Existence and uniqueness of solutions for the system of first-order nonlinear differential equations with three-point and integral boundary conditions

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Abstract. In the paper, the existence and uniqueness of the solutions for the system of the nonlinear first-order ordinary differential equations with three-point and integral boundary conditions are studied. The Green function is constructed and the considered problem is reduced to the equivalent integral equation. The existence and uniqueness of the solutions for the given problem are analyzed by using the Banach contraction principle. The Schaefer's fixed point theorem is then used to prove the existence of the solutions. Finally, the examples are given to verify the given theorems.

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1. Introduction and Problem statement

The multipoint and integral boundary value problems for ODEs and their systems play an important role both in theory and application. Boundary value problems with nonlocal boundary conditions for the nonlinear differential equations arise in several branches of physics and applied mathematics. Some examples in application to heat conduction, thermo-elasticity, chemical engineering, plasma physics, and underground water flow can be reduced to nonlocal problems with integral boundary conditions (see [4, 6, 12, 22]).

At present, first-order differential equations with nonlocal conditions have been little studied [2, 4, 6, 13, 19–25, 27], and especially for the second-order differential equations with nonlocal boundary conditions [7–11, 14–16, 18, 26] and the references therein.

The existence and uniqueness of solutions to boundary value problems for the nonlinear system of first-order ordinary differential equations with three-point boundary conditions have been studied by a broad range of techniques [18, 19, 23, 25, 27].

The theory of boundary-value problems with integral boundary conditions for differential equations has become an important area of investigation in recent years. There are many results on the existence and uniqueness of solutions for boundary value problems with integral boundary conditions.They include two, three, and multipoint and nonlocal boundary value problems as special cases. We refer the reader to [2, 5, 7–11, 13–17, 28] and the references therein.

Note that numerical methods for multipoint and integral boundary problems for first-order ordinary differential equations were developed in [1, 3].

In this paper, we concerned the existence and uniqueness of the system of nonlinear differential equations of the type

$$\dot{x}(t) = f(t, x(t)) \text{ for } t \in [0, T], \tag{1}$$

subject to three-point and integral boundary conditions

$$Ax(0) + Bx(\tau) + Cx(T) + \int_{0}^{T} n(t)x(t)dt = d, \tag{2}$$

where A, B, C are constant square matrices of order n such that $\det N \neq 0$, $N = A + B + C + \int_{0}^{T} n(t)dt$; $f : [0, T] \times R^n \to R^n$ and $n : [0, T] \to R^{n \times n}$ are given matrix-functions; $d \in R^n$ is a given vector; and τ satisfies the condition $0 < \tau < T$.

We denote by $C([0, T]; R^n)$ the Banach space of all continuous functions $x(t)$ from $[0, T]$ to R^n with the norm

$$\|x\| = \max \left\{ |x(t)| : t \in [0, T] \right\}$$

where $|\cdot|$ is the norm in the space R^n.

This paper is organized as follows. In Section 2, we introduce definition and lemmas which are the key tools for our main result. Section 3 focuses the theorems on the existence and uniqueness of the solution of problem (1)-(2) established under some sufficient conditions on the nonlinear terms. In Section 4, the given examples are verified to show the effectiveness of the proposed method.

2. Preliminaries

We define the solution of problem (1)-(2) as follows:

Definition 1. *A function $x \in C([0, T], R^n)$ is said to be a solution of problem (1)-(2) if $\dot{x}(t) = f(t, x(t))$ for each $t \in [0, T]$, and boundary conditions (2) are satisfied.*

For the sake of simplicity, we can consider the following problem:

$$\dot{x} = f(t), \quad t \in [0, T], \tag{3}$$

$$Ax(0) + Bx(\tau) + Cx(T) + \int_0^T n(t)x(t)dt = d. \tag{4}$$

Lemma 1. *Let $f(t) \in C([0, T], R^n)$. Then the unique solution $x(t) \in C([0, T], R^n)$ of the boundary value problem for differential equation* (3) *with boundary conditions* (4) *is given by*

$$x(t) = D + \int_0^T G(t, s)f(s)ds, \tag{5}$$

for $t \in [0, T]$ where

$$G(t, s) = \begin{cases} G_1(t, s), & 0 \le t \le \tau \\ G_2(t, s), & \tau < t \le T \end{cases}$$

$$D = N^{-1}d$$

with

$$G_1(t, s) = \begin{cases} N^{-1}\left(A + \int_0^s n(\xi)d\xi\right), & 0 \le s \le \tau, \\ -N^{-1}\left(B + C + \int_s^T n(\xi)d\xi\right), & t < s \le \tau, \\ -N^{-1}\left(C + \int_s^T n(\xi)d\xi\right), & \tau < s \le T, \end{cases}$$

and

$$G_2(t, s) = \begin{cases} N^{-1}\left(A + \int_0^s n(\xi)d\xi\right), & 0 \le s \le \tau, \\ N^{-1}\left(A + B + \int_0^s n(\xi)d\xi\right), & \tau < s \le t, \\ -N^{-1}\left(C + \int_s^T n(\xi)d\xi\right), & t < s \le T, \end{cases}$$

Proof. Assume that $x(t)$ is a solution of boundary value problem (1)-(2), then for $t \in [0, T]$

$$x(t) = x(0) + \int_0^t f(\xi)d\xi. \tag{6}$$

In order the formula (6) satisfy condition (4), we get

$$\left(A + B + C + \int_0^T n(t)dt\right)x(0) = d - B\int_0^\tau f(\xi)d\xi - C\int_0^T f(\xi)d\xi - \int_0^T n(t)\int_0^t f(\xi)d\xi \tag{7}$$

Let us denote $N = A + B + C + \int_0^T n(t)dt$ and from equality (7) we determine $x(0)$ as follows

$$x(0) = D - N^{-1}B \int_0^\tau f(\xi)d\xi - N^{-1}C \int_0^T f(\xi)d\xi - N^{-1} \int_0^T n(t) \int_0^t f(\xi)d\xi dt.$$

Since equality

$$\int_0^T n(t) \int_0^t f(\xi)d\xi dt = \int_0^T \int_t^T n(\xi) \, d\xi f(t) \, dt,$$

is satisfied we can rewrite the above equality as below:

$$x(0) = D - N^{-1}B \int_0^\tau f(\xi)d\xi - N^{-1}C \int_0^T f(\xi)d\xi - N^{-1} \int_0^T \int_t^T n(\xi)d\xi f(t)dt. \qquad (8)$$

Now substituting the value $x(0)$ determined from equality (8) into (6), we obtain

$$x(t) = D - N^{-1}B \int_0^\tau f(\xi)d\xi - N^{-1}C \int_0^T f(\xi)d\xi$$

$$- N^{-1} \int_0^T \int_t^T n(\xi)d\xi f(t)dt + \int_0^t f(\xi)d\xi. \qquad (9)$$

Now consider that $t \in [0, \tau]$. Then we can rewrite equality (9) as follows:

$$x(t) = D - N^{-1}B \int_0^t f(\xi)d\xi - N^{-1}B \int_t^\tau f(\xi)d\xi - N^{-1}C \int_0^t f(\xi)d\xi$$

$$- N^{-1}C \int_t^\tau f(\xi)d\xi - N^{-1}C \int_\tau^T f(\xi)d\xi - N^{-1} \int_0^t \int_s^T n(\xi)d\xi f(s)ds$$

$$- N^{-1} \int_t^\tau \int_s^T n(\xi)d\xi f(s)ds - N^{-1} \int_\tau^T \int_s^T n(\xi)d\xi f(s)ds + \int_0^t f(\xi)d\xi.$$

In the above formula, grouping similar terms, and then simplifying we get

$$x(t) = D + \int_0^t \left(E - N^{-1}B - N^{-1}C - N^{-1} \int_s^T n(\xi)d\xi \right) f(s)ds$$

$$+ \int_t^\tau \left(-N^{-1}B - N^{-1}C - N^{-1} \int_s^T n(\xi)d\xi \right) f(s)ds + \int_\tau^T \left(-N^{-1}C - N^{-1} \int_s^T n(\xi)d\xi \right) f(s)ds$$

$$= D + \int_0^t N^{-1} \left(A + \int_0^s n(\xi)d\xi \right) f(s)ds - \int_t^\tau N^{-1} \left(B + C + \int_s^T n(\xi)d\xi \right) f(s)ds$$

$$- \int_\tau^T N^{-1} \left(C + \int_s^T n(\xi)d\xi \right) f(s)ds. \tag{10}$$

Let us define the new function as follows:

$$G_1(t,s) = \begin{cases} N^{-1} \left(A + \int_0^s n(\xi)d\xi \right), & 0 \le s \le t, \\ -N^{-1} \left(B + C + \int_s^T n(\xi)d\xi \right), & t < s \le \tau, \\ -N^{-1} \left(C + \int_s^T n(\xi)d\xi \right), & \tau < s \le T. \end{cases}$$

Using above equality as in (10) we obtain the following result

$$x(t) = D + \int_0^T G_1(t,s)f(s)ds.$$

For the case, $t \in (\tau, T]$ we can write equality (9) as follows

$$x(t) = D - N^{-1}B \int_0^\tau f(\xi)d\xi - N^{-1}C \int_0^\tau f(\xi)d\xi - N^{-1}C \int_\tau^t f(\xi)d\xi$$

$$- N^{-1}C \int_t^T f(\xi)d\xi - N^{-1} \int_0^\tau \int_t^T n(\xi)d\xi f(s)ds - N^{-1} \int_\tau^t \int_t^T n(\xi)d\xi f(s)ds$$

$$- N^{-1} \int_t^T \int_s^T n(\xi)d\xi f(s)ds + \int_0^\tau f(\xi)d\xi + \int_\tau^t f(\xi)d\xi$$

$$= D + \int_0^\tau \left(E - N^{-1}B - N^{-1}C - N^{-1} \int_s^T n(\xi)d\xi \right) f(s)ds$$

$$+ \int_\tau^t \left(E - N^{-1}C - N^{-1} \int_s^T n(\xi)d\xi \right) f(s)ds + \int_t^T \left(-N^{-1}C - N^{-1} \int_s^T n(\xi)d\xi \right) f(s)ds$$

$$= D + \int\limits_0^\tau N^{-1}\left(A + \int\limits_0^s n(\xi)d\xi\right) f(s)ds + \int\limits_\tau^t N^{-1}\left(A + B + \int\limits_0^s n(\xi)d\xi\right) f(s)ds$$

$$- \int\limits_t^T N^{-1}\left(C + \int\limits_s^T n(\xi)d\xi\right) f(s)ds$$

Hence, we introduce the new function

$$G_2(t,s) = \begin{cases} N^{-1}\left(A + \int\limits_0^s n(\xi)d\xi\right), & 0 \le s \le \tau, \\[2mm] N^{-1}\left(A + B + \int\limits_0^s n(\xi)d\xi\right), & \tau < s \le t, \\[2mm] -N^{-1}\left(C + \int\limits_s^T n(\xi)d\xi\right), & t < s \le T. \end{cases}$$

Thus, for each $t \in (\tau, T]$ we have

$x(t) = D + \int\limits_0^T G_2(t,s)f(s)ds.$ As a result, we deduce that the solution of boundary-value problem (3)-(4) is in the form

$$x(t) = D + \int\limits_0^T G(t,s)f(s)ds. \tag{11}$$

The proof is completed.

The following remark follows from the proved lemma.

Remark 1. *From solution* (11) *we get:*

(i) The constant function $x(t) = D$ *is a solution to the following boundary value problem:*

$$\dot{x} = 0, \quad t \in [0, T],$$

$$Ax(0) + Bx(\tau) + Cx(T) + \int\limits_0^T n(t)x(t)dt = d;$$

(ii) The function $x(t) = \int\limits_0^T G(t,s)f(s)ds$ *is the solution of* $\dot{x} = f(t), \quad t \in [0, T]$ *with the following boundary conditions:*

$$Ax(0) + Bx(\tau) + Cx(T) + \int\limits_0^T n(t)x(t)dt = 0,$$

here $G(t,s)$ is the Green function of the boundary value problem (3)-(4).

Lemma 2. *Assume that* $f \in C([0,T] \times R^n, R^n)$. *Then the function* $x(t)$ *is a solution of the boundary-value problem* (1)-(2) *if and only if* $x(t)$ *is a solution of the integral equation*

$$x(t) = D + \int_0^T G(t,s)f(s,x(s))ds \qquad (12)$$

Proof. Let $x(t)$ be a solution of the boundary-value problem (1)-(2). Then in the same way as in Lemma 2.1, we can prove that it is also a solution of the integral equation (12). Obviously, the solution of integral equation (12) satisfies the boundary-value problem (1)-(2).

Lemma 2.2 is proved.

3. Main results

Let P be an operator such that, $P : C([0,T]; R^n) \to C([0,T]; R^n)$ as

$$(Px)(t) = D + \int_0^T G(t,s)f(s,x(s))ds.$$

It is known that the problem (1)-(2) is equivalent to the fixed point problem. Hence, the problem (1)-(2) has a solution if and only if the operator P has a fixed point.

We now present existence and uniqueness result for nonlinear problem (1)-(2) applying the Banach fixed point theorem.

Theorem 1. *Assume that the following assumption holds*
(H1) There exists a continuous function $M(t) > 0$ *such that*

$$|f(t,x) - f(t,y)| \le M(t)|x - y|$$

for each $t \in [0,T]$ *and all* $x,y \in R^n$ *and*

$$L = TSM < 1, \qquad (13)$$

where

$$M = \max_{[0,T]} M(t),$$

$$S = \max_{[0,T] \times [0,T]} \|G(t,s)\|.$$

Then boundary-value problem (1)-(2) *has a unique solution on* $[0,T]$.

Proof. We denote $\max_{[0,T]} |f(t,0)| = M_f$ and choose $r \geq \frac{|D|+M_fTS}{1-L}$. We show that $PB_r \subset B_r$, where

$$B_r = \{x \in C([0,T]; R^n) : \|x\| \leq r\} \; .$$

For $x \in B_r$, we have

$$\|(Px)(t)\| \leq |D| + \int_0^T |G(t,s)| \, |f(s,x(s)) - f(s,0)| + |f(s,0)| \, ds$$

$$\leq |D| + S \int_0^T (M\,|x| + M_f)dt \leq |D| + SMrT + M_fTS \leq |D| + Lr + M_fTS \leq r.$$

Thus, we obtain $P : B_r \to B_r$.

For any $x, y \in B_r$, it is true

$$|Px - Py| \leq \int_0^T |G(t,s)(f(s,x(s)) - f(s,y(s))| \, ds$$

$$\leq \int_0^T |G(t,s)| \, |f(s,x(s)) - f(s,y(s))| \, ds$$

$$\leq S \int_0^T M(t)\,|x(t) - y(t)| \, dt \leq SMT \max_{[0,T]} |x(t) - y(t)| \leq SMT \, \|x - y\|$$

or

$$\|Px - Py\| \leq L \, \|x - y\| \, .$$

Hence, P is contraction by condition (13), and boundary-value problem (1)-(2) has a unique solution.

We are now in a position to state and prove our existence result by using Schafer's fixed point theorem for the problem (1)-(2).

Theorem 2. *Assume that the following assumptions are satisfied:*
(H2) The function $f : [0,T] \times R^n \to R^n$ is continuous;
(H3)There exists a constant $N_1 > 0$ such that $|f(t,x)| \leq N_1$ for each $t \in [0,T]$ and all $x \in R^n$.
Then the boundary-value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. We will show that P has a fixed point, by applying Theorem 3.2. The proof will be given in several steps. We first will show that P is completely continuous.

Step 1. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,T); R^n])$. Then for any $t \in [0,T]$

$$|P(x_n)(t) - P(x)(t)| = \left| \int_0^T G(t,s) \left(f(s,x_n(s) - f(s,x(s))) \right) ds \right|$$

$$\leq S \int_0^T |f(s,x_n(s) - f(s,x(s))| \, ds \leq ST \max_{[0,T]} |f(s,x_n(s) - f(s,x(s)).|$$

Since F continuous, then
$\|P(x_n)(t) - P(x)(t)\| \to 0$, as $n \to \infty$.

Step 2. Here we prove that the operator P maps bounded sets from $C([0,T); R^n])$. Indeed, it is enough to show that for any $\eta > 0$ there exists a positive constant l such that for each $\{x \in C([0,T); R^n) : \|x\|\} \leq \eta$, it is true $\|P(x)\| \leq l$. For each $t \in [0,T]$, by (H3) we have

$$|P(x)(t)| \leq |D| + \int_0^T |G(t,s)| \, |f(s,x(s))| \, ds$$

Hence,

$$|P(x)(t)| \leq |D| + SN_1 T.$$

In particular,

$$\|P(x)(t)\| \leq |D| + SN_1 T = l.$$

Step 3. P maps bounded sets into equicontinuous sets of $C([0,T); R^n])$.
Let $s_1, s_2 \in [0,T]$, $s_1 < s_2$, β_n be a bounded set in $C([0,T); R^n])$ as in step 2. We assume that $x \in B_r$.

Case 1:
Let $s_1, s_2 \in [0,\tau]$. Then

$$P(x)(s_2) - P(x)(s_1) =$$

$$\int_0^{s_2} N^{-1} \left(A + \int_0^s n(\xi)d\xi \right) f(s,x(s)) ds - \int_{s_2}^\tau N^{-1} \left(B + C + \int_s^T n(\xi)d\xi \right) f(s,x(s)) ds$$

$$- \int_\tau^T N^{-1} \left(C + \int_s^T n(\xi)d\xi \right) f(s,x(s)) ds - \int_0^{s_1} N^{-1} \left(A + \int_0^s n(\xi)d\xi \right) f(s,x(s)) ds$$

$$+ \int_{s_1}^\tau N^{-1} \left(B + C + \int_s^T n(\xi)d\xi \right) f(s,x(s)) ds + \int_\tau^T N^{-1} \left(C + \int_s^T n(\xi)d\xi \right) f(s.x(s)) ds$$

$$= \int_{s_1}^{s_2} N^{-1} \left(A + \int_0^s n(\xi)d\xi \right) f(s,x(s)) ds + \int_{s_1}^{s_2} N^{-1} \left(B + C + \int_s^T n(\xi)d\xi \right) f(s,x(s)) ds = \int_{s_1}^{s_2} f(s,x(s)) ds.$$

Hence we can write

$$|P(x)(s_2) - P(x)(s_1)| = \left| \int_{s_1}^{s_2} f(s, x(s))ds \right| \leq \int_{s_1}^{s_2} |f(s, x(s))| \, ds.$$

Case 2:

Let $s_1 \in [0, \tau]$, and $s_2 \in (\tau, T]$.

$$P(x)(s_2) - P(x)(s_1) =$$

$$= \int_0^\tau N^{-1} \left(A + \int_0^s n(\xi)d\xi \right) f(s, x(s))ds + \int_\tau^{s_2} N^{-1} \left(A + B + \int_0^s n(\xi)d\xi \right) f(s, x(s))ds$$

$$- \int_{s_2}^T N^{-1} \left(C + \int_s^T n(\xi)d\xi \right) f(s, x(s))ds - \int_0^{s_1} \left(A + \int_0^s n(\xi)d\xi \right) f(s, x(s))ds$$

$$+ \int_{s_1}^\tau N^{-1} \left(B + C + \int_s^T n(\xi)d\xi \right) f(s, x(s))ds + \int_\tau^T N^{-1} \left(C + \int_s^T n(\xi)d\xi \right) f(s, x(s))ds$$

$$= \int_{s_1}^\tau N^{-1} \left(A + \int_0^s n(\xi)d\xi \right) f(s, x(s))ds + \int_\tau^{s_2} N^{-1} \left(C + \int_s^T n(\xi)d\xi \right) f(s, x(s))ds$$

$$+ \int_\tau^{s_2} N^{-1} \left(A + B + \int_0^s n(\xi)d\xi \right) f(s, x(s))ds + \int_{s_1}^\tau N^{-1} \left(B + C + \int_s^T n(\xi)d\xi \right) f(s, x(s))ds$$

$$= \int_{s_1}^\tau f(s, x(s))ds + \int_\tau^{s_2} f(s, x(s))ds = \int_{s_1}^{s_2} f(s, x(s))ds.$$

Hence we can write

$$|P(x)(s_2) - P(x)(s_1)| = \left| \int_{s_1}^{s_2} f(s, x(s))ds \right| \leq \int_{s_1}^{s_2} |f(s, x(s))| \, ds.$$

Case 3:

If $s_1, s_2 \in [\tau, T]$ and $s_1 < s_2$, then similarly to case 1, we can write

$$|P(x)(s_2) - P(x)(s_1)| \leq \int_{s_1}^{s_2} |f(s, x(s))ds.|$$

As $s_1 \to s_2$ the right-hand side of the above inequalities for all three cases tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that the mapping
$P : C([0,T]; R^n) \to C([0,T]; R^n)$ is completely continuous.
Step 4. Here we prove the necessary apriory bounds. Indeed, we show that the set
$\Omega = \{x \in C([0,T]; R^n) : x = \lambda P(x), for\ some\ 0 < \lambda < 1\}$ is bounded.
Then for each $t \in [0,T]$, we have

$$x(t) = \lambda D + \lambda \int_0^T G(t,s) f(s, x(s)) ds.$$

Using (H3), we get for each $t \in [0,T]$,

$$|P(x)(t)| \le |D| + SN_1 T.$$

Thus

$$\|x\| \le |D| + SN_1 T.$$

This shows that Ω is bounded.

As a consequence of Theorem 3.2, we deduce that P has a fixed point which is a solution of (1)-(2).

4. Examples

Now we give some examples to illustrate the main results obtained in this paper.
Example 4.1. Let us consider the following nonlocal boundary value problem for a system of nonlinear differential equations:

$$\begin{aligned} \dot{x}_1 &= 0.2 \cos x_2(t), & t \in (0,2), \\ \dot{x}_2 &= 0.2 \sin x_1(t), & t \in (0,2), \end{aligned} \tag{14}$$

with

$$\begin{cases} \frac{1}{2} x_1(0) - \frac{1}{2} x_2(0) + x_2(1) - \frac{1}{2} x_2(2) + \int_0^2 \frac{1}{4} x_1(t)\, dt = 1, \\ -\frac{1}{2} x_1(0) + x_1(1) - \frac{1}{2} x_1(2) + \frac{1}{2} x_2(2) + \int_0^2 \frac{1}{4} x_2(t)\, dt = 1. \end{cases} \tag{15}$$

Evidently, $\int_0^2 n(t) dt = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$,

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad d = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$N = A + B + C + \int_0^2 n(t)\, dt = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that, the Green function for boundary value problem (14)-(15)

$$G(t,s) = \begin{cases} G_1(t,s), & t \in [0,1] \\ G_2(t,s), & t \in (1,2] \end{cases}$$

is as follows:

$$G_1(t,s) = \begin{cases} \begin{pmatrix} 0.5 + 0.25s & -0.5 \\ -0.5 & 0.25s \end{pmatrix}, & 0 \le s \le t, \\ \begin{pmatrix} 0.25(s-2) & -0.5 \\ -0.5 & 0.25(s-2) + 0.5 \end{pmatrix}, & t < s \le 1, \\ \begin{pmatrix} 0.25(s-2) & 0.5 \\ 0.5 & 0.25(s-2) - 0.5 \end{pmatrix}, & 1 < s \le 2, \end{cases}$$

and

$$G_2(t,s) = \begin{cases} \begin{pmatrix} 0.5 + 0.25s & -0.5 \\ -0,5 & 0.25s \end{pmatrix}, & 0 \le s \le 1, \\ \begin{pmatrix} 0.5 + 0.25s & 0.5 \\ 0.5 & 0.25s \end{pmatrix}, & t < s \le 1, \\ \begin{pmatrix} 0.25(s-2) & 0.5 \\ 0.5 & 0.25(s-2) - 0.5 \end{pmatrix}, & 1 < s \le 2. \end{cases}$$

This implies that

$$\|G\| < 2.$$

Here $S < 2$, $M = 0.2$. Thus, the conditions (H1) and (H2) hold with $M = 0.2$. We can easily see that condition (13) is satisfied:

$$L = SMT = 0.2 \cdot 2 \cdot 2 = 0.8 < 1.$$

Example 4.2. Let us consider the following boundary value problem on $[0,2]$.

$$\begin{cases} \dot{x}_1 = \frac{1}{1+x_2^2}, & t \in (0,2), \\ \dot{x}_2 = \frac{1}{1+x_1^2}, & t \in (0,2), \end{cases} \tag{16}$$

with

$$\begin{array}{l} x_1(0) + x_2(1) - x_1(2) + \int_0^2 tx_2(t)dt = -1, \\ x_2(0) + x_1(1) - 2x_2(2) - \int_0^2 t^2 x_1(t)dt = 1. \end{array} \tag{17}$$

Since

$$n(t) = \begin{pmatrix} 0 & t \\ t^2 & 0 \end{pmatrix}, \int_0^2 n(t)dt = \begin{pmatrix} 0 & 2 \\ \frac{3}{8} & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad d = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

$$N = A + B + C + \int_0^2 n(t)\,dt = \begin{pmatrix} 0 & 3 \\ \frac{11}{8} & -1 \end{pmatrix}$$

and the function $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+x_2^2} \\ \frac{1}{1+x_1^2} \end{pmatrix}$ is continuous and bounded, it follows that conditions (H2)-(H3) hold. By Theorem 3.2, the boundary-value problem (16)-(17) has at least one solution on $[0,2]$.

5. Conclusion

The boundary conditions considered in this paper are general enough and can be used extensively in a wide class of problems. In this work, the existence and uniqueness of the solutions for the first-order nonlinear differential equations with three-point and integral boundary conditions are established under sufficient conditions. Note that, given here methods can be used in similar multi-point problems for the ordinary differential equations as follows:

$\dot{x}(t) = f(t, x(t)) \text{for} t \in [0, T],$

$$\sum_{j=0}^{m} L_j x(t_j) + \int_0^T n(t) x(t) dt = \alpha.$$

Here $0 = t_0 < t_1 ... < t_{m-1} < t_m = T$; $n(t) \in R^{n \times n}$ is a given function; $L_j \in R^{n \times n}$ are given matrices; $\alpha \in R^n$ is a given vector

$$\det N \neq 0, \quad N = \sum_{j=0}^{m} L_j + \int_0^T n(t) dt.$$

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