



MHD flows of second grade fluid through the moving porous cylindrical domain

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Abstract. The flows of Magnetohydrodynamics(MHD) second grade fluid between two infinite porous coaxial circular cylinders are studied. At time $t = 0^+$, the inner cylinder begins to rotate around its axis and to slide along the same axis due to torsional and longitudinal time dependent shear stresses and the outer cylinder is also rotate around its axis and to slide along the same axis with acceleration. The exact solutions obtained with the help of discrete Laplace and finite Hankel transform, satisfy all imposed initial and boundary conditions. The solution presented in convolution product of Laplace transform . The corresponding solutions for second grade and Newtonian fluids are also obtained as limiting cases with and without MHD effect. Finally, the influence of pertinent parameters on the velocity components and shear stresses, as well as a comparison among, second grade and Newtonian fluids with and without MHD is also analyzed by graphical illustrations.

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Key Words and Phrases: Second grade fluid, MHD flows, velocity field, time-dependent shear stresses, Hankel transform

1. Introduction

Now-a-days, the non-Newtonian fluids are more important and significant in science and technological applications than the Newtonian fluids. In several sectors, like polymer and petroleum, food and agriculture, chemical and manufacturing, pharmaceutical and biotechnology industries non-Newtonian fluids are more important, the fluids are the solution of either artificial or natural material with other Newtonian fluids as water, oils, red cells and different substances having long chain molecules; the resulting fluids have the non-Newtonian features [5, 6, 10, 16, 18, 20]. There are three famous solution other than exact solutions, can be applicable, those are analytical solutions, numerical solutions

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and experimental solutions [21]. Analytical solution is the general information and closed-form solution, but it has simple geometry / physics and it can solve only linear problem. Numerical solution is used for complicated physical and nonlinear problems, but its main disadvantages are truncation errors and inappropriate modeling while experimental solution is most realistic and reliable but due to equipment / operation costs, scaling problem, measurement difficulties are not much appreciated [12–14, 17].

The flow issue in porous medium like tube, channel, pipes and space are considered in many research literatures but the only few researcher used the porous medium in the magnetohydrodynamic (MHD) flow of second grade fluid in an infinite cylindrical domain, which is much more realistic and interested when the Reynold's number is same with any system, such as in blood circulation, digestive and urinary systems in any body, water and other chemical sewerage filtration and purification plants, underground tube wheel, MHD generators for generating electricity and its MHD pumps and other engineering applications [6–8, 10, 12, 16, 18].

The Navier-Stokes equation is used to solve the problem of Newtonian fluid but the governing equations of non-Newtonian fluids are more convoluted and having the higher order than the Navier-Stokes equation for Newtonian fluids, therefore Navier-Stokes equations are inadequate and insufficient to designate their behavior [1, 4], so we used some mathematical techniques and transforms to solve the flow problem of the non-Newtonian fluid. In cylindrical domain, the Hankel and Laplace transform are the one of the best transform to determine the exact or analytical solution [6, 9, 10, 12, 16].

The objective of this paper is to extend the concepts of boundary conditions and identities of Hankel transformation used in [2, 9] to second grade fluid with more generalized boundary conditions from which we recover many general solutions as special solution to our general solutions. We also point out that the boundary conditions and identities used in mentioned paper [9, 15] are very rare in literature. Therefore motivated by these facts, we study, the MHD flow of second grade fluid between two infinite porous coaxial circular cylinders. At time $t = 0^+$, the inner cylinder begins to rotate around its axis and to slide along the same axis due to torsional and longitudinal time dependent shear stresses $f_1 t^p$ and $g_1 t^p$ the outer cylinder is also rotate around its axis and to slide along the same axis with velocities $f_2 t^p$ and $g_2 t^p$. The exact solutions obtained with the help of Laplace and finite Hankel transform, presented in convolution product of Laplace transform, which satisfy all imposed initial and boundary conditions. As the limiting and special cases, we find solutions for second grade with and without MHD/porous effects, Newtonian with and without MHD/porous effects and discussed graphically. Furthermore the solutions for the special case for $p = 1$ is also presented.

2. Formation of the problem

Suppose that an incompressible second grade MHD fluid at rest is situated in the annular region between two infinite coaxial circular cylinders of radii R_1 and $R_2 (> R_1)$. At time $t = 0^+$ the inner cylinder begins to rotate around its axis and to slide along the same axis due to the time-dependent shear stresses

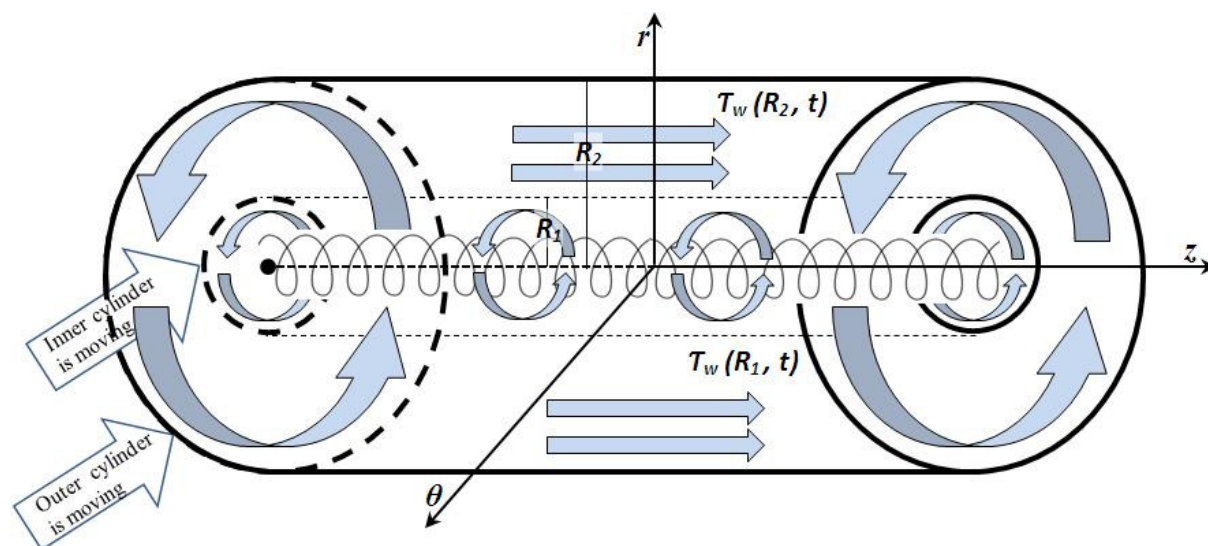


Figure 1: Geometry of the problem for rotational and translational flows of fluid through an annulus

$$\tau_\omega(R_1, t) = f_1 t^p \quad \text{and} \quad \tau_v(R_1, t) = g_1 t^p, \quad \text{where } p \geq 0, \tag{1}$$

and the outer cylinder also starts to rotate around its axis and slide along same axis with rotational velocity $f_2 t^p$ and translated velocity $g_2 t^p$, where f_1, f_2, g_1 and g_2 are constants. Due to the shear, the fluid is gradually moved its velocity and extra stress tensor are considered to be

$$\mathbf{V} = \mathbf{V}(r, t) = w(r, t)\mathbf{e}_\theta + v(r, t)\mathbf{e}_z, \quad \mathbf{S} = \mathbf{S}(r, t), \tag{2}$$

where \mathbf{e}_θ and \mathbf{e}_z are unit vectors in the θ and z -directions of the cylindrical coordinate system r, θ and z . The governing equations are given by [11, 19]

$$\frac{\partial}{\partial t} \omega(r, t) = \left(\vartheta + \alpha \frac{\partial}{\partial t} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \omega(r, t) - M \omega(r, t) - \Phi \left(\vartheta + \alpha \frac{\partial}{\partial t} \right) \omega(r, t); \tag{3}$$

where $r \in [R_1, R_2], \quad t > 0,$

$$\frac{\partial}{\partial t} v(r, t) = \left(\vartheta + \alpha \frac{\partial}{\partial t} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v(r, t) - M v(r, t) - \Phi \left(\vartheta + \alpha \frac{\partial}{\partial t} \right) v(r, t); \quad (4)$$

where $r \in [R_1, R_2], \quad t > 0,$

$$\tau_\omega = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \omega(r, t), \quad (5)$$

$$\tau_v = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \frac{\partial \bar{v}(r, q)}{\partial r}, \quad (6)$$

where $\tau_\omega = s_{r\theta}$ and $\tau_v = s_{rz}$ are the shear stresses that are different of zero and $\nu = \mu/\rho$ is the kinematic viscosity, μ is the dynamic viscosity, ρ is the density of the fluid and $\Phi = \frac{\phi}{k}$ and $M = \frac{\sigma B_0^2}{\rho}$ are magnetic and porosity constants, where ϕ is the porosity and κ is the permeability of the porous medium, B_0 is the magnitude of applied magnetic field and σ is the electrically conductivity of fluid, while the appropriate initial and boundary conditions are

$$w(r, 0) = \frac{\partial w(r, 0)}{\partial t} = v(r, 0) = \frac{\partial v(r, 0)}{\partial t} = 0 \quad \text{and} \quad \tau_1(r, 0) = \tau_2(r, 0) = 0; \quad r \in [R_1, R_2], \quad (7)$$

respectively,

$$\tau_\omega(R_1, t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \omega(r, t) \Big|_{r=R_1} = f_1 t^p, \quad t > 0, \quad (8)$$

and

$$\tau_v(R_1, t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) v(r, t) \Big|_{r=R_1} = g_1 t^p, \quad t > 0, \quad (9)$$

$$\omega(R_2, t) = f_2 t^p \quad \text{and} \quad v(R_2, t) = g_2 t^p; \quad t > 0. \quad (10)$$

In order to solve this problem we shall use Laplace Transforms and the finite Hankel transforms.

2.1. Estimation of the velocity field

Applying the Laplace transform to eqs. (3) and (4) and having in mind the initial conditions (7), since the image functions $\bar{w}(r, q)$ and $\bar{v}(r, q)$ be inverse Laplace transform, we find that as

$$q\bar{w}(r, q) = \vartheta \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \bar{w}(r, q) + \alpha \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) q\bar{w}(r, q) - M\bar{w}(r, q) - \Phi\vartheta - \Phi\alpha q\bar{w}(r, q); \quad r \in (R_1, R_2), \quad (11)$$

$$q\bar{v}(r, q) = \vartheta \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \bar{v}(r, q) + \alpha \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) q\bar{v}(r, q) - M\bar{v}(r, q) - \Phi\vartheta - \Phi\alpha q\bar{v}(r, q); \quad r \in (R_1, R_2), \quad (12)$$

Similarly applying Laplace transform on eqs. (8), (9) and (10), we get

$$\left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{w}(r, t) \Big|_{r=R_1} = \frac{f_1 p!}{q^{p+1} (\mu + \alpha_1 q)}; \quad t > 0, \quad (13)$$

and

$$\frac{\partial}{\partial r} \bar{v}(r, q) \Big|_{r=R_1} = \frac{g_1 p!}{q^{p+1} (\mu + \alpha_1 q)}; \quad t > 0, \quad (14)$$

also

$$\bar{w}(R_2, t) = \frac{f_2 p!}{q^{p+1}} \quad \text{and} \quad \bar{v}(R_2, t) = \frac{g_2 p!}{q^{p+1}}. \quad (15)$$

The Hankel transformation with respect to r is defined as [2, 3]

$$w_H(r, s) = \int_{R_1}^{R_2} r \bar{w}(r, s) B_w(r, r_m) dr, \quad m = 1, 2, 3, \dots \quad (16)$$

$$v_H(r, s) = \int_{R_1}^{R_2} r \bar{v}(r, s) B_v(r, r_n) dr; \quad n = 1, 2, 3, \dots \quad (17)$$

where

$$B_w(r, r_m) = J_1(rr_m)Y_2(R_1r_m) - J_2(R_1r_m)Y_1(rr_m),$$

$$B_v(r, r_n) = J_0(rr_n)Y_1(R_1r_n) - J_1(R_1r_n)Y_0(rr_n),$$

also

$$\int_{R_1}^{R_2} r \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \bar{w}(r, s) B_w(r, r_m) dr = -r_m^2 w_{Hm} + \frac{2}{\pi r_m} \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{w}(r, s) \Big|_{r=R_1} + r_m R_2 \bar{w}(r, s) D_w(R_1, R_2, r_m) \tag{18}$$

$$\int_{R_1}^{R_2} r \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \bar{v}(r, t) B_v(r, r_n) dr = -r_n^2 v_{Hn} + \frac{2}{\pi r_n} \frac{\partial}{\partial r} \bar{v}(r, s) \Big|_{r=R_1} + r_n R_2 \bar{v}(r, s) D_v(R_1, R_2, r_n) \tag{19}$$

where

$$D_w(R_1, R_2, r_m) = J_2(R_2r_m)Y_2(R_1r_m) - J_2(R_1r_m)Y_2(R_2r_m),$$

$$D_v(R_1, R_2, r_n) = J_1(R_2r_n)Y_1(R_1r_n) - J_1(R_1r_n)Y_1(R_2r_n).$$

Multiplying Eqs. (11) and (12) by $rB_w(rr_m)$ and $rB_v(rr_n)$, respectively, integrating the results with respect to r from R_1 to R_2 and using the boundary conditions (13), (14) and (15) and the identities (18), (19), we find that

$$\bar{w}_H(r, q) = \frac{2f_1 p!}{\pi r_m q^{p+1} \rho (\vartheta + \alpha q)} \frac{\vartheta + \alpha q}{q + M + \Phi(\vartheta + \alpha q) + r_m^2 (\vartheta + \alpha q)} + f_2 R_2 r_m p! D_w(R_1, R_2, r_m) \frac{1}{q^{p+1}} \frac{\vartheta + \alpha q}{q + M + \Phi(\vartheta + \alpha q) + r_m^2 (\vartheta + \alpha q)}, \tag{20}$$

$$\bar{v}_H(r, q) = \frac{2g_1 p!}{\pi r_n q^{p+1} \rho (\vartheta + \alpha q)} \frac{\vartheta + \alpha q}{q + M + \Phi(\vartheta + \alpha q) + r_n^2 (\vartheta + \alpha q)}$$

$$+g_2 R_2 r_n p! D_v(R_1, R_2, r_n) \frac{1}{q^{p+1}} \frac{\vartheta + \alpha q}{q + M + \Phi(\vartheta + \alpha q) + r_n^2(\vartheta + \alpha q)}. \tag{21}$$

After modifying the aforesaid outcomes (20) and (21) in more appropriate interchangeable pattern, we get

$$\begin{aligned} \bar{\omega}_H(r, q) &= \frac{2f_1}{\pi \mu r_m^3} \frac{p!}{q^{p+1}} - \frac{2f_1}{\pi \mu r_m^3} \frac{p!}{q^p} \frac{[1 + \alpha(\Phi + r_m^2)] + (M + \Phi\vartheta)q^{-1}}{[1 + \alpha(\Phi + r_m^2)] + [M + \vartheta(\Phi + r_m^2)]} \\ &+ f_2 R_2 r_m D_\omega(R_1, R_2, r_m) \left[\frac{p!}{q^{p+1}} + \frac{p!}{q^{p+1}} \frac{[M + \vartheta(\Phi + r_m^2 - 1)] + [1 + \alpha(\Phi + r_m^2 - 1)]q}{[1 + \alpha(\Phi + r_m^2)]q + [M + \vartheta(\Phi + r_m^2)]} \right], \end{aligned} \tag{22}$$

$$\begin{aligned} \bar{v}_H(r, q) &= \frac{2g_1}{\pi \mu r_n^3} \frac{p!}{q^{p+1}} - \frac{2g_1}{\pi \mu r_n^3} \frac{p!}{q^p} \frac{[1 + \alpha(\Phi + r_n^2)] + (M + \Phi\vartheta)q^{-1}}{[1 + \alpha(\Phi + r_n^2)] + [M + \vartheta(\Phi + r_n^2)]} \\ &+ g_2 R_2 r_n D_v(R_1, R_2, r_n) \left[\frac{p!}{q^{p+1}} + \frac{p!}{q^{p+1}} \frac{[M + \vartheta(\Phi + r_n^2 - 1)] + [1 + \alpha(\Phi + r_n^2 - 1)]q}{[1 + \alpha(\Phi + r_n^2)]q + [M + \vartheta(\Phi + r_n^2)]} \right]. \end{aligned} \tag{23}$$

Additionally, we have the relation of discrete inverse Hankel transform, and apply the discrete inverse Hankel transform formulae to (22) and (23), then we get the equations

$$\begin{aligned} \bar{\omega}(r, q) &= \frac{f_1}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) \frac{p!}{q^{p+1}} - \frac{f_1 \pi}{\mu} \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) B_w(r r_m)}{r_m [J_2^2(R_1 r_m) - J_1^2(R_2 r_m)]} \frac{1}{1 + \alpha(\Phi + r_m^2)} \\ &\times \left[\frac{p!}{q^p} \frac{1 + \alpha(\Phi + r_m^2)}{q + \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)}} + \frac{p!}{q^{p+1}} \frac{M + \Phi \vartheta}{q + \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)}} \right] + \frac{f_2 R_2 \pi^2}{2} \sum_{m=1}^{\infty} \frac{r_m^3 J_1^2(R_2 r_m) B_w(r r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \frac{D_\omega(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \\ &\times \left[\frac{p!}{q^{p+1}} - \frac{1}{1 + \alpha(\Phi + r_m^2)} \left\{ \frac{p!}{q^{p+1}} \frac{M + \vartheta(\Phi + r_m^2 - 1)}{q + \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)}} + \frac{p!}{q^p} \frac{1 + \alpha(\Phi + r_m^2 - 1)}{q + \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)}} \right\} \right], \end{aligned} \tag{24}$$

$$\bar{v}(r, t) = \frac{g_1}{2\mu} R_1 \ln \left(\frac{r}{R_2} \right) \frac{p!}{q^{p+1}} - \frac{g_1 \pi}{\mu} \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) B_v(r r_n)}{r_n [J_1^2(R_1 r_n) - J_0^2(R_2 r_n)]} \frac{1}{1 + \alpha(\Phi + r_n^2)}$$

$$\begin{aligned} & \times \left[\frac{p!}{q^p} \frac{1 + \alpha(\Phi + r_n^2)}{q + \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)}} + \frac{p!}{q^{p+1}} \frac{M + \Phi \vartheta}{q + \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)}} \right] + \frac{g_2 R_2 \pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^3 J_0^2(R_2 r_n) B_v(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} D_v(R_1, R_2, r_n) \\ & \times \left[\frac{p!}{q^{p+1}} - \frac{1}{1 + \alpha(\Phi + r_n^2)} \left\{ \frac{p!}{q^{p+1}} \frac{M + \vartheta(\Phi + r_n^2 - 1)}{q + \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)}} + \frac{p!}{q^p} \frac{1 + \alpha(\Phi + r_n^2 - 1)}{q + \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)}} \right\} \right]. \end{aligned} \tag{25}$$

In order to get the result of velocity field for rotational and longitude cases as $\omega(r, t) = \mathcal{L}[\bar{\omega}(r, q)]$ and $v(r, t) = \mathcal{L}[\bar{v}(r, q)]$ respectively, applying the convolution theorem of inverse Laplace transformation on eqs. (24) and (25), we get the velocity field for rotational case

$$\begin{aligned} \omega(r, t) &= \frac{f_1}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2}{r} \right) t^p - \frac{f_1 \pi}{\mu} \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) B_w(r r_m)}{r_m [J_2^2(R_1 r_m) - J_1^2(R_2 r_m)]} \frac{1}{1 + \alpha(\Phi + r_m^2)} \\ & \times \int_0^t (t - u)^p \left[\frac{p(1 + \alpha(\Phi + r_m^2))}{t - u} + M + \Phi \vartheta \right] \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} u \right] du \\ & + \frac{f_2 R_2 \pi^2}{2} \sum_{m=1}^{\infty} \frac{r_m^3 J_1^2(R_2 r_m) B_w(r r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} D_\omega(R_1, R_2, r_m) \left\{ t^p - \frac{1}{1 + \alpha(\Phi + r_m^2)} \int_0^t (t - u)^p \right. \\ & \times \left. \left[[M + \vartheta(\Phi + r_m^2 - 1)] + \frac{p[1 + \alpha(\Phi + r_m^2 - 1)]}{t - u} \right] \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} u \right] du \right\}, \end{aligned} \tag{26}$$

and for longitude case, we have

$$\begin{aligned} v(r, q) &= \frac{g_1}{2\mu} R_1 \ln \left(\frac{r}{R_2} \right) t^p - \frac{g_1 \pi}{\mu} \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) B_v(r r_n)}{r_n [J_1^2(R_1 r_n) - J_0^2(R_2 r_n)]} \frac{1}{1 + \alpha(\Phi + r_n^2)} \\ & \times \int_0^t (t - u)^p \left[\frac{p(1 + \alpha(\Phi + r_n^2))}{t - u} + M + \Phi \vartheta \right] \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} u \right] du \\ & + \frac{g_2 R_2 \pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^3 J_0^2(R_2 r_n) B_v(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} D_v(R_1, R_2, r_n) \left\{ t^p - \frac{1}{1 + \alpha(\Phi + r_n^2)} \int_0^t (t - u)^p \right. \\ & \times \left. \left[[M + \vartheta(\Phi + r_n^2 - 1)] + \frac{p[1 + \alpha(\Phi + r_n^2 - 1)]}{t - u} \right] \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} u \right] du \right\}. \end{aligned} \tag{27}$$

2.2. Estimation of the shear stresses

In order to determine the shear stresses $\tau_\omega(r, t)$ and $\tau_v(r, t)$, with the initial conditions, Taking eqs. (5) and (6) and applying the Laplace transformation, we get

$$\bar{\tau}_\omega(r, q) = (\mu + \alpha_1 q) \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{\omega}(r, q), \tag{28}$$

and

$$\bar{\tau}_v(r, q) = (\mu + \alpha_1 q) \frac{\partial \bar{v}(r, q)}{\partial r}. \tag{29}$$

Now taking eqs. (20) and (21), after modifying the outcomes in more appropriate interchangeable pattern for satisfying the initial and boundary conditions and applying the inverse Hankel transform, then we get the results of

$$\begin{aligned} \left(\frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{\omega}(r, q) &= \frac{f_1}{\rho} \left(\frac{R_1}{r} \right)^2 \frac{p!}{q^{p+1}(\vartheta + \alpha q)} + \frac{f_1 \pi}{\rho} \\ &\times \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) \bar{B}_\omega(r r_m)}{r_m [J_2^2(R_1 r_m) - J_1^2(R_2 r_m)]} \frac{1}{\vartheta + \alpha q} \frac{p!}{q^p} \left\{ \frac{1 + [M + \Phi(\vartheta + \alpha q)] q^{-1}}{[1 + \alpha(\Phi + r_m^2)] q + [M + \vartheta(\Phi + r_m^2)]} \right\} - \frac{f_2 R_2 \pi^2}{2} \\ &\times \sum_{m=1}^{\infty} \frac{r_m^3 J_1^2(R_2 r_m) \bar{B}_\omega(r r_m) D_\omega(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \left\{ \frac{(\vartheta + \alpha q)}{[1 + \alpha(\Phi + r_m^2)] q + [M + \vartheta(\Phi + r_m^2)]} \right\}, \tag{30} \end{aligned}$$

$$\begin{aligned} \frac{\partial \bar{v}(r, q)}{\partial r} &= \frac{g_1}{\rho} R_1 \ln \left(\frac{r}{R_2} \right) \frac{p!}{q^{p+1}(\vartheta + \alpha q)} + \frac{g_1 \pi}{\rho} \\ &\times \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) \bar{B}_v(r r_n)}{r_n [J_1^2(R_1 r_n) - J_0^2(R_2 r_n)]} \frac{1}{\vartheta + \alpha q} \frac{p!}{q^p} \left\{ \frac{1 + [M + \Phi(\vartheta + \alpha q)] q^{-1}}{[1 + \alpha(\Phi + r_n^2)] q + [M + \vartheta(\Phi + r_n^2)]} \right\} - \frac{g_2 R_2 \pi^2}{2} \\ &\times \sum_{n=1}^{\infty} \frac{r_n^3 J_0^2(R_2 r_n) \bar{B}_v(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \left\{ \frac{(\vartheta + \alpha q)}{[1 + \alpha(\Phi + r_n^2)] q + [M + \vartheta(\Phi + r_n^2)]} \right\}. \tag{31} \end{aligned}$$

Put (30) and (31) in the eqs. (28) and (29) respectively, then the sample equivalent form, which are

$$\begin{aligned} \bar{\tau}_\omega(r, q) &= f_1 \left(\frac{R_1}{r} \right)^2 \frac{p!}{q^{p+1}} + f_1 \pi \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) \bar{B}_\omega(r r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \frac{1}{1 + \alpha(\Phi + r_m^2)} \\ &\times \left\{ \frac{(p-1)!}{q^p} \frac{p}{q + \frac{M+\vartheta(\Phi+r_m^2)}{1+\alpha(\Phi+r_m^2)}} + \frac{p!}{q^{p+1}} \frac{M + \Phi \vartheta}{q + \frac{M+\vartheta(\Phi+r_m^2)}{1+\alpha(\Phi+r_m^2)}} + \frac{(p-1)!}{q^p} \frac{\Phi \alpha p}{q + \frac{M+\vartheta(\Phi+r_m^2)}{1+\alpha(\Phi+r_m^2)}} \right\} \\ &+ \frac{f_2 R_2 \pi^2}{2} \sum_{m=1}^{\infty} \frac{r_m^4 J_1^2(R_2 r_m) \bar{B}_\omega(r r_m) D_\omega(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \left\{ \frac{p!}{q^{p+1}} \frac{\vartheta^2}{q + \frac{M+\vartheta(\Phi+r_m^2)}{1+\alpha(\Phi+r_m^2)}} \right. \\ &\left. + \frac{(p-1)!}{q^p} \frac{2\alpha \vartheta p}{q + \frac{M+\vartheta(\Phi+r_m^2)}{1+\alpha(\Phi+r_m^2)}} + \frac{(p-2)!}{q^{p-1}} \frac{\alpha^2 p (p-1)}{q + \frac{M+\vartheta(\Phi+r_m^2)}{1+\alpha(\Phi+r_m^2)}} \right\}, \end{aligned} \tag{32}$$

$$\begin{aligned} \bar{\tau}_v(r, q) &= g_1 \left(\frac{R_1}{r} \right) \frac{p!}{q^{p+1}} + g_1 \pi \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) \bar{B}_v(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \frac{1}{1 + \alpha(\Phi + r_n^2)} \\ &\times \left\{ \frac{(p-1)!}{q^p} \frac{p}{q + \frac{M+\vartheta(\Phi+r_n^2)}{1+\alpha(\Phi+r_n^2)}} + \frac{p!}{q^{p+1}} \frac{M + \Phi \vartheta}{q + \frac{M+\vartheta(\Phi+r_n^2)}{1+\alpha(\Phi+r_n^2)}} + \frac{(p-1)!}{q^p} \frac{\Phi \alpha p}{q + \frac{M+\vartheta(\Phi+r_n^2)}{1+\alpha(\Phi+r_n^2)}} \right\} \\ &+ \frac{g_2 R_2 \pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^4 J_0^2(R_2 r_n) \bar{B}_v(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \left\{ \frac{p!}{q^{p+1}} \frac{\vartheta^2}{q + \frac{M+\vartheta(\Phi+r_n^2)}{1+\alpha(\Phi+r_n^2)}} \right. \\ &\left. + \frac{(p-1)!}{q^p} \frac{2\alpha \vartheta p}{q + \frac{M+\vartheta(\Phi+r_n^2)}{1+\alpha(\Phi+r_n^2)}} + \frac{(p-2)!}{q^{p-1}} \frac{\alpha^2 p (p-1)}{q + \frac{M+\vartheta(\Phi+r_n^2)}{1+\alpha(\Phi+r_n^2)}} \right\}. \end{aligned} \tag{33}$$

After applying the convolution theorem of inverse Laplace transformation on eqs. (32) and (33), we get the shear stress for rotational case

$$\tau_\omega(r, t) = f_1 \left(\frac{R_1}{r} \right) t^p + f_1 \pi \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) \bar{B}_\omega(r r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \frac{1}{1 + \alpha(\Phi + r_m^2)} \int_0^t (t - u)^p$$

$$\begin{aligned}
& \times \left[\frac{p(1 + \alpha \Phi)}{t - u} + M + \Phi \vartheta \right] \mathbf{Exp} \left[- \frac{M + \vartheta (\Phi + r_m^2)}{1 + \alpha (\Phi + r_m^2)} u \right] du - f_2 \frac{\pi^2 \rho R_2}{2} \\
& \times \sum_{m=1}^{\infty} \frac{r_m^4 J_1^2(R_2 r_m) \bar{B}_\omega(r r_m) D_w(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \frac{1}{1 + \alpha (\Phi + r_m^2)} \int_0^t (t - u)^p \\
& \times \left[\vartheta^2 + \frac{2\alpha \vartheta P}{t - u} + \frac{\alpha^2 p(p - 1)}{(t - u)^2} \right] \mathbf{Exp} \left[- \frac{M + \vartheta (\Phi + r_m^2)}{1 + \alpha (\Phi + r_m^2)} u \right] du, \tag{34}
\end{aligned}$$

and for longitudinal case

$$\begin{aligned}
\tau_v(r, t) &= g_1 \left(\frac{R_1}{r} \right) t^p + g_1 \pi \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) \bar{B}_v(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \frac{1}{1 + \alpha (\Phi + r_n^2)} \int_0^t (t - u)^p \\
& \times \left[\frac{p(1 + \alpha \Phi)}{t - u} + M + \Phi \vartheta \right] \mathbf{Exp} \left[- \frac{M + \vartheta (\Phi + r_n^2)}{1 + \alpha (\Phi + r_n^2)} u \right] du - g_2 \frac{\pi^2 \rho R_2}{2} \\
& \times \sum_{n=1}^{\infty} \frac{r_n^4 J_0^2(R_2 r_n) \bar{B}_\omega(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \frac{1}{1 + \alpha (\Phi + r_n^2)} \int_0^t (t - u)^p \\
& \times \left[\vartheta^2 + \frac{2\alpha \vartheta P}{t - u} + \frac{\alpha^2 p(p - 1)}{(t - u)^2} \right] \mathbf{Exp} \left[- \frac{M + \vartheta (\Phi + r_n^2)}{1 + \alpha (\Phi + r_n^2)} u \right] du. \tag{35}
\end{aligned}$$

3. The limiting and special cases

3.1. Second grade with porous (without MHD effect)

Making $\mathbf{M} \rightarrow 0$ into (26), (27), (34) and (35), we obtain the velocity field components and shear stresses for Second grade fluid with porous effect and without MHD.

3.2. MHD Second grade (without porous effect)

Putting $\Phi \rightarrow 0$ into resultant eqs. (26), (27), (34) and (35), we obtain the velocity field components and the shear stresses for MHD Second grade without porous effects.

3.3. Second grade fluid (without MHD and porous effects)

Applying $\mathbf{M} \rightarrow 0$ & $\Phi \rightarrow 0$ into (26), (27), (34) and (35), we get the solutions for velocity fields components respectively the shear stresses for Second grade fluid without MHD and porous effects.

3.4. MHD Newtonian with porous effect

Employing $\alpha \rightarrow 0$ into eqs. (26), (27), (34) and (35), we get

$$\begin{aligned} \omega(r, t) = & \frac{f_1}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) t^p - \frac{f_1 \pi}{\mu} \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) B_w(r r_m)}{r_m [J_2^2(R_1 r_m) - J_1^2(R_2 r_m)]} \int_0^t (t-u)^p \left[\frac{p}{t-u} + M + \Phi \vartheta \right] \\ & \times \mathbf{Exp} \left[- \{ M + \vartheta (\Phi + r_m^2) \} u \right] du + \frac{f_2 R_2 \pi^2}{2} \sum_{m=1}^{\infty} \frac{r_m^3 J_1^2(R_2 r_m) B_w(r r_m) D_\omega(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \left\{ t^p \right. \\ & \left. - \int_0^t (t-u)^p \left[M + \vartheta (\Phi + r_m^2 - 1) + \frac{p}{t-u} \right] \mathbf{Exp} \left[- \{ M + \vartheta (\Phi + r_m^2) \} u \right] du \right\}, \quad (36) \end{aligned}$$

$$\begin{aligned} v(r, q) = & \frac{g_1}{2\mu} R_1 \ln \left(\frac{r}{R_2} \right) t^p - \frac{g_1 \pi}{\mu} \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) B_v(r r_n)}{r_n [J_1^2(R_1 r_n) - J_0^2(R_2 r_n)]} \int_0^t (t-u)^p \left[\frac{p}{t-u} + M + \Phi \vartheta \right] \\ & \times \mathbf{Exp} \left[- \{ M + \vartheta (\Phi + r_n^2) \} u \right] du + \frac{g_2 R_2 \pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^3 J_0^2(R_2 r_n) B_v(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \left\{ t^p \right. \\ & \left. - \int_0^t (t-u)^p \left[M + \vartheta (\Phi + r_n^2 - 1) + \frac{p}{t-u} \right] \mathbf{Exp} \left[- \{ M + \vartheta (\Phi + r_n^2) \} u \right] du \right\}, \quad (37) \end{aligned}$$

the solution for the velocity field components, and

$$\begin{aligned} \tau_\omega(R_1, t) = & f_1 \left(\frac{R_1}{r} \right) t^p + f_1 \pi \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) \bar{B}_\omega(r r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \int_0^t (t-u)^p \left[\frac{p}{t-u} + M + \Phi \vartheta \right] \\ & \times \mathbf{Exp} \left[- \{ M + \vartheta (\Phi + r_m^2) \} u \right] du - f_2 \frac{\pi^2 \rho R_2 \vartheta^2}{2} \sum_{m=1}^{\infty} \frac{r_m^4 J_1^2(R_2 r_m) \bar{B}_\omega(r r_m) D_\omega(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \end{aligned}$$

$$\times \int_0^t (t-u)^p \mathbf{Exp} \left[- \{M + \vartheta (\Phi + r_m^2)\} u \right] du, \quad (38)$$

$$\begin{aligned} \tau_v(R_1, t) = & g_1 \left(\frac{R_1}{r} \right) t^p + g_1 \pi \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) \bar{B}_v(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \int_0^t (t-u)^p \left[\frac{p}{t-u} + M + \Phi \vartheta \right] \\ & \times \mathbf{Exp} \left[- \{M + \vartheta (\Phi + r_n^2)\} u \right] du - g_2 \frac{\pi^2 \rho R_2 \vartheta^2}{2} \sum_{n=1}^{\infty} \frac{r_n^4 J_0^2(R_2 r_n) \bar{B}_v(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \\ & \times \int_0^t (t-u)^p \mathbf{Exp} \left[- \{M + \vartheta (\Phi + r_n^2)\} u \right] du, \end{aligned} \quad (39)$$

the shear stresses for MHD Newtonian with porous effect.

3.5. Newtonian with porous effect

Making $M \rightarrow 0$ in Eqs. (36), (37), (38) and (39), we get solutions for Newtonian fluid with porous effect and without MHD.

3.6. MHD Newtonian

Similarly, putting $M \rightarrow 0$ in Eqs. (36), (37), (38) and (39), we get solutions for MHD Newtonian fluid without porous effects.

3.7. Newtonian

By vanishing as $\mathbf{M} \rightarrow 0$ & $\Phi \rightarrow 0$ into the Eqs. (36), (37), (38) and (39), of velocity field components and shear stresses, we get the Newtonian solutions,

$$\begin{aligned} \omega(r, t) = & \frac{f_1}{2\mu} \left(\frac{R_1}{R_2} \right)^2 \left(r - \frac{R_2^2}{r} \right) t^p - \frac{f_1 \pi}{\mu} \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) B_w(r r_m)}{r_m [J_2^2(R_1 r_m) - J_1^2(R_2 r_m)]} \int_0^t (t-u)^{p-1} \\ & \times \mathbf{Exp}(-\vartheta r_m^2 u) du + \frac{f_2 R_2 \pi^2}{2} \sum_{m=1}^{\infty} \frac{r_m^3 J_1^2(R_2 r_m) B_w(r r_m) D_\omega(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \left\{ t^p \right. \\ & \left. - \int_0^t (t-u)^p \left[\vartheta (r_m^2 - 1) + \frac{p}{t-u} \right] \mathbf{Exp}(-\vartheta r_m^2 u) du \right\}, \end{aligned} \quad (40)$$

$$\begin{aligned}
 v(r, q) = & \frac{g_1}{2\mu} R_1 \ln\left(\frac{r}{R_2}\right) t^p - \frac{g_1 p}{\mu} \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) B_v(r r_n)}{r_n [J_1^2(R_1 r_n) - J_0^2(R_2 r_n)]} \int_0^t (t-u)^{p-1} \\
 & \times \mathbf{Exp}(-\vartheta r_n^2 u) du + \frac{g_2 R_2 \pi}{2} \sum_{n=1}^{\infty} \frac{r_n^3 J_0^2(R_2 r_n) B_v(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \left\{ t^p \right. \\
 & \left. - \int_0^t (t-u)^p \left[\vartheta (r_n^2 - 1) + \frac{p}{t-u} \right] \mathbf{Exp}(-\vartheta r_n^2 u) du \right\}, \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 \tau_{\omega}(R_1, t) = & f_1 \left(\frac{R_1}{r}\right) t^p + f_1 \pi \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) \bar{B}_{\omega}(r r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \int_0^t (t-u)^p \left(\frac{p}{t-u}\right) \\
 & \times \mathbf{Exp}(-\vartheta r_m^2 u) du - f_2 \frac{\pi^2 \rho R_2 \vartheta^2}{2} \sum_{m=1}^{\infty} \frac{r_m^4 J_1^2(R_2 r_m) \bar{B}_{\omega}(r r_m) D_{\omega}(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \\
 & \times \int_0^t (t-u)^p \mathbf{Exp}(-\vartheta r_m^2 u) du, \tag{42}
 \end{aligned}$$

$$\begin{aligned}
 \tau_v(R_1, t) = & g_1 \left(\frac{R_1}{r}\right) t^p + g_1 \pi \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) \bar{B}_v(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \int_0^t (t-u)^p \left(\frac{p}{t-u}\right) \\
 & \times \mathbf{Exp}(-\vartheta r_n^2 u) du - g_2 \frac{\pi^2 \rho R_2 \vartheta^2}{2} \sum_{n=1}^{\infty} \frac{r_n^4 J_0^2(R_2 r_n) \bar{B}_v(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \\
 & \times \int_0^t (t-u)^p \mathbf{Exp}(-\vartheta r_n^2 u) du. \tag{43}
 \end{aligned}$$

3.8. The special solution for $p = 1$

Finally, switch $p = 1$ into the resultant equations of velocity field components as well as shear stress (26), (27), (34) and (35), we get longitudinal and rotational cases result by solving the integral from 0 to t

$$\begin{aligned} \omega(r, t) = & \frac{f_1}{2\mu} \left(\frac{R_1}{R_2}\right)^2 \left(r - \frac{R_2}{r}\right) t - \frac{f_1[M + \vartheta(\Phi + r_m^2)]}{\mu[1 + \alpha(\Phi + r_m^2)]^3} \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) B_w(r r_m)}{r_m [J_2^2(R_1 r_m) - J_1^2(R_2 r_m)]} \\ & \times \left[\vartheta r_m^2 [1 + \alpha(\Phi + r_m^2)] \left(1 - \mathbf{Exp} \left[-\frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} \right] \right) + t[M(M + \vartheta(2\Phi + r_m^2)) + \Phi \vartheta^2(\Phi + r_m^2)] \right] \\ & + \frac{f_2 R_2 \pi}{2[1 + \alpha(\Phi + r_m^2)]} \sum_{m=1}^{\infty} \frac{r_m^3 J_1^2(R_2 r_m) B_w(r r_m) D_\omega(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \left\{ t - \frac{1}{[M + \vartheta(\Phi + r_m^2)]^2} \left[\vartheta \right. \right. \\ & \left. \left. - \alpha[M + \vartheta(\Phi + r_m^2 - 1)] - t\vartheta[M + \vartheta(\Phi + r_m^2)(\Phi + r_m^2 - 1)] \right] \left(1 - \mathbf{Exp} \left[-\frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} \right] \right) \right. \\ & \left. + t \left[M^2 + \vartheta^2(\Phi + r_m^2)(\Phi + r_m^2 - 1) \mathbf{Exp} \left[-\frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} \right] \right] \right\}, \quad (44) \end{aligned}$$

$$\begin{aligned} v(r, q) = & \frac{g_1}{2\mu} R_1 \ln \left(\frac{r}{R_2}\right) t - \frac{g_1[M + \vartheta(\Phi + r_n^2)]}{\mu[1 + \alpha(\Phi + r_n^2)]^3} \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) B_v(r r_n)}{r_n [J_1^2(R_1 r_n) - J_0^2(R_2 r_n)]} \\ & \times \left[\vartheta r_n^2 [1 + \alpha(\Phi + r_n^2)] \left(1 - \mathbf{Exp} \left[-\frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} \right] \right) + t[M(M + \vartheta(2\Phi + r_n^2)) + \Phi \vartheta^2(\Phi + r_n^2)] \right] \\ & + \frac{g_2 R_2 \pi}{2[1 + \alpha(\Phi + r_n^2)]} \sum_{n=1}^{\infty} \frac{r_n^3 J_1^2(R_2 r_n) B_v(r r_n) D_v(R_1, R_2, r_n)}{J_2^2(R_1 r_n) - J_1^2(R_2 r_n)} \left\{ t - \frac{1}{[M + \vartheta(\Phi + r_n^2)]^2} \left[\vartheta \right. \right. \\ & \left. \left. - \alpha[M + \vartheta(\Phi + r_n^2 - 1)] - t\vartheta[M + \vartheta(\Phi + r_n^2)(\Phi + r_n^2 - 1)] \right] \left(1 - \mathbf{Exp} \left[-\frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} \right] \right) \right. \\ & \left. + t \left[M^2 + \vartheta^2(\Phi + r_n^2)(\Phi + r_n^2 - 1) \mathbf{Exp} \left[-\frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} \right] \right] \right\}, \quad (45) \end{aligned}$$

$$\begin{aligned}
 \tau_{\omega}(R_1, t) &= f_1 \left(\frac{R_1}{r} \right) t + f_1 \sum_{m=1}^{\infty} \frac{J_1^2(R_2 r_m) \bar{B}_{\omega}(r r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \frac{1}{1 + \alpha(\Phi + r_m^2)} \left[[1 + \alpha \Phi] \frac{1 + \alpha(\Phi + r_m^2)}{M + \vartheta(\Phi + r_m^2)} \right. \\
 &\times \left\{ 1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} t \right] \right\} + [M + \Phi \vartheta] \frac{1 + \alpha(\Phi + r_m^2)}{[M + \vartheta(\Phi + r_m^2)]^2} \left\{ t[M + \vartheta(\Phi + r_m^2)] \right. \\
 &\quad \left. \left. - [1 + \alpha(\Phi + r_m^2)] \left[1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} t \right] \right] \right\} - f_2 \frac{\pi \rho R_2}{2[1 + \alpha(\Phi + r_m^2)]} \right. \\
 &\times \sum_{m=1}^{\infty} \frac{r_m^4 J_1^2(R_2 r_m) \bar{B}_{\omega}(r r_m) D_w(R_1, R_2, r_m)}{J_2^2(R_1 r_m) - J_1^2(R_2 r_m)} \left[\vartheta^2 \frac{1 + \alpha(\Phi + r_m^2)}{[M + \vartheta(\Phi + r_m^2)]^2} \left\{ t[M + \vartheta(\Phi + r_m^2)] \right. \right. \\
 &\quad \left. \left. - [1 + \alpha(\Phi + r_m^2)] \left[1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} t \right] \right] \right\} + 2\alpha\vartheta \frac{1 + \alpha(\Phi + r_m^2)}{M + \vartheta(\Phi + r_m^2)} \right. \\
 &\quad \left. \left. \times \left\{ 1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_m^2)}{1 + \alpha(\Phi + r_m^2)} t \right] \right\} \right], \tag{46}
 \end{aligned}$$

$$\begin{aligned}
 \tau_v(R_1, t) &= g_1 \left(\frac{R_1}{r} \right) t + g_1 \sum_{n=1}^{\infty} \frac{J_0^2(R_2 r_n) \bar{B}_v(r r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \frac{1}{1 + \alpha(\Phi + r_n^2)} \left[[1 + \alpha \Phi] \frac{1 + \alpha(\Phi + r_n^2)}{M + \vartheta(\Phi + r_n^2)} \right. \\
 &\times \left\{ 1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} t \right] \right\} + [M + \Phi \vartheta] \frac{1 + \alpha(\Phi + r_n^2)}{[M + \vartheta(\Phi + r_n^2)]^2} \left\{ t[M + \vartheta(\Phi + r_n^2)] \right. \\
 &\quad \left. \left. - [1 + \alpha(\Phi + r_n^2)] \left[1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} t \right] \right] \right\} - g_2 \frac{\pi \rho R_2}{2[1 + \alpha(\Phi + r_n^2)]} \right. \\
 &\times \sum_{n=1}^{\infty} \frac{r_n^4 J_0^2(R_2 r_n) \bar{B}_v(r r_n) D_v(R_1, R_2, r_n)}{J_1^2(R_1 r_n) - J_0^2(R_2 r_n)} \left[\vartheta^2 \frac{1 + \alpha(\Phi + r_n^2)}{[M + \vartheta(\Phi + r_n^2)]^2} \left\{ t[M + \vartheta(\Phi + r_n^2)] \right. \right. \\
 &\quad \left. \left. - [1 + \alpha(\Phi + r_n^2)] \left[1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} t \right] \right] \right\} + 2\alpha\vartheta \frac{1 + \alpha(\Phi + r_n^2)}{M + \vartheta(\Phi + r_n^2)} \right. \\
 &\quad \left. \left. \times \left\{ 1 - \mathbf{Exp} \left[- \frac{M + \vartheta(\Phi + r_n^2)}{1 + \alpha(\Phi + r_n^2)} t \right] \right\} \right]. \tag{47}
 \end{aligned}$$

4. Numerical results and discussion

In this part of the article, the obtained exact solutions are studied numerically in order to determine the effects of several involved parameter such as material parameter α , kinematic viscosity ν , magnetic parameter M , porosity parameter Φ , parameter p , time t and radial distance r .

Comparison between two different fluid are also presented for understanding the presented flow problem.

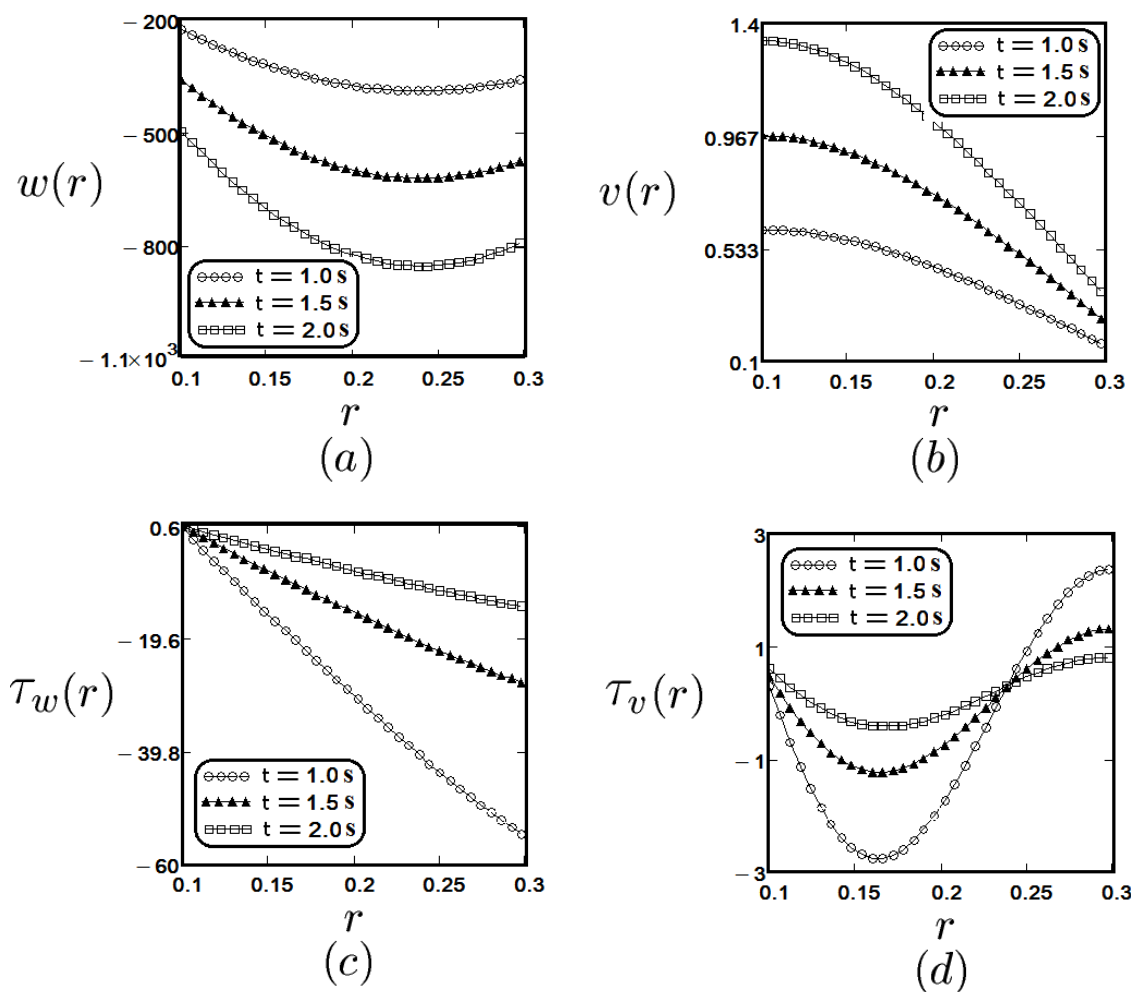


Figure 2: Profiles of the velocity field components for MHD Second grade fluid with porous effect $w(r, t)$, $u(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ given by eqs. (26), (27), (34) and (35), for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\nu = 0.6355$, $\mu = 12.26$, $\alpha = 0.2$, $M = 2$, $\Phi = 2$, $p = 1$ and different values of time t .

Figs. 2 exhibit the influence of time on the fluid motion. The velocity components $w(r, t)$, $v(r, t)$ and rotational shear stresses $\tau_w(r, t)$ are increasing function of time t . However the amplitude of the longitudinal shear stress $\tau_v(r, t)$ decreases with time.

These numerical values demonstrate the influence of above parameter on velocity components and shear stresses profiles for the flow generated through the inner cylinder that is subject to a rotational and longitudinal time dependent shear stresses and the outer cylinder subject to a rotational and longitudinal velocities along its axes.

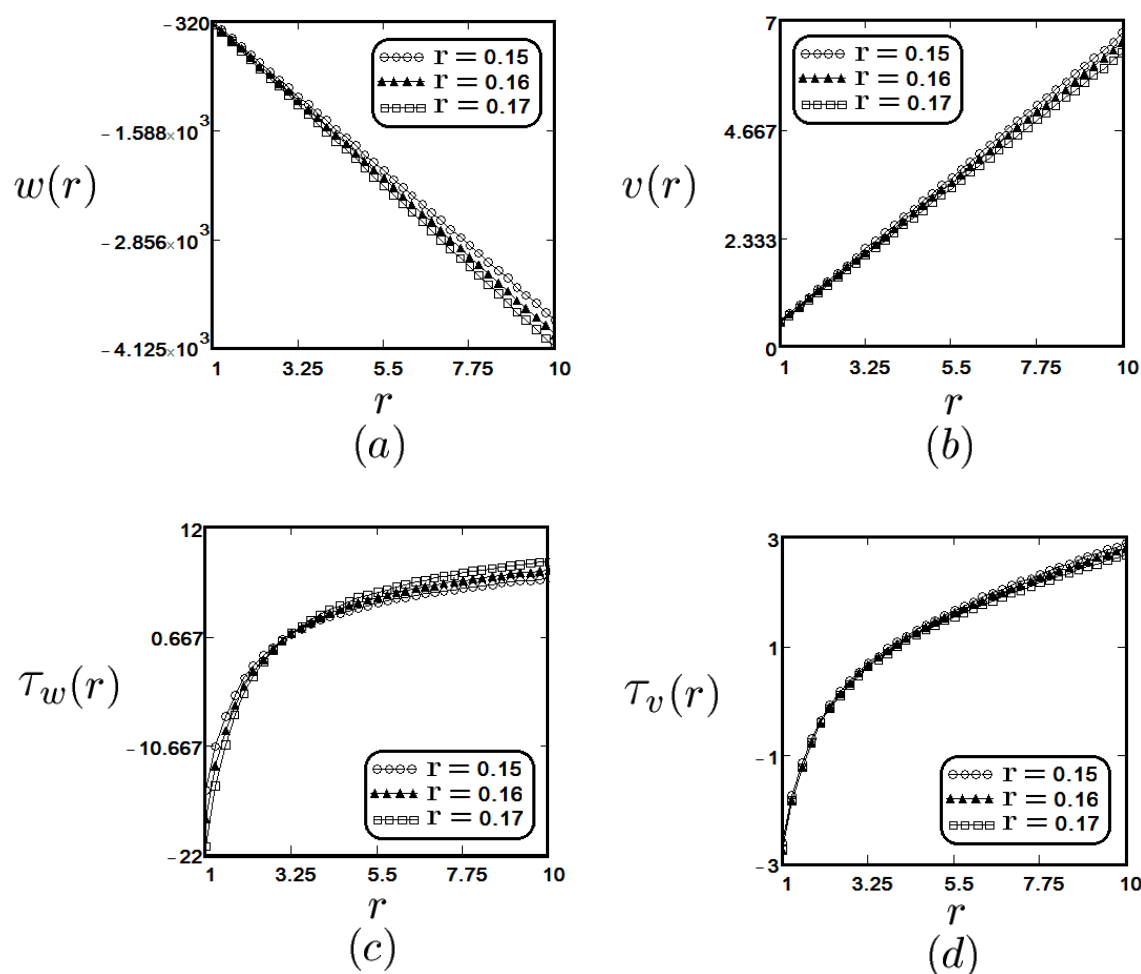


Figure 3: Profiles of the velocity field components for MHD Second grade fluid with porous effect $w(r, t)$, $u(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ given by eqs. (26), (27), (34) and (35), for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\nu = 0.6355$, $\mu = 12.26$, $\alpha = 0.2$, $M = 2$, $\Phi = 2$, $p = 1$ and different values of radius r .

The influence of the annular distance from R_1 to R_2 is shown in Figs. 3. It is found

that rotational pair $w(r, t)$, $\tau_w(r, t)$ are increases and longitudinal pair $v(r, t)$, $\tau_v(r, t)$ decreases along the annular distance.

These diagrams of the velocity field components $w(r, t)$, $v(r, t)$ and the shear stress $\tau_w(r, t)$, $\tau_v(r, t)$ were presented against r for different values of t and of the above mentioned parameter.

All these comments are demonstrated through absolute sense where in the diagrams negative values are appeared.

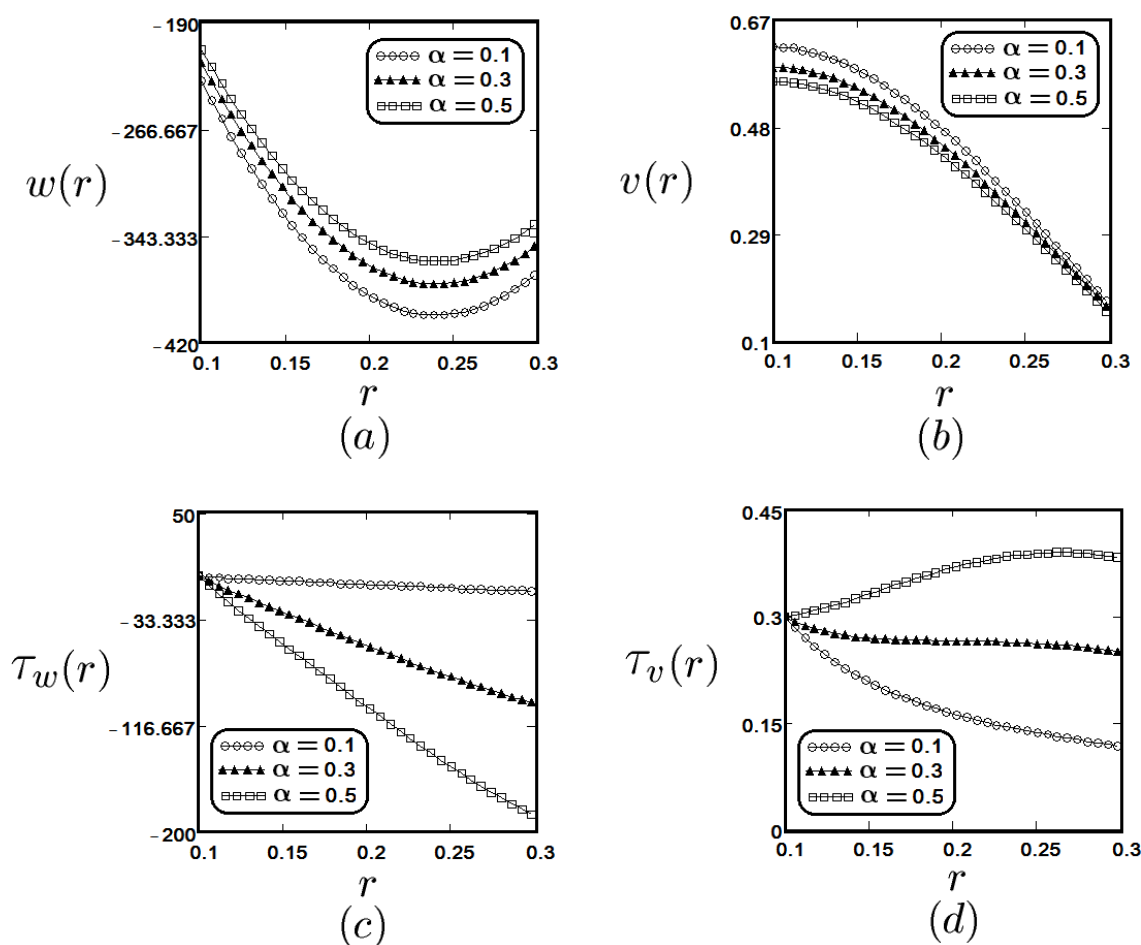


Figure 4: Profiles of the velocity field components for MHD Second grade fluid with porous effect $w(r, t)$, $u(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ given by eqs. (26), (27), (34) and (35), for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\nu = 0.6355$, $\mu = 12.26$, $M = 2$, $\Phi = 2$, $p = 1$, $t = 1s$ and different values of material parameter α .

Figs. 4 are plotted to show the effects of material parameter on velocities and shear

stresses profiles. It can be observed that both velocity components and rotational shear stress are decreasing function and longitudinal shear stress is decreasing function of the parameter α when other parameter are fixed.

Viscous effects are important for both Newtonian and non-Newtonian fluids. In Figs. 5, the velocities and shear stresses profiles are depicted for different values of the kinematic viscosity ν . Here the values of kinematic viscosity are chosen 0.3, 0.5 and 0.9.

It is observed that velocity components and shear stresses are decay with enhancing the viscous effects, which is natural phonomania.

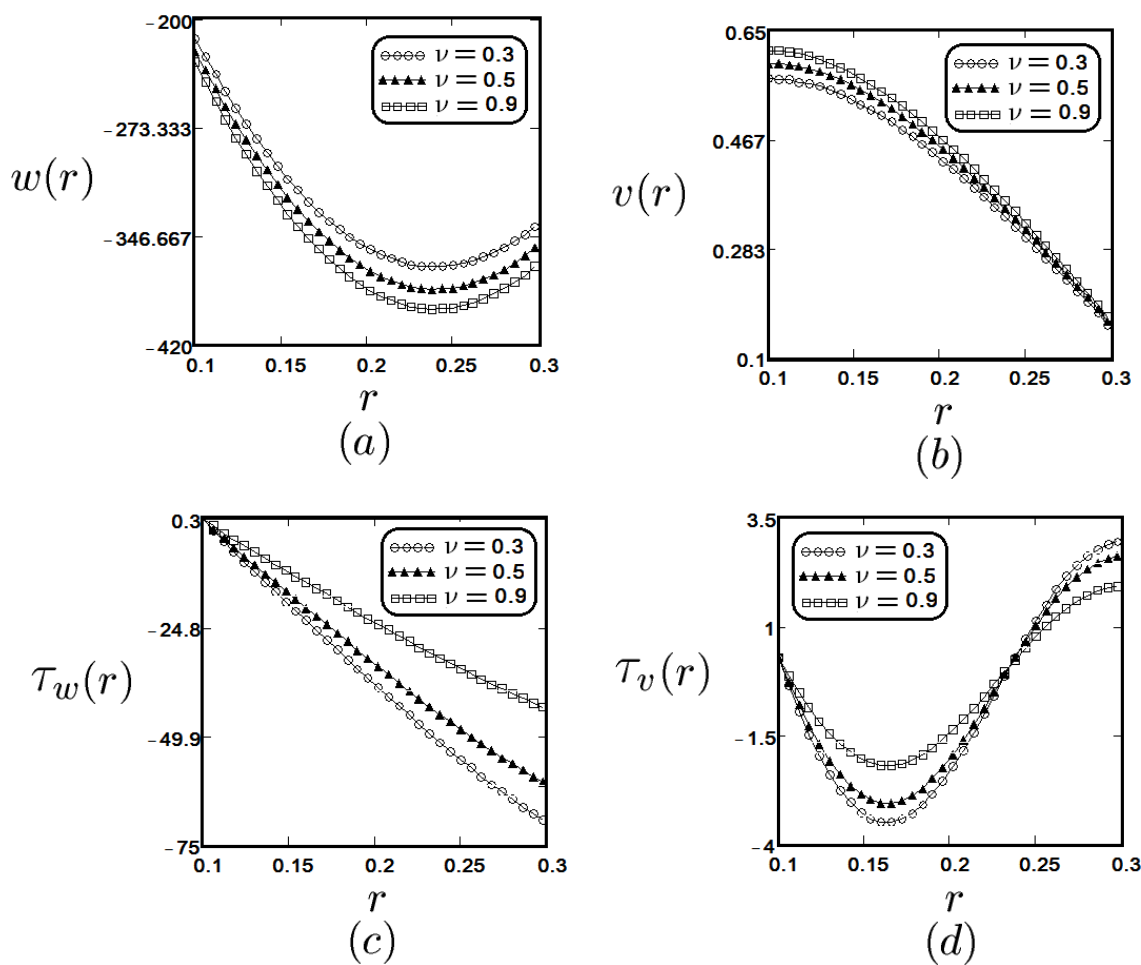


Figure 5: Profiles of the velocity field components for MHD Second grade fluid with porous effect $w(r, t)$, $u(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ given by eqs. (26), (27), (34) and (35), for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\rho = 19.292$, $\alpha = 0.2$, $M = 2$, $\Phi = 2$, $p = 1$, $t = 0.2s$ and different values of kinematic viscosity ν .

Figs. 6 present the nature of velocities and shear stresses profiles respectively for the variation of magnetic parameter M . The range of the magnetic parameter is taken from 0.1 to 12.

It come to the observation that the higher the value of magnetic parameter rise the velocities shear stresses values.

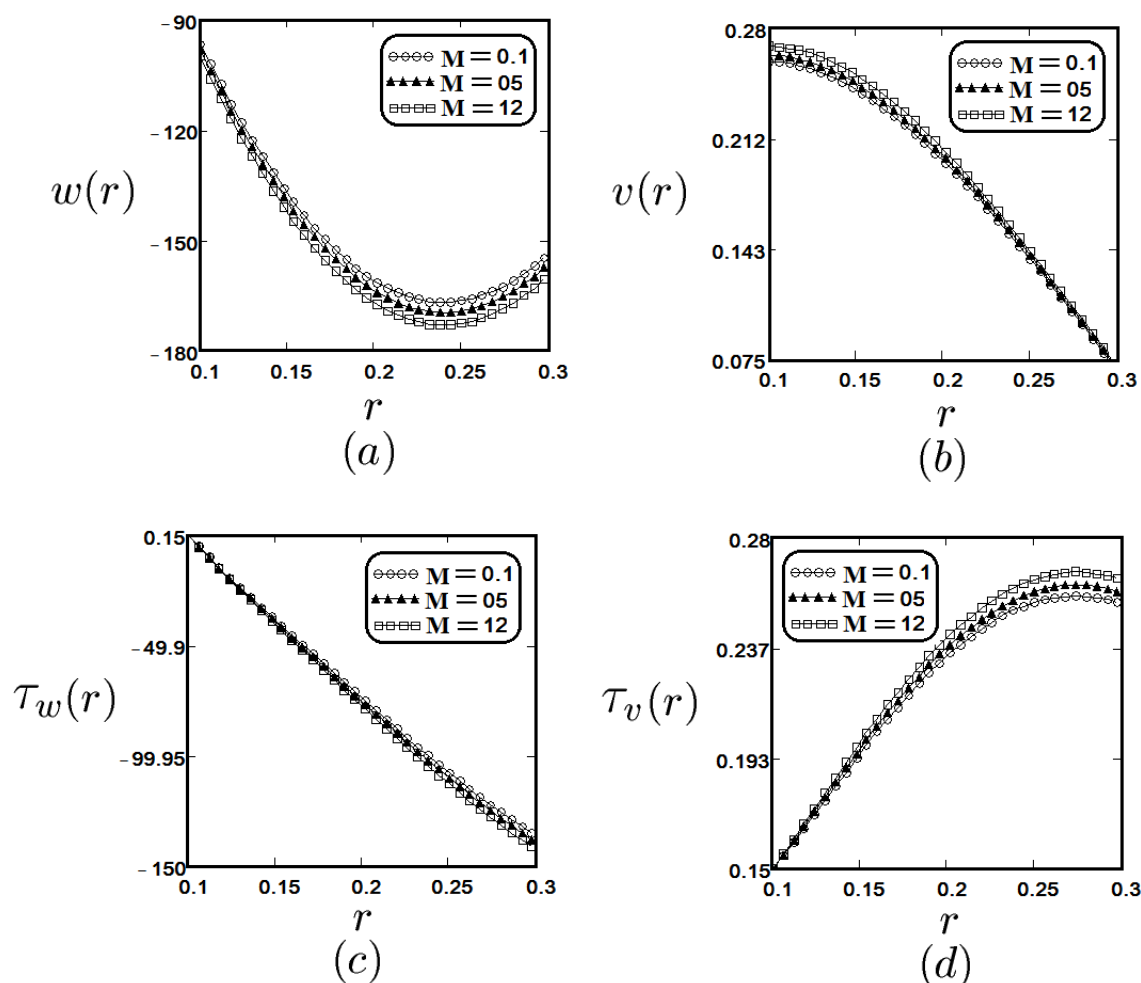


Figure 6: Profiles of the velocity field components for MHD Second grade fluid with porous effect $w(r, t)$, $u(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ given by eqs. (26), (27), (34) and (35), for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\nu = 0.6355$, $\mu = 12.26$, $\alpha = 0.2$, $\Phi = 2$, $p = 1$, $t = 0.5s$ and different values of magnetic parameter M .

Figs. 7 illustrates the effects of the porosity parameter Φ on fluid motion. These figures show the variations of velocities as well as shear stresses profiles for increasing values of porosity parameter Φ when other parameters are kept constant.

Figs. 7 (a), (b) clearly indicates that porosity parameter have negligible effects on velocity field components.

However shear stresses have strong influence on the shear stresses profiles. The rotational shear stress $\tau_w(r, t)$ decreases and longitudinal shear stress $\tau_v(r, t)$ decreases with the strength of porosity parameter.

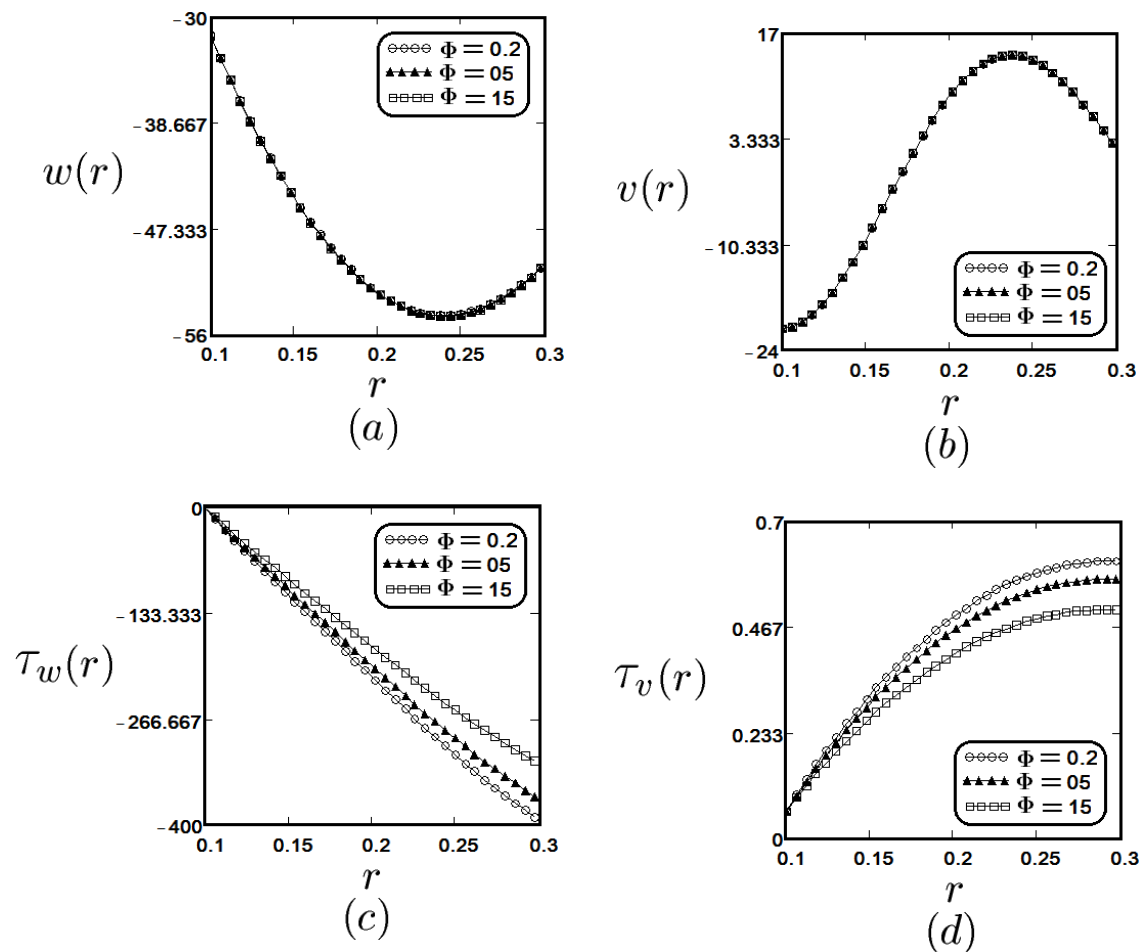


Figure 7: Profiles of the velocity field components for MHD Second grade fluid with porous effect $w(r, t)$, $u(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ given by eqs. (26), (27), (34) and (35), for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\nu = 0.6355$, $\mu = 12.26$, $\alpha = 0.2$, $M = 2$, $p = 1$, $t = 0.2s$ and different values of porosity Φ .

One of the significant parameter for the flow problem is considered, the power p , that appeared in the boundary conditions (22)-(24).

We consider only integer values of power $p = 1, 2, 3$ in Figs 8. It is brought to knowledge that increase in the power parameter p causes a increase in rotational pair and decreases the longitudinal pair of quantities for the flow problem under consideration.

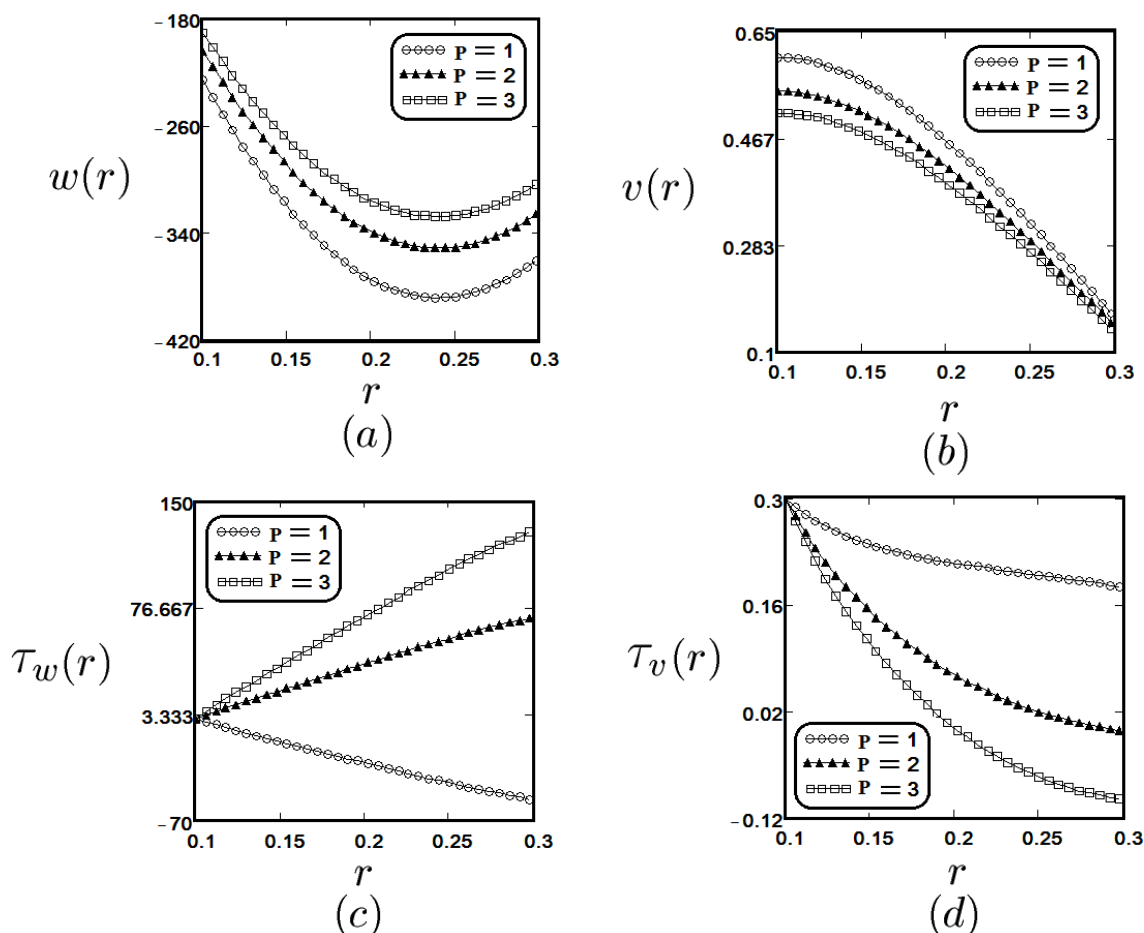


Figure 8: Profiles of the velocity field components for MHD Second grade fluid with porous effect $w(r, t)$, $u(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ given by eqs. (26), (27), (34) and (35), for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\nu = 0.6355$, $\mu = 12.26$, $\alpha = 0.2$, $M = 2$, $\Phi = 2$, $t = 1s$ and different values of the power p .

In the end, for comparison the velocities and shear stresses profiles corresponding to two models MHD second grade and MHD Newtonian in porous medium are together depicted in Figs. 9 for same value of t , α , μ , ν , M and Φ .

It is found that MHD Newtonian fluid flows faster than the MHD second grade fluid in porous medium. However the shear stresses are larger of MHD second grade fluid are larger than the MHD Newtonian fluid.

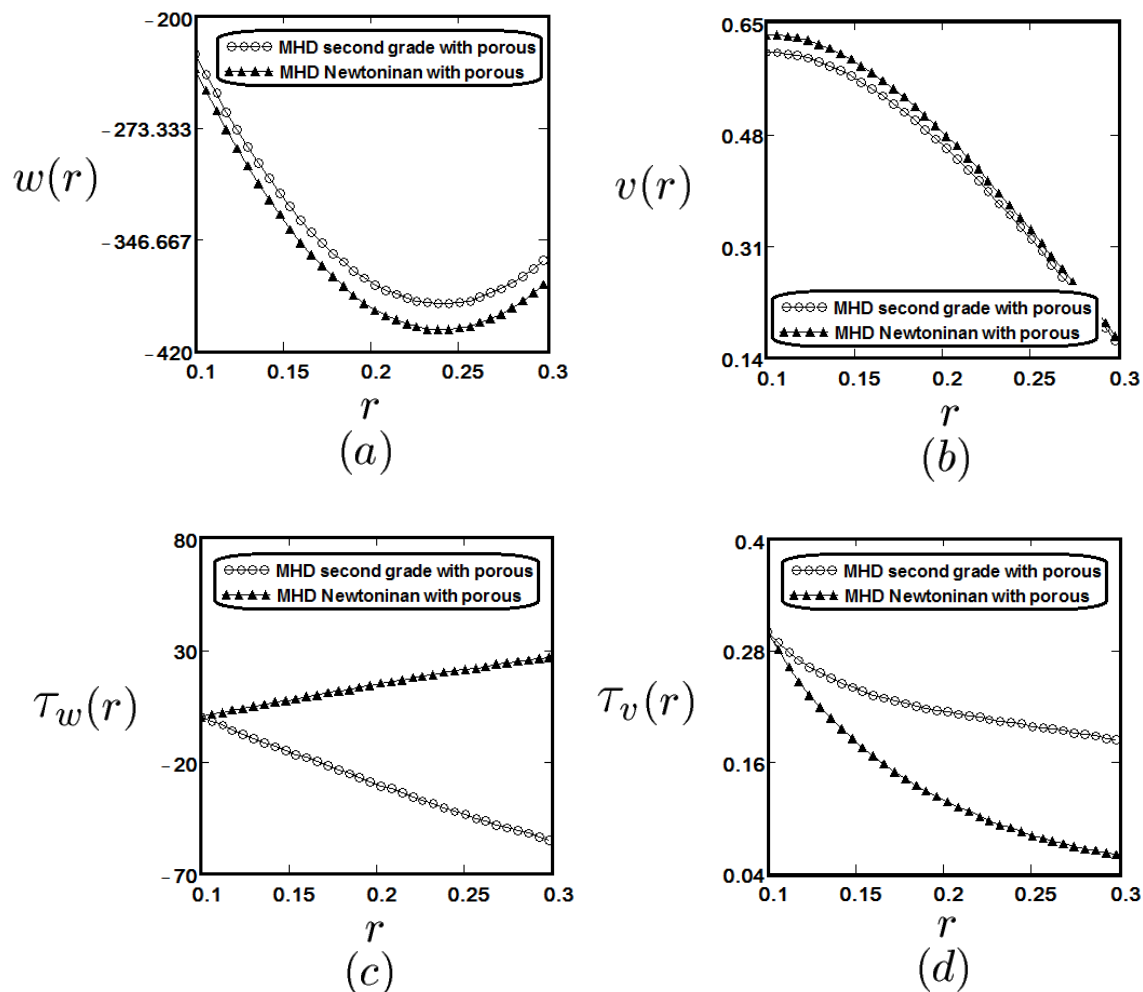


Figure 9: Profiles of the velocity field components for MHD Second grade fluid and MHD Newtonian with porous effect $w(r,t)$, $u(r,t)$ and the shear stresses $\tau_w(r,t)$, $\tau_v(r,t)$ given by eqs. (26), (27), (34), (35), (36), (37), (38) and (39) for $R_1 = 0.1$, $R_2 = 0.3$, $f_1 = g_1 = 0.3$, $f_2 = g_2 = 0.2$, $\nu = 0.6355$, $\mu = 12.26$, $\alpha = 0.2$, $M = 2$, $\Phi = 2$, $p = 1$ and $t = 1s$.

These comparison is completely agreement the Figs. 4 which also shows that when $\alpha \rightarrow 0$ the MHD second grade fluid flows fast in porous medium. SI units are considered for material parameters in Figs. 2 - 9 and $(2m - 1)\pi/2(R_2 - R_1)$ and $(2n - 1)\pi/2(R_2 - R_1)$ are used as approximated values of the roots r_m and r_n .

5. Concluding remarks

The purpose of this chapter is to present the unsteady flow of second grade fluid between two infinite coaxial circular porous cylinders in the presence of magnetic field is studied by techniques of the discrete Laplace and finite Hankel transforms. The flow of the fluid is generated by the inner cylinder that is subject to a rotational and longitudinal time dependent shear stresses and the outer cylinder also subject to a rotational and longitudinal time dependent velocities along its axes when time $t = 0^+$. The exact analytical solutions are obtained for the velocity field components $w(r, t)$, $v(r, t)$ and the shear stresses $\tau_w(r, t)$, $\tau_v(r, t)$ in series form and presented in term of convolution product of Laplace transforms, satisfy all imposed initial and boundary conditions. The corresponding solutions for, second grade, MHD Newtonian and Newtonian fluids in porous medium are also obtained from general solutions for $M \rightarrow 0$ and $\alpha \rightarrow 0$. In the special case when $\Phi \rightarrow 0$, the general solution reduces to the solutions for second grade and Newtonian fluids in the absence of porous medium. The following conclusions are extracted for this study:

The general solutions for velocities and shear stresses are written in simple forms in term of convolution product of Laplace transforms.

In absolute sense velocity field components are increasing functions of time. The rotational velocity increases and longitudinal velocity decreases along radial direction.

The effects of material parameter α and kinematic viscosity ν are opposite on velocity profiles. For instance rotational velocity is increasing and longitudinal velocity is decreasing functions of α .

The increasing values of magnetic values enhance the velocity and shear stress quantities.

The porosity parameter Φ have ignorable impact on the velocity profiles.

When both porous cylindrical helically moved, the MHD Newtonian fluid flows faster than MHD second grade fluid.

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References

- [1] M. R. Salimpour A. F. Dizaji and F. Jam. Flow field of a third grade non newtonian fluid in the annulus of rotating concentric cylinders in the presence of magnetic field. *J. Math. Anal. Appl.*, 337:632–645, 2008.
- [2] X. Zhang D. Tong and Xinhong Zhang. Unsteady helical flows of a generalized oldroyd-b fluid. *Journal of Non-Newtonian Fluid Mechanics*, 156:75–83, 2009.

- [3] L. Debnath and D. Bhatta. *Integral transforms and their applications (second ed.)*. Chapman and Hall/ CRC Press, Boca-Raton-London-New York, 2007.
- [4] Asset Durmagambetov. Global estimation of the cauchy problem solution and blow up the navier-stokes equation. *European Journal of Pure and Applied Mathematic*, 10:763–785, 2017.
- [5] Zakia Hammouch. Multiple solutions of steady mhd flow of dilatant fluids. *European Journal of Pure and Applied Mathematics*, 1:11–20, 2008.
- [6] S. E. E. Hamza. Effect of uniform suction on mhd flow through porous medium due to a rotating disk. *International Journal of Scientific & Engineering Research*, 5:896, 2014.
- [7] O. S. Iyiola and S. B. Folarin. Approximate analytical study of fingero-imbibition phenomena of time-fractional type in double phase flow through porous media. *European Journal of Pure and Applied Mathematics*, 7:210–229, 2014.
- [8] B. K. Jha and C. A. Apere. Time-dependent mhd couette flow in a porous annulus. *Communications in Nonlinear Science and Numerical Simulation*, 18:1959–1969, 2013.
- [9] K. Khandelwal and V. Mathur. Exact solutions for an unsteady flow of viscoelastic fluid in cylindrical domains using the fractional maxwell model. *International Journal of Applied and Computational Mathematics*, 1:143–156, 2015.
- [10] J. Hatami M. Hatami and D. D. Ganji. Computer simulation of mhd blood conveying gold nano particles as a third grade non newtonian nano fluid in a hollow porous vessel. *Computer methods and programs in biomedicine*, 113:632–641, 2014.
- [11] C. Fetecau M. Jamil, A. Rauf and N. A. Khan. Helical flows of second grade fluid due to constantly accelerated shear stresses. *Comm in Nonlinear Sc and NumSimul*, 16:1959–1969, 2011.
- [12] C. Fetecau M. Khan, E. Naheed and T. Hayat. Exact solutions of starting flows for second grade fluid in a porous medium. *International Journal of Non-Linear Mechanics*, 43:868–879, 2008.
- [13] F. M. Mahomed R. Ellahi, T. Hayat and A. Zeeshan. Analytic solutions for mhd flow in an annulus. *Commun Nonlinear Sci Numer Simulat*, 15:1224–1227, 2010.
- [14] GH. R. Mehdizadeh Ahangar S. E. Ghasemi, M. Hatami and D. D. Ganji. Electrohydrodynamic flow analysis in a circular cylindrical conduit using least square method. *Journal of Electrostatics*, 72:47–52, 2014.
- [15] S. Bajwa N. A. Khan M. Tanveer S. Ullah, N. A. Khan and K. Liaqat. Some exact solutions for the rotational flow of oldroyd-b fluid between two circular cylinders. *Advances in Mechanical Engineering*, 9:1–15, 2017.

- [16] C. Fetecaua T. Hayat and M. Sajid. Analytic solution for mhd transient rotating flow of a second grade fluid in a porous space. *Nonlinear Analysis: Real World Applications*, 9:1619–1627, 2008.
- [17] M. Qasim T. Hayat, S. A. Shehzad and S. Obaidat. Flow of a second grade fluid with convective boundary conditions. *Thermal Science*, 15:253–261, 2011.
- [18] N. Ahmed T. Hayat and M. Sajid. Analytic solution for mhd flow of a third order fluid in a porous channel. *Journal of Computational and Applied Mathematics*, 214:572 – 582, 2008.
- [19] S. B. Khan T. Hayat and M. Khan. Exact solution for rotating flows of a generalized burgers fluid in a porous space. *Applied Mathematical Modelling*, 32:749–760, 2008.
- [20] Z. Abbas T. Hayat and M. Sajid. Heat and mass transfer analysis on the flow of a second grade fluid in the presence of chemical reaction. *Physics Letters, A* 372:2400–2408, 2008.
- [21] C. Truesdell and K.R. Rajagopal. *An Introduction to the Mechanics of Fluids*. Springer Science+Business Media, LLC, 233 Spring Street, New York, NY, 10013, USA, 1999.