + h\varphi_2(hA)(g(t_n + h) - g(t_n)), \\ \|u_{n+1} - u_{n2}\| &\leq Ch\|u_{n2} - u_n\| + Ch^2, \end{aligned} \tag{31}$$

since  $\|g(t_n + h) - g(t_n)\| \leq Ch$ . Now

$$\begin{aligned} u_{n2} - u_n &= u_{n2} - u_{n+1} + u_{n+1} - u_n, \\ \|u_{n2} - u_n\| &\leq \|u_{n+1} - u_n\| + \|u_{n+1} - u_{n2}\|. \end{aligned} \tag{32}$$

From (6) we can write for  $n \geq 1$

$$\begin{aligned} u_{n+1} - u_n &= e^{hA}u_n - u_n + \mathcal{O}(h), \\ &= (e^{hA} - 1)u_n + \mathcal{O}(h) = hA\varphi_1(hA)u_n + \mathcal{O}(h), \\ &= hA\varphi_1(hA)(u(t_n) + \epsilon_n) + \mathcal{O}(h), \text{ since } u_n = u(t_n) + u_n - u(t_n), \\ &= hA\varphi_1(hA)\epsilon_n + hA\varphi_1(hA)u(t_n) + \mathcal{O}(h), \\ \|u_{n+1} - u_n\| &\leq C\|\epsilon_n\| + Ch \cdot \frac{C}{t_n} + \mathcal{O}(h), \end{aligned} \tag{33}$$

since  $\|Au(t_n)\| \leq \frac{C}{t_n}$  for  $n > 0$ .

Using (33) in (32) and then (32) in (31) yields

$$\begin{aligned} \|u_{n+1} - u_{n2}\| &\leq Ch\|u_{n+1} - u_{n2}\| + Ch\|\epsilon_n\| + Ch^2 \cdot \frac{1}{t_n} + Ch^2, \\ (1 - hC)\|u_{n+1} - u_{n2}\| &\leq Ch\|\epsilon_n\| + Ch^2 \cdot \frac{1}{t_n} + Ch^2, \end{aligned} \tag{34}$$

$h$  small gives that  $hC \leq \frac{1}{2}$ , therefore for  $n \geq 1$

$$\|u_{n+1} - u_{n2}\| \leq Ch\|\epsilon_n\| + \frac{Ch^2}{t_n} + Ch^2. \tag{35}$$

Using (35) in (30) and then (30) in (28) gives for  $n \geq 1$

$$\|\epsilon_{n+1}(u)\| \leq Ch\|\epsilon_n\| + Ch\|\epsilon_{n+1}\| + Ch^2\|\epsilon_n\| + \frac{Ch^3}{t_n}$$

$$+ Ch^3 + \|R_{n+1}\|. \tag{36}$$

For  $n = 0$ , we use (30) to obtain

$$\|u(t_1) - u_{02}\| \leq \|\epsilon_1\| + \|u_1 - u_{02}\|,$$

with the help of (31), we get

$$\|u(t_1) - u_{02}\| \leq \|\epsilon_1\| + Ch\|u_{02} - u_0\| + Ch^2. \tag{37}$$

As

$$\|u_{02} - u_0\| \leq \|u_{02}\| + \|u_0\| \leq C, \tag{38}$$

Using (38) in (37) and then (37) in (28) gives for  $n = 0$

$$\|\epsilon_1(u)\| \leq Ch\|\epsilon_0\| + Ch\|\epsilon_1\| + Ch^2 + Ch^3 + \|R_1\|. \tag{39}$$

Now we want to prove  $\|R_{n+1}\|$  is bounded. For this, we can write  $u(\tau)$  and  $u(t_{n+1})$  by using the same concept of (24) in the following form

$$u(\tau) = u(t_n) + (\tau - t_n)u'(t_n) + \int_{t_n}^{\tau} (\tau - s)u''(s)ds, \tag{40}$$

$$u(t_{n+1}) = u(t_n) + hu'(t_n) + \int_{t_n}^{t_{n+1}} (t_{n+1} - s)u''(s)ds. \tag{41}$$

Substituting the expressions for  $u(\tau)$  from (40) and  $u(t_n + h) = u(t_{n+1})$  from (41) in (29) and simplifying, we get

$$R_{n+1} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}B \left( \int_{t_n}^{\tau} (\tau - s)u''(s)ds - \frac{\tau - t_n}{h} \int_{t_n}^{t_{n+1}} (t_{n+1} - s)u''(s)ds \right) d\tau. \tag{42}$$

$$R_{n+1} = R_{n+1,2} + R_{n+1,3} \tag{43}$$

where

$$R_{n+1,2} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}B \left( \int_{t_n}^{\tau} (\tau - s)u''(s)ds \right) d\tau, \tag{44}$$

and

$$R_{n+1,3} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}B \left( \frac{\tau - t_n}{h} \int_{t_n}^{t_{n+1}} (t_{n+1} - s)u''(s)ds \right) d\tau. \tag{45}$$

Now to solve (44), we use the identity  $AA^{-1} = I$  and get

$$R_{n+1,2} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}B \left( \int_{t_n}^{\tau} (\tau - s)AA^{-1}u''(s)ds \right) d\tau. \tag{46}$$

$A$  can commute with  $B$ , i.e.,  $AB = BA$ , if  $B$  is a convolution integral, therefore

$$R_{n+1,2} = A \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A}B \left( \int_{t_n}^{\tau} (\tau - s)A^{-1}u''(s)ds \right) d\tau. \tag{47}$$



Since

$$\begin{aligned} \epsilon_{n+1} &= u(t_{n+1}) - u_{n+1}, \\ &= e^{hA}\epsilon_n + \epsilon_{n+1}(g) + \epsilon_{n+1}(u). \end{aligned} \tag{48}$$

Therefore from (6) and (14) and using (22) and (36), we get from (48) the error recursion for  $n > 0$

$$\begin{aligned} \|\epsilon_{n+1}\| &\leq \|e^{hA}\|\|\epsilon_n\| + Ch\|\epsilon_n\| + Ch\|\epsilon_{n+1}\| + Ch^2\|\epsilon_n\| \\ &\quad + \frac{Ch^3}{t_n} + Ch^3 + \|R_{n+1}\|. \end{aligned} \tag{49}$$

$h$  small gives that  $hC \leq \frac{1}{2}$ , therefore

$$\begin{aligned} \|\epsilon_{n+1}\| &\leq C\|e^{hA}\|\|\epsilon_n\| + Ch\|\epsilon_n\| + Ch^2\|\epsilon_n\| + \frac{Ch^3}{t_n} + Ch^3 + \|R_{n+1}\|, \\ &\quad \vdots \\ \|\epsilon_n\| &\leq C\|e^{nhA}\|\|\epsilon_0\| + Ch \sum_{j=0}^{n-1} \|e^{(n-j-1)hA}\|\|\epsilon_j\| + Ch^2 \sum_{j=0}^{n-1} \|e^{(n-j-1)hA}\|\|\epsilon_j\| \\ &\quad + Ch^3 \sum_{j=1}^{n-1} \|e^{(n-j-1)hA}\|\|\frac{1}{t_j}\| + Ch^2 + Ch^3 \sum_{j=0}^{n-1} \|e^{(n-j-1)hA}\| \\ &\quad + \sum_{j=0}^{n-1} \|e^{(n-j-1)hA}R_{j+1}\|. \end{aligned} \tag{50}$$

Using Lemma 2 given in Gondal [5] and fact that  $t_j = jh$ , we get

$$\begin{aligned} \|\epsilon_n\| &\leq C\|\epsilon_0\| + Ch \sum_{j=0}^{n-1} \|\epsilon_j\| + Ch^2 \sum_{j=0}^{n-1} \|\epsilon_j\| + Ch^2 \sum_{j=1}^{n-1} \|\frac{1}{j}\| + Ch^3 \cdot n \\ &\quad + \sum_{j=0}^{n-1} \|e^{(n-j-1)hA}R_{j+1}\|. \end{aligned} \tag{51}$$

Since we know that  $nh = T$ ,  $Ch^2 \leq Ch$  and  $\|\epsilon_0\| = 0$ , and using the result (??), we get

$$\begin{aligned} \|\epsilon_n\| &\leq Ch \sum_{j=0}^{n-1} \|\epsilon_j\| + Ch^2(C + |\log h|) + Ch^2T + \sum_{j=0}^{n-1} \|e^{(n-j-1)hA}R_{j+1}\|, \\ &\leq Ch \sum_{j=0}^{n-1} \|\epsilon_j\| + Ch^2(1 + |\log h|) + \sum_{j=0}^{n-1} \|e^{(n-j-1)hA}R_{j+1}\|. \end{aligned} \tag{52}$$

After this, it is proved that  $\sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1}\|$  is bounded. For this we can write

$$\sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1}\| = \sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1,2}\| + \sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1,3}\|. \tag{53}$$

From (47) we can write

$$\begin{aligned} & \sum_{j=1}^{n-2} \|e^{(n-j-1)hA} R_{j+1,2}\| \tag{54} \\ &= \sum_{j=1}^{n-2} \|Ae^{(n-j-1)hA} \int_{t_j}^{t_{j+1}} e^{(t_{j+1}-\tau)A} B \left( \int_{t_j}^{\tau} (\tau-s)A^{-1}u''(s)ds \right) d\tau\|, \\ &\leq \sum_{j=1}^{n-2} \|Ae^{(n-j-1)hA}\| \int_{t_j}^{t_{j+1}} \|e^{(t_{j+1}-\tau)A}\| \|B\| \left( \int_{t_j}^{\tau} (\tau-s)\|A^{-1}u''(s)\|ds \right) d\tau. \end{aligned}$$

Using Lemma 2 given in Gondal [5] and Lemma 1 in above equation and then integrating, we get

$$\begin{aligned} \sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1,2}\| &\leq \sum_{j=1}^{n-2} \frac{C}{t_{n-j-1}} \cdot h^3 \cdot \frac{C}{t_j} + \text{term for } j = 0 + \text{term for } j = n - 1, \\ &= Ch^3 \sum_{j=1}^{n-2} \frac{1}{t_{n-j-1}t_j} + \|e^{(n-1)hA} R_{1,2}\| + \|R_{n,2}\|, \\ &= Ch^3 \sum_{j=1}^{[n/2]} \frac{1}{t_{n-j-1}t_j} + Ch^3 \sum_{j=[n/2]+1}^{n-2} \frac{1}{t_{n-j-1}t_j} + \|e^{(n-1)hA} R_{1,2}\| \\ &\quad + \|R_{n,2}\|, \\ &\leq \frac{Ch^3}{t_n} \sum_{j=1}^{[n/2]} \frac{1}{t_j} + \frac{Ch^3}{t_n} \sum_{j=[n/2]+1}^{n-2} \frac{1}{t_{n-j-1}} + \|e^{(n-1)hA} R_{1,2}\| + \|R_{n,2}\|, \\ &\leq \frac{2Ch^2}{t_n} (1 + |\log h|) + \|e^{(n-1)hA} R_{1,2}\| + \|R_{n,2}\|. \tag{55} \end{aligned}$$

Now for  $j = 0$  and  $j = n - 1$  term, we first rewrite  $R_{n+1}$  by using

$$u(\tau) = u(t_n) + \int_{t_n}^{\tau} u'(s)ds, \tag{56}$$

and

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} u'(s)ds, \tag{57}$$

in (29) and simplifying, we get

$$R_{n+1} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} B \left( \int_{t_n}^{\tau} u'(s)ds - \frac{\tau-t_n}{h} \int_{t_n}^{t_{n+1}} u'(s)ds \right) d\tau. \tag{58}$$

$$R_{n+1} = R_{n+1,2} + R_{n+1,3} \tag{59}$$

where

$$R_{n+1,2} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} B \left( \int_{t_n}^{\tau} u'(s) ds \right) d\tau, \tag{60}$$

and

$$R_{n+1,3} = \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-\tau)A} B \left( \frac{\tau - t_n}{h} \int_{t_n}^{t_{n+1}} u'(s) ds \right) d\tau. \tag{61}$$

By substituting  $n = 0$  in (60) and using the identity  $AA^{-1} = I$  and  $AB = BA$ , we can write the term for  $j = 0$  as

$$\begin{aligned} \|e^{(n-1)hA} R_{1,2}\| &= \|Ae^{(n-1)hA} \int_0^h e^{(h-\tau)A} B \left( \int_0^{\tau} A^{-1} u'(s) ds \right) d\tau\|, \\ &\leq \|Ae^{(n-1)hA}\| \int_0^h \|e^{(h-\tau)A}\| \|B\| \left( \int_0^{\tau} \|A^{-1} u'(s)\| ds \right) d\tau, \\ &\leq \frac{C}{t_{n-1}} \cdot C \cdot h^2, \\ &\leq \frac{Ch^2}{t_{n-1}}. \end{aligned} \tag{62}$$

Now by substituting  $n = n - 1$  in (60) we can write the term for  $j = n - 1$  as

$$\begin{aligned} \|R_{n,2}\| &= \left\| \int_{t_{n-1}}^{t_n} e^{(t_n-\tau)A} B \left( \int_{t_{n-1}}^{\tau} u'(s) ds \right) d\tau \right\|, \\ &\leq \int_{t_{n-1}}^{t_n} \|e^{(t_n-\tau)A}\| \|B\| \left( \int_{t_{n-1}}^{\tau} \|u'(s)\| ds \right) d\tau, \\ &\leq \frac{Ch^2}{t_{n-1}}. \end{aligned} \tag{63}$$

Substituting (62) and (63) in (55), we get

$$\sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1,2}\| \leq \frac{Ch^2}{t_n} (1 + |\log h|) + \frac{Ch^2}{t_{n-1}}. \tag{64}$$

Similarly we can prove that  $\sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1,3}\|$  is bounded and we get

$$\sum_{j=0}^{n-1} \|e^{(n-j-1)hA} R_{j+1,3}\| \leq \frac{Ch^2}{t_n} (1 + |\log h|) + \frac{Ch^2}{t_{n-1}}. \tag{65}$$

Now substitute (64) and (65) in (53) and then (53) in (52) and simplifying we get

$$\|\epsilon_n\| \leq Ch \sum_{j=0}^{n-1} \|\epsilon_j\| + \frac{Ch^2}{t_n} (1 + |\log h|) + \frac{Ch^2}{t_{n-1}}. \tag{66}$$

Note that  $\frac{Ch^2}{t_{n-1}} = \frac{Ch^2}{t_n} \cdot \frac{t_{n-1}+h}{t_{n-1}} \leq \frac{Ch^2}{t_n}$ . By using the Lemma 6.2(Gronwall lemma) given in [15], we get

$$\|\epsilon_n\| \leq \frac{Ch^2}{t_n} (|\log h| + 1). \tag{67}$$

### 4. Numerical experiments

This section deals with the numerical experiments for the verification of our calculated error bounds. Lets assume the linear parabolic problem, called as partial integro-differential equations, that arise in financial mathematics. This was studied by Tangman, Gopaul, & Bhuruth [3]

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{1}{2}\sigma^2 - \lambda\kappa) \frac{\partial u}{\partial x} - (r + \lambda)u + \lambda \int_{\mathbb{R}} b(x - y)u(y, \tau)dy. \tag{68}$$

with

$$b(z) = \frac{1}{\sqrt{2\pi\gamma}} e^{-(z-\mu)^2/(2\gamma^2)}. \tag{69}$$

Where we considers parameters  $r, \sigma, \lambda, \gamma, \kappa, \mu$ . Equation (68) indicates the European option pricing problem in Mertons jump-diffusion model. The initial condition associated with the European call option price

$$u(x, 0) = \max(Ee^x - E, 0) \tag{70}$$

and boundary conditions suggested in [3] are

$$u_\tau(x, \tau) = -ru(x, \tau), \quad x \rightarrow -\infty, \tag{71}$$

$$u_{xx}(x, \tau) = u_x(x, \tau), \quad x \rightarrow \infty. \tag{72}$$

#### 4.1. Space discretization

The discretization for the problem (68), using finite difference schemes will be given here. We require to truncate the infinite  $x$ -domain to finite  $x$ -domain, for instance,  $x_{\min} \leq x \leq x_{\max}$  for a finite difference discretization of the spatial derivatives. Hence

$$-1.5 = x_{\min} = x_0 < x_1 < x_2 < x_3 < \dots < x_M < x_{M+1} = x_{\max} = 1.5,$$

with grid points  $x_i = x_{i-1} + \delta x_i$  and  $\delta x_i = x_i - x_{i-1}$

Here we need the first-order and second-order finite difference approximations for the discretization of (68) on a non-equidistant grid, which are given as

$$\frac{\partial u}{\partial x}(x_i) \cong \frac{u(x_{i+1}) - u(x_{i-1}))}{\delta x_i + \delta x_{i+1}}, \tag{73}$$

$$\frac{\partial^2 u}{\partial x^2}(x_i) \cong \frac{2u(x_{i+1})}{\delta x_{i+1}(\delta x_{i+1} + \delta x_i)} - \frac{2u(x_i)}{\delta x_i \delta x_{i+1}} + \frac{2u(x_{i-1})}{\delta x_i(\delta x_i + \delta x_{i+1})}, \tag{74}$$

The integral term in (68) is discretized in such a way that the infinite integral will split into three parts. See [1].

$$\int_{-\infty}^{\infty} b(x-y)u(y, t)dy = \int_{-\infty}^a b(x-y)u(y, t)dy + \int_a^c b(x-y)u(y, t)dy + \int_c^{\infty} b(x-y)u(y, t)dy, \tag{75}$$

in above equation  $[a, c] = [y_{\min}, y_{\max}]$  and  $y_{\min} = x_{\min}, y_{\max} = x_{\max}$ .  
 With the help of the composite trapezoidal rule, one can write  $\int_a^c b(x-y)u(y, t)dy$  in the form of

$$\begin{aligned} Bu(t) &\approx \lambda \int_a^c b(x_i - y)u(y, t)dy, \\ &\approx \lambda \left[ \frac{1}{2} \delta x_1 b(x_i - y_1)u(y_1, t) + \frac{1}{2} \delta x_{M-1} b(x_i - y_M)u(y_M, t) \right. \\ &\quad \left. + \sum_{j=2}^{M-1} \frac{\delta x_j + \delta x_{j-1}}{2} b(x_i - y_j)u(y_j, t) \right]. \end{aligned} \tag{76}$$

To compute a European call option, [1] proposed the replacement of the integrand  $u(x, \tau)$  over  $(-\infty, a)$  and  $(c, \infty)$  by using the following approximations

$$\begin{aligned} u(x, \tau) &\rightarrow Ee^x - Ee^{-r\tau}, \quad \text{as } x \rightarrow +\infty, \\ u(x, \tau) &\rightarrow 0, \quad \text{as } x \rightarrow -\infty. \end{aligned}$$

Hence, the other part of integral can be written as

$$\begin{aligned} g(t) = \lambda \int_{-\infty}^a b(x-y)u(y, t)dy + \lambda \int_c^{\infty} b(x-y)u(y, t)dy &= \lambda Ee^{x+\mu+\frac{\gamma^2}{2}} \phi\left(\frac{x_i - x_{\max} + \mu + \gamma^2}{\gamma}\right) \\ &\quad - \lambda Ee^{-rt} \phi\left(\frac{x_i - x_{\max} + \mu}{\gamma}\right), \end{aligned} \tag{77}$$

with

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{\beta^2 \gamma}{2}} d\beta.$$

We can write equation (68) in abstract form as

$$u'(t) = Au(t) + Bu(t) + g(t). \tag{78}$$

Above equation (78) is a parabolic equation with

$$A = A_4 + A_3 + A_2,$$

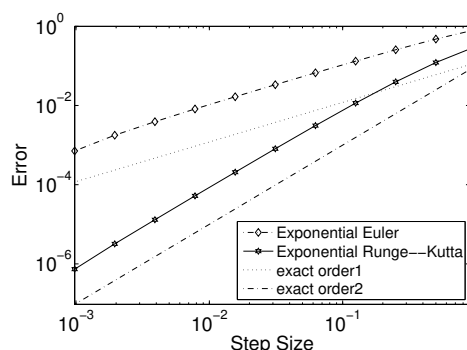


Figure 1: The error of the exponential Euler method of order one and the exponential Runge–Kutta method of order two when applied to (68) with 200 grid points. For comparison, we added lines with slope one and two.

Table 1: The Table clearly demonstrates the numerically observed temporal orders of convergence in the  $L^2$  norm with  $M$  grid points and  $h = 1/128$ . Here  $r = 0.05$ ,  $E = 100$ ,  $\sigma = 0.2$ ,  $\lambda = 2$ ,  $T = 1$ .

$M$	Exponential Euler method	Exponential Runge–Kutta method
50	1.0660	2.0159
100	1.0650	2.0149
200	1.0644	2.0141

where

$$A_4 = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}, \quad A_3 = \left(r - \frac{1}{2}\sigma^2 - \lambda\kappa\right) \frac{\partial u}{\partial x}, \quad A_2 = -(r + \lambda)u.$$

Figure 1 clearly elucidates the convergence of computed first order exponential Euler method and second order exponential Runge-Kutta method for constant time steps with 200 grid points. The computed solution for exponential Euler converges at a first-order and for exponential RungeKutta as second-order rate, as one can see undoubtedly from Figure 1. The errors are measured in the  $L^2$  norm. For comparison, we added the lines with slope one and slope two.

The numerical values are shown in Table 1 for the temporal orders of convergence in the  $L^2$  norm with  $M$  grid points and  $h = 1/128$ , for the PIDE (68) in case of the exponential Euler method of order one and the exponential RungeKutta method of order two.

### 5. Concluding remarks

The current paper deals with the convergence analysis for the non-smooth initial data, discussed in Sections 3. For the applications in financial mathematics, these error estimates are very essential. Optimal convergence is proved for second order convergence for a two stage explicit exponential Runge-Kutta method (Theorem 1). The results added significantly to the known convergence results of exponential integrators. The error bounds were previously experienced only for Runge-Kutta methods and Rosenbrock methods. We developed them for their exponential counterparts for non-smooth initial data.

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