EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 3, 2019, 906-943
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Intuitionistic Fuzzy Ideals with Thresholds $(\alpha, \beta]$ in LA-rings

Nasreen Kausar[1,*], Badar ul Islam[2], Syed Amjad Ahmad[3], Muhammad Azam Waqar[4]

[1] *Department of Mathematics, University of Agriculture, FSD, Pakistan*
[2] *Department of Electrical Engineering, NFC IEFR FSD, Pakistan*
[3] *Department of Mechanical Engineering, NFC IEFR FSD, Pakistan*
[4] *Department of School of Business Manegement, NFC IEFR FSD, Pakistan*

Abstract. In this paper, we give characterizations of regular (intra-regular, both regular and intra-regular) LA-rings by the properties of intuitionistic fuzzy (left, right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

2010 Mathematics Subject Classifications: 03F55, 08A72, 20N25

Key Words and Phrases: Intuitionistic fuzzy left (right, interior, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$, regular (intra-regular) LA-rings.

---

1. Introduction

In ternary operations, the commutative law is given by $abc = cba$. Kazim et al [18], have generalized this notion by introducing the paranthesis on the left side of this equation to get a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is called a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law. An LA-semigroup is a midway structure between a commutative semigroup and a groupoid. Ideals in LA-semigroups have been investigated by Protic et al [24].

In [12] (resp. [8]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [18], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al [24] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [14], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup G is called a left almost group, if there exists a left identity

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v12i3.3441

Email addresses: kausar.nasreen@gmail.com (K. Nasreen), badar.utp@gmail.com (I. Badar) samjadahmad67@yahoo.com (S. A. Ahmad), azamwaqar4@gmail.com (W. Azam)

$e \in G$ such that $ea = a$ for all $a \in G$ and for every $a \in G$ there exists $b \in G$ such that $ba = e$.

Shah et al [25], discussed the left almost ring (abbreviated as LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

A non-empty subset A of R is called an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R if it is both a left ideal and a right ideal of R. A non-empty subset A of R is called an interior ideal of R if $(A, +)$ is an LA-group and $(RA)R \subseteq A$. A non-empty subset A of R is called a quasi-ideal of R if $(A, +)$ is an LA-group and $AR \cap RA \subseteq A$. An LA-subring A of R is called a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of R is called a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$.

We will introduce the concept intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$ of an LA-ring R. We will establish a study by describing the different properties in terms of such ideals, which will be very useful for the characterizations of regular (intra-regular, both regular and intra-regular) LA-rings in terms of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

2. Intuitionistic Fuzzy Ideals with Thresholds $(\alpha, \beta]$

After the introduction of fuzzy set by Zadeh [31], several researchers explored on the generalization of the notion of fuzzy set. The concept of intuitionistic fuzzy set was introduced by Atanassov [1, 2], as a generalization of the notion of fuzzy set.

Liu [20], introduced the concept of fuzzy subrings and fuzzy ideals of a ring. Many authors have explored the theory of fuzzy rings (for example [11, 19, 21, 22, 29]). Gupta et al [11], gave the idea of intrinsic product of fuzzy subsets of a ring. Kuroki [19], characterized regular (intra-regular, both regular and intra-regular) rings in terms of fuzzy left (right, quasi, bi-) ideals.

An intuitionistic fuzzy set (briefly, IFS) A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where the functions $\mu_A \colon X \to [0, 1]$ and $\gamma_A \colon X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ [1, 2].

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ in X can be identified to be an ordered pair (μ_A, γ_A) in $I^X \times I^X$, where I^X is the set of all functions from X to $[0, 1]$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS

$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

Banerjee et al [3] and Hur et al [10], initiated the notion of intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring. Subsequently many authors studied the intuitionistic fuzzy subrings and intuitionistic fuzzy ideals of a ring by describing the different properties (see [9]). Shah et al [26], have initiated the concept of intuitionistic fuzzy normal LA-subrings of an LA-ring.

Bhakat et al [4–6], introduced the notion of (α, β)-fuzzy subgroups. It is a generalization of Rosenfeld fuzzy subgroups as $(\in, \in \vee q)$-fuzzy subgroups. Then many authors studied the algebraic structures by employing the idea of (α, β)-fuzzy subsets (for example [7, 13, 23]). Yuan et al [30], initiated the concept of fuzzy subgroups with thresholds. Shabir et al [28], gave the idea of fuzzy ideals with thresholds in semigroups.

Now we initiate the concept of intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta]$ and intuitionistic fuzzy left (resp. right, interior, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$ of an LA-ring R.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R if

(1) $max\{\mu_A(x-y), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}$,

(2) $min\{\gamma_A(x-y), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$,

(3) $max\{\mu_A(xy), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}$,

(4) $min\{\gamma_A(xy), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$ for all $x, y \in R$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R if

(1) $max\{\mu_A(x-y), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}$,

(2) $min\{\gamma_A(x-y), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$,

(3) $max\{\mu_A(xy), \alpha\} \geq min\{\mu_A(y), \beta\}$,

(4) $min\{\gamma_A(xy), (1-\alpha)\} \leq max\{\gamma_A(y), (1-\beta)\}$ for all $x, y \in R$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R if

(1) $max\{\mu_A(x-y), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}$,

(2) $min\{\gamma_A(x-y), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$,

(3) $max\{\mu_A(xy), \alpha\} \geq min\{\mu_A(x), \beta\}$,

(4) $min\{\gamma_A(xy), (1-\alpha)\} \leq max\{\gamma_A(x), (1-\beta)\}$ for all $x, y \in R$ and $\alpha, \beta \in (0, 1]$ such that $\alpha < \beta$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R if it is both an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ and an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R.

Every intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R, but converse is not true in general.

Example 1. *Let* $R = \{0,1,2,3,4,5,6,7\}$. *Define* $+$ *and* $\cdot$ *in* R *as follows* :

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

and

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

Then R is an LA-ring and $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. We define $(\alpha = 0.1, \beta = 0.7)$ $\mu_A(0) = \mu_A(4) = 0.7$, $\mu_A(1) = \mu_A(2) = \mu_A(3) = \mu_A(5) = \mu_A(6) = \mu_A(7) = 0.1$ and $\gamma_A(0) = \gamma_A(4) = 0.1$, $\gamma_A(1) = \gamma_A(2) = \gamma_A(3) = \gamma_A(5) = \gamma_A(6) = \gamma_A(7) = 0.7$. Since

$$\begin{aligned}
max\{\mu_A(41), \alpha\} &= max\{\mu_A(3), \alpha\} = max\{0.1, 0.1\} = 0.1. \\
min\{\mu_A(4), \beta\} &= min\{0.7, 0.7\} = 0.7. \\
&\Rightarrow max\{\mu_A(41), \alpha\} \ngeq min\{\mu_A(4), \beta\}. \\
\text{and } min\{\gamma_A(41), (1-\alpha)\} &= min\{\gamma_A(3), (1-\alpha)\} = min\{0.7, 0.9\} = 0.7. \\
max\{\gamma_A(4), (1-\beta)\} &= max\{0.1, 0.3\} = 0.3. \\
&\Rightarrow min\{\gamma_A(41), (1-\alpha)\} \nleq max\{\gamma_A(4), (1-\beta)\}.
\end{aligned}$$

Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R, but not an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R if

(1) $max\{\mu_A(x-y), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}$,

(2) $min\{\gamma_A(x-y), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$,

(3) $max\{\mu_A((xy)z), \alpha\} \geq min\{\mu_A(y), \beta\}$,

(4) $min\{\gamma_A((xy)z), (1-\alpha)\} \leq max\{\gamma_A(y), (1-\beta)\}$ for all $x, y, z \in R$ and $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R if

(1) $max\{\mu_A(x-y), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}$,

(2) $min\{\gamma_A(x-y), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$,

(3) $max\{\mu_A(x), \alpha\} \geq min\{(\mu_A \circ R)(x), (R \circ \mu_A)(x), \beta\}$,

(4) $min\{\gamma_A(x), (1-\alpha)\} \leq max\{(\gamma_A \circ R)(x), (R \circ \gamma_A)(x), (1-\beta)\}$ for all $x, y \in R$ and $\alpha, \beta \in (0,1]$ such that $\alpha < \beta$.

An intuitionistic fuzzy LA-subring $A = (\mu_A, \gamma_A)$ with thresholds $(\alpha, \beta]$ of an LA-ring R is called an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R if

(1) $max\{\mu_A\left((xy)z\right),\alpha\} \geq min\{\mu_A\left(x\right),\mu_A\left(z\right),\beta\}$,

(2) $min\{\gamma_A\left((xy)z\right),(1-\alpha)\} \leq max\{\gamma_A\left(x\right),\gamma_A\left(z\right),(1-\beta)\}$ for all $x,y,z \in R$ and $\alpha,\beta \in (0,1]$ such that $\alpha < \beta$.

An IFS $A = (\mu_A, \gamma_A)$ of an LA-ring R is called an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha,\beta]$ of R if

(1) $max\{\mu_A\left(x-y\right),\alpha\} \geq min\{\mu_A\left(x\right),\mu_A(y),\beta\}$,

(2) $min\{\gamma_A\left(x-y\right),(1-\alpha)\} \leq max\{\gamma_A\left(x\right),\gamma_A(y),(1-\beta)\}$,

(3) $max\{\mu_A\left((xy)z\right),\alpha\} \geq min\{\mu_A\left(x\right),\mu_A\left(z\right),\beta\}$,

(4) $min\{\gamma_A\left((xy)z\right),(1-\alpha)\} \leq max\{\gamma_A\left(x\right),\gamma_A\left(z\right),(1-\beta)\}$ for all $x,y,z \in R$ and $\alpha,\beta \in (0,1]$ such that $\alpha < \beta$.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets of an LA-ring R, then the product of A and B is denoted by $A \circ B = (\mu_A \circ \mu_B, \gamma_A \circ \gamma_B)$ and defined by:

$$(\mu_A \circ \mu_B)(x) = \begin{cases} \vee_{x=\sum\limits_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_B(b_i)\}\} & \text{if } x = \sum\limits_{i=1}^{n} a_i b_i,\ a_i, b_i \in R \\ 0 & \text{if } x \neq \sum\limits_{i=1}^{n} a_i b_i \end{cases}$$

$$\text{and } (\gamma_A \circ \gamma_B)(x) = \begin{cases} \wedge_{x=\sum\limits_{i=1}^{n} a_i b_i} \{\vee_{i=1}^{n}\{\gamma_A(a_i) \vee \gamma_B(b_i)\}\} & \text{if } x = \sum\limits_{i=1}^{n} a_i b_i,\ a_i, b_i \in R \\ 1 & \text{if } x \neq \sum\limits_{i=1}^{n} a_i b_i \end{cases}$$

Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R and $\alpha,\beta \in (0,1]$ such that $\alpha < \beta$. We define an intuitionistic fuzzy set A_α^β of R as follow: $(\mu_A)_\alpha^\beta(x) = (\mu_A(x) \wedge \beta) \vee \alpha$ and $(\gamma_A)_\alpha^\beta(x) = (\gamma_A(x) \vee (1-\beta)) \wedge (1-\alpha)$ for all $x \in R$.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets of an LA-ring R. We define intuitionistic fuzzy sets $A \wedge_\alpha^\beta B$, $A \vee_\alpha^\beta B$, $A \circ_\alpha^\beta B$ and $A -_\alpha^\beta B$ of R as follows:

$$\begin{aligned} (\mu_A \wedge_\alpha^\beta \mu_B)(x) &= \{(\mu_A \wedge \mu_B)(x) \wedge \beta\} \vee \alpha \\ \text{and } (\gamma_A \vee_\alpha^\beta \gamma_B)(x) &= \{(\gamma_A \vee \gamma_B)(x) \vee (1-\beta)\} \wedge (1-\alpha). \end{aligned}$$

$$\begin{aligned} (\mu_A \vee_\alpha^\beta \mu_B)(x) &= \{(\mu_A \vee \mu_B)(x) \wedge \beta\} \vee \alpha \\ \text{and } (\gamma_A \wedge_\alpha^\beta \gamma_B)(x) &= \{(\gamma_A \wedge \gamma_B)(x) \vee (1-\beta)\} \wedge (1-\alpha). \end{aligned}$$

$$\begin{aligned} (\mu_A \circ_\alpha^\beta \mu_B)(x) &= \{(\mu_A \circ \mu_B)(x) \wedge \beta\} \vee \alpha \\ \text{and } (\gamma_A \circ_\alpha^\beta \gamma_B)(x) &= \{(\gamma_A \circ \gamma_B)(x) \vee (1-\beta)\} \wedge (1-\alpha). \end{aligned}$$

$$(\mu_A -_\alpha^\beta \mu_B)(x) = \{(\mu_A - \mu_B)(x) \wedge \beta\} \vee \alpha$$

$$\text{and } (\gamma_A -^{\beta}_{\alpha} \gamma_B)(x) \quad = \quad \{(\gamma_A - \gamma_B)(x) \vee (1-\beta)\} \wedge (1-\alpha),$$

for all $x \in R$.

Now we are giving the central properties of such ideals of an LA-ring R, which will be very helpful for further sections.

Lemma 1. *Let A and B be two intuitionistic fuzzy sets of an LA-ring R. Then the following properties holds.*

(1) $A \wedge^{\beta}_{\alpha} B = A^{\beta}_{\alpha} \wedge B^{\beta}_{\alpha}$.

(2) $A \vee^{\beta}_{\alpha} B = A^{\beta}_{\alpha} \vee B^{\beta}_{\alpha}$.

(3) $A \circ^{\beta}_{\alpha} B \geq A^{\beta}_{\alpha} \circ B^{\beta}_{\alpha}$.

If every element x of R is expressible as $x = \sum\limits_{i=1}^{n} a_i b_i$, then $A \circ^{\beta}_{\alpha} B = A^{\beta}_{\alpha} \circ B^{\beta}_{\alpha}$.

If $\chi_A = (\mu_{\chi_A}, \gamma_{\chi_A})$ is an intuitionistic characteristic function of A, then $(\chi_A)^{\beta}_{\alpha}$ is defined as

$$(\mu_{\chi_A})^{\beta}_{\alpha}(x) = \begin{cases} \beta \text{ if } x \in A \\ \alpha \text{ if } x \notin A \end{cases} \text{ and } (\gamma_{\chi_A})^{\beta}_{\alpha}(x) = \begin{cases} \alpha \text{ if } x \in A \\ \beta \text{ if } x \notin A \end{cases}$$

Lemma 2. *Let R be an LA-ring. Then the following properties hold.*

(1) $(A \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} C = (C \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} A$,

(2) $(A \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} (C \circ^{\beta}_{\alpha} D) = (A \circ^{\beta}_{\alpha} C) \circ^{\beta}_{\alpha} (B \circ^{\beta}_{\alpha} D)$ *for all intuitionistic fuzzy sets A, B, C and D of R.*

Proof. Let $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be intuitionistic fuzzy sets of an LA-ring R. We have to show that $(A \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} C = (C \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} A$. Now

$$\begin{aligned} ((A \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} C)(x) &= \{((A \circ B) \circ C)(x) \wedge \beta\} \vee \alpha \\ &= \{((C \circ B) \circ A)(x) \wedge \beta\} \vee \alpha \\ &= ((C \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} A)(x). \end{aligned}$$

In same lines, we can prove (2) .

Proposition 1. *Let R be an LA-ring with left identity e. Then the following assertions hold.*

(1) $A \circ^{\beta}_{\alpha} (B \circ^{\beta}_{\alpha} C) = B \circ^{\beta}_{\alpha} (A \circ^{\beta}_{\alpha} C)$,

(2) $(A \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} (C \circ^{\beta}_{\alpha} D) = (D \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} (C \circ^{\beta}_{\alpha} A)$,

(3) $(A \circ^{\beta}_{\alpha} B) \circ^{\beta}_{\alpha} (C \circ^{\beta}_{\alpha} D) = (D \circ^{\beta}_{\alpha} C) \circ^{\beta}_{\alpha} (B \circ^{\beta}_{\alpha} A)$ *for all intuitionistic fuzzy sets A, B, C and D of R.*

Proof. Same as Lemma 2.

Theorem 1. *Let A and B be two non-empty subsets of an LA-ring R. Then the following conditions hold.*

(1) $\chi_A \circ_\alpha^\beta \chi_B = (\chi_{AB})_\alpha^\beta$.
(2) $\chi_A \vee_\alpha^\beta \chi_B = (\chi_{A\cup B})_\alpha^\beta$.
(3) $\chi_A \wedge_\alpha^\beta \chi_B = (\chi_{A\cap B})_\alpha^\beta$.

Proof. Straight forward.

Theorem 2. *Let A be a non-empty subset of an LA-ring R. Then the following properties hold.*

(1) *A is an LA-subring of R if and only if χ_A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.*

(2) *A is a left (resp. right, two-sided) ideal of R if and only if χ_A is an intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. (1) Let A be an LA-subring of an LA-ring R and $x, y \in R$. If $x, y \notin A$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(x) = 0 = \mu_{\chi_A}(y)$ and $\gamma_{\chi_A}(x) = 1 = \gamma_{\chi_A}(y)$. Thus

$$\begin{aligned} \mu_{\chi_A}(x-y) &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ &\Rightarrow \mu_{\chi_A}(x-y) \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ &\Rightarrow max\{\mu_{\chi_A}(x-y), \alpha\} \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ \text{and } \mu_{\chi_A}(xy) &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y)\} = min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ &\Rightarrow \mu_{\chi_A}(xy) \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ &\Rightarrow max\{\mu_{\chi_A}(xy), \alpha\} \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} min\{\gamma_{\chi_A}(x-y), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1-\beta)\} \\ \text{and } min\{\gamma_{\chi_A}(xy), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1-\beta)\}. \end{aligned}$$

In same lines, we have

$$\begin{aligned} max\{\mu_{\chi_A}(x-y), \alpha\} &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}, \\ max\{\mu_{\chi_A}(xy), \alpha\} &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}, \\ min\{\gamma_{\chi_A}(x-y), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1-\beta)\}, \\ min\{\gamma_{\chi_A}(xy), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1-\beta)\}, \end{aligned}$$

when $x, y \in A$. Hence the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.

Conversely, suppose that the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of an LA-ring R. Let $x, y \in A$, then by definition $\mu_{\chi_A}(x) = 1 = \mu_{\chi_A}(y)$ and $\gamma_{\chi_A}(x) = 0 = \gamma_{\chi_A}(y)$. Since

$$max\{\mu_{\chi_A}(x-y), \alpha\} \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} = \beta,$$

$$\begin{aligned} max\{\mu_{\chi_A}(xy),\alpha\} &\geq min\{\mu_{\chi_A}(x),\mu_{\chi_A}(y),\beta\}=\beta,\\ min\{\gamma_{\chi_A}(x-y),(1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x),\gamma_{\chi_A}(y),(1-\beta)\}=1-\beta,\\ min\{\gamma_{\chi_A}(xy),(1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x),\gamma_{\chi_A}(y),(1-\beta)\}=1-\beta, \end{aligned}$$

χ_A being an intuitionistic fuzzy LA-subring with thresholds $(\alpha,\beta]$ of R. Thus

$$\begin{aligned} max\{\mu_{\chi_A}(x-y),\alpha\} &\geq \beta \text{ and } max\{\mu_{\chi_A}(xy),\alpha\}\geq\beta.\\ min\{\gamma_{\chi_A}(x-y),(1-\alpha)\} &\leq 1-\beta \text{ and } min\{\gamma_{\chi_A}(xy),(1-\alpha)\}\leq 1-\beta. \end{aligned}$$

This implies that $\mu_{\chi_A}(x-y)=1=\mu_{\chi_A}(xy)$ and $\gamma_{\chi_A}(x-y)=0=\gamma_{\chi_A}(xy)$, i.e., $x-y$ and $xy\in A$. Hence A is an LA-subring of R.

(2) Let A be a left ideal of an LA-ring R and $x,y\in R$. If $y\notin A$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(y)=0$ and $\gamma_{\chi_A}(y)=1$. Thus

$$\begin{aligned} \mu_{\chi_A}(xy) &\geq \mu_{\chi_A}(y)=min\{\mu_{\chi_A}(y),\beta\}\\ &\Rightarrow \mu_{\chi_A}(xy)\geq min\{\mu_{\chi_A}(y),\beta\}\\ &\Rightarrow max\{\mu_{\chi_A}(xy),\alpha\}\geq min\{\mu_{\chi_A}(y),\beta\}\\ \text{and } \gamma_{\chi_A}(xy) &\leq \gamma_{\chi_A}(y)=max\{\gamma_{\chi_A}(y),(1-\beta)\}\\ &\Rightarrow \gamma_{\chi_A}(xy)\leq max\{\gamma_{\chi_A}(y),(1-\beta)\}\\ &\Rightarrow min\{\gamma_{\chi_A}(xy),(1-\alpha)\}\leq max\{\gamma_{\chi_A}(y),(1-\beta)\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} max\{\mu_{\chi_A}(xy),\alpha\} &\geq min\{\mu_{\chi_A}(y),\beta\},\\ min\{\gamma_{\chi_A}(xy),(1-\alpha)\} &\leq max\{\gamma_{\chi_A}(y),(1-\beta)\}, \end{aligned}$$

when $y\in A$. Therefore the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy left ideal with thresholds $(\alpha,\beta]$ of R.

Conversely, assume that the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy left ideal with thresholds $(\alpha,\beta]$ of an LA-ring R. Let $y\in A$ and $z\in R$, then by definition $\mu_{\chi_A}(y)=1$ and $\gamma_{\chi_A}(y)=0$. Since

$$\begin{aligned} max\{\mu_{\chi_A}(zy),\alpha\} &\geq min\{\mu_{\chi_A}(y),\beta\}=\beta,\\ min\{\gamma_{\chi_A}(zy),(1-\alpha)\} &\leq max\{\gamma_{\chi_A}(y),(1-\beta)\}=1-\beta, \end{aligned}$$

χ_A being an intuitionistic fuzzy left ideal with thresholds $(\alpha,\beta]$ of R. Thus

$$max\{\mu_{\chi_A}(zy),\alpha\}\geq\beta \text{ and } min\{\gamma_{\chi_A}(zy),(1-\alpha)\}\leq 1-\beta.$$

This implies that $\mu_{\chi_A}(zy)=1$ and $\gamma_{\chi_A}(zy)=0$, i.e., $zy\in A$. Therefore A is a left ideal of R.

Remark 1. (i) *A is an additive LA-subgroup of R if and only if χ_A is an intuitionistic fuzzy additive LA-subgroup with thresholds $(\alpha,\beta]$ of R.*

(ii) *A is an LA-subsemigroup of R if and only if χ_A is an intuitionistic fuzzy LA-subsemigroup with thresholds $(\alpha,\beta]$ of R.*

Theorem 3. *Let A be an IFS of an LA-ring R. Then the following assertions hold.*

(1) A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R if and only if $A \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$ and $A -_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$.

(2) A is an intuitionistic fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta]$ of R if and only if $R \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$ (resp. $A \circ_{\alpha}^{\beta} R \subseteq A_{\alpha}^{\beta}$) and $A -_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$.

Proof. (1) Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of an LA-ring R and $x \in R$. For $A \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$. If $(A \circ_{\alpha}^{\beta} A)(x) = 0$, then obvious $A \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$, otherwise we have

$$\begin{aligned}
(\mu_A \circ_{\alpha}^{\beta} \mu_A)(x) &= \{(\mu_A \circ \mu_A)(x) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\leq \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\mu_A(a_i b_i)\}\right) \wedge \beta\} \vee \alpha \\
&= \{(\mu_A(x) \wedge \beta)\} \vee \alpha = (\mu_A)_{\alpha}^{\beta}(x). \\
&\Rightarrow \mu_A \circ_{\alpha}^{\beta} \mu_A \subseteq (\mu_A)_{\alpha}^{\beta}.
\end{aligned}$$

Similarly, we have $\gamma_A \circ_{\alpha}^{\beta} \gamma_A \supseteq (\gamma_A)_{\alpha}^{\beta}$. Thus $A \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$.

Now for $A -_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$. If $(A -_{\alpha}^{\beta} A)(x) = 0$, then obvious $A -_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$, otherwise we have

$$\begin{aligned}
(\mu_A -_{\alpha}^{\beta} \mu_A)(x) &= \{(\mu_A - \mu_A)(x) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i - b_i}\{\wedge_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\leq \{\left(\vee_{x=\sum_{i=1}^{n} a_i - b_i}\{\wedge_{i=1}^{n}\mu_A(a_i - b_i)\}\right) \wedge \beta\} \vee \alpha \\
&= \{(\mu_A(x) \wedge \beta)\} \vee \alpha = (\mu_A)_{\alpha}^{\beta}(x). \\
&\Rightarrow \mu_A -_{\alpha}^{\beta} \mu_A \subseteq (\mu_A)_{\alpha}^{\beta}.
\end{aligned}$$

Similarly, we have $\gamma_A -_{\alpha}^{\beta} \gamma_A \supseteq (\gamma_A)_{\alpha}^{\beta}$. Thus $A -_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$.

Conversely, assume that $A \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$ and $A -_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$. Let $x, y \in R$ such that $a = xy$. Now

$$\begin{aligned}
max\{\mu_A(xy), \alpha\} &= max\{\mu_A(a), \alpha\} = max\{min\{\mu_A(a), \beta\}, \alpha\} \\
&= (\mu_A)_{\alpha}^{\beta}(a) \geq (\mu_A \circ_{\alpha}^{\beta} \mu_A)(a) = \{(\mu_A \circ \mu_A)(a) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{a=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\geq \{(\mu_A(x) \wedge \mu_A(y)) \wedge \beta\} \vee \alpha \\
&= (\mu_A(x) \wedge \mu_A(y)) \wedge \beta = min\{\mu_A(x), \mu_A(y), \beta\}. \\
&\Rightarrow max\{\mu_A(xy), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}.
\end{aligned}$$

Similarly, we have $min\{\gamma_A(xy), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$. Now we set $a = x - y$ and

$$max\{\mu_A(x-y), \alpha\} = max\{\mu_A(a), \alpha\} = max\{min\{\mu_A(a), \beta\}, \alpha\}$$

$$
\begin{aligned}
&= \mu_{\alpha}^{\beta}(a) \geq (\mu_A -_{\alpha}^{\beta} \mu_A)(a) = \{(\mu_A - \mu_A)(a) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{a=\sum_{i=1}^{n} a_i - b_i}\{\wedge_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\geq \{(\mu_A(x) \wedge \mu_A(y)) \wedge \beta\} \vee \alpha \\
&= (\mu_A(x) \wedge \mu_A(y)) \wedge \beta = min\{\mu_A(x), \mu_A(y), \beta\}. \\
&\Rightarrow max\{\mu_A(x-y), \alpha\} \geq min\{\mu_A(x), \mu_A(y), \beta\}.
\end{aligned}
$$

Similarly, we have $min\{\gamma_A(x-y), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(y), (1-\beta)\}$. Hence A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.

(2) Assume that A is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of an LA-ring R and $x \in R$. If $(R \circ_{\alpha}^{\beta} A)(x) = 0$, then obvious $R \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$, otherwise we have

$$
\begin{aligned}
(R \circ_{\alpha}^{\beta} \mu_A)(x) &= \{(R \circ \mu_A)(x) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{R(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{1 \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\mu_A(b_i)\}\right) \wedge \beta\} \vee \alpha \\
&\leq \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\mu_A(a_i b_i)\}\right) \wedge \beta\} \vee \alpha \\
&= (\mu_A(x) \wedge \beta) \vee \alpha = (\mu)_{\alpha}^{\beta}(x). \\
&\Rightarrow R \circ_{\alpha}^{\beta} \mu_A \subseteq (\mu_A)_{\alpha}^{\beta}.
\end{aligned}
$$

Similarly, we have $R \circ_{\alpha}^{\beta} \gamma_A \supseteq (\gamma_A)_{\alpha}^{\beta}$. Thus $R \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$.

Conversely, suppose that $R \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$. Let $y, z \in R$ such that $x = yz$. Now

$$
\begin{aligned}
max\{\mu_A(yz), \alpha\} &= max\{\mu_A(x), \alpha\} = max\{min\{\mu_A(x), \beta\}, \alpha\} \\
&= (\mu_A)_{\alpha}^{\beta}(x) \geq (R \circ_{\alpha}^{\beta} \mu_A)(x) = \{(R \circ \mu_A)(x) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{R(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\geq ((R(y) \wedge \mu_A(z)) \wedge \beta) \vee \alpha \\
&= (1 \wedge \mu_A(z)) \wedge \beta = min\{\mu_A(z), \beta\}. \\
&\Rightarrow max\{\mu_A(yz), \alpha\} \geq min\{\mu_A(z), \beta\}.
\end{aligned}
$$

Similarly, we have $min\{\gamma_A(yz), (1-\alpha)\} \leq max\{\gamma_A(z), (1-\beta)\}$. Therefore A is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 3. *If A and B are two intuitionistic fuzzy LA-subrings (resp. (left, right, two-sided) ideals) with thresholds $(\alpha, \beta]$ of an LA-ring R, then $A \wedge_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy LA-subring (resp. (left, right, two-sided) ideal) with thresholds $(\alpha, \beta]$ of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta]$ of an LA-ring R. We have to show that $A \wedge_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R. Now

$$\begin{aligned}
& max\{(\mu_A \wedge_\alpha^\beta \mu_B)(x-y), \alpha\} \\
= \; & max\{\{\{(\mu_A \wedge \mu_B)(x-y) \wedge \beta\} \vee \alpha\}, \alpha\} \\
= \; & \{(\mu_A \wedge \mu_B)(x-y) \wedge \beta\} \vee \alpha \\
= \; & \{\mu_A(x-y) \wedge \mu_B(x-y) \wedge \beta\} \vee \alpha \\
\geq \; & \{\mu_A(x) \wedge \mu_A(y) \wedge \mu_B(x) \wedge \mu_B(y) \wedge \beta\} \vee \alpha \\
= \; & \{\mu_A(x) \wedge \mu_B(x) \wedge \mu_A(y) \wedge \mu_B(y) \wedge \beta\} \vee \alpha \\
= \; & \{(\mu_A \wedge \mu_B)(x) \wedge (\mu_A \wedge \mu_B)(y) \wedge \beta \wedge \beta \wedge \beta\} \vee \alpha \\
= \; & \{((\mu_A \wedge \mu_B)(x) \wedge \beta) \wedge ((\mu_A \wedge \mu_B)(y) \wedge \beta) \wedge \beta\} \vee \alpha \\
= \; & (\{(\mu_A \wedge \mu_B)(x) \wedge \beta\} \vee \alpha) \wedge (\{(\mu_A \wedge \mu_B)(y) \wedge \beta\} \vee \alpha) \wedge (\beta \vee \alpha) \\
= \; & (\mu_A \wedge_\alpha^\beta \mu_B)(x) \wedge (\mu_A \wedge_\alpha^\beta \mu_B)(y) \wedge \beta \\
= \; & min\{(\mu_A \wedge_\alpha^\beta \mu_B)(x), (\mu_A \wedge_\alpha^\beta \mu_B)(y), \beta\}.
\end{aligned}$$

Thus $max\{(\mu_A \wedge_\alpha^\beta \mu_B)(x-y), \alpha\} \geq min\{(\mu_A \wedge_\alpha^\beta \mu_B)(x), (\mu_A \wedge_\alpha^\beta \mu_B)(y), \beta\}$. Similarly, we have $max\{(\mu_A \wedge_\alpha^\beta \mu_B)(xy), \alpha\} \geq min\{(\mu_A \wedge_\alpha^\beta \mu_B)(x), (\mu_A \wedge_\alpha^\beta \mu_B)(y), \beta\}$. In same lines we have $min\{(\gamma_A \vee_\alpha^\beta \gamma_B)(x-y), (1-\alpha)\} \leq max\{(\gamma_A \vee_\alpha^\beta \gamma_B)(x), (\gamma_A \vee_\alpha^\beta \gamma_B)(y), (1-\beta)\}$ and $min\{(\gamma_A \vee_\alpha^\beta \gamma_B)(xy), (1-\alpha)\} \leq max\{(\gamma_A \vee_\alpha^\beta \gamma_B)(x), (\gamma_A \vee_\alpha^\beta \gamma_B)(y), (1-\beta)\}$.

Hence $A \wedge_\alpha^\beta B$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.

Lemma 4. *If A and B are two intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta]$ of an LA-ring R, then $A \circ_\alpha^\beta B$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta]$ of an LA-ring R. We have to show that $A \circ_\alpha^\beta B$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R. Now

$$\begin{aligned}
(\mu_A \circ_\alpha^\beta \mu_B)^2 &= (\mu_A \circ_\alpha^\beta \mu_B) \circ_\alpha^\beta (\mu_A \circ_\alpha^\beta \mu_B) \\
&= (\mu_A \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta (\mu_B \circ_\alpha^\beta \mu_B) \subseteq (\mu_A)_\alpha^\beta \circ_\alpha^\beta (\mu_B)_\alpha^\beta = \mu_A \circ_\alpha^\beta \mu_B \\
\text{and } (\gamma_A \circ_\alpha^\beta \gamma_B)^2 &= (\gamma_A \circ_\alpha^\beta \gamma_B) \circ_\alpha^\beta (\gamma_A \circ_\alpha^\beta \gamma_B) \\
&= (\gamma_A \circ_\alpha^\beta \gamma_A) \circ_\alpha^\beta (\gamma_B \circ_\alpha^\beta \gamma_B) \supseteq (\gamma_A)_\alpha^\beta \circ_\alpha^\beta (\gamma_B)_\alpha^\beta = \gamma_A \circ_\alpha^\beta \gamma_B.
\end{aligned}$$

Since $\mu_B -_\alpha^\beta \mu_B \subseteq (\mu_B)_\alpha^\beta$ and $\gamma_B -_\alpha^\beta \gamma_B \supseteq (\gamma_B)_\alpha^\beta$, $B = (\mu_B, \gamma_B)$ being an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$. This implies that $\mu_A \circ_\alpha^\beta (\mu_B -_\alpha^\beta \mu_B) \subseteq \mu_A \circ_\alpha^\beta \mu_B$ and $\gamma_A \circ_\alpha^\beta (\gamma_B -_\alpha^\beta \gamma_B) \supseteq \gamma_A \circ_\alpha^\beta \gamma_B$, i.e., $\mu_A \circ_\alpha^\beta \mu_B -_\alpha^\beta \mu_A \circ_\alpha^\beta \mu_B \subseteq \mu_A \circ_\alpha^\beta \mu_B$ and $\gamma_A \circ_\alpha^\beta \gamma_B -_\alpha^\beta \gamma_A \circ_\alpha^\beta \gamma_B \supseteq \gamma_A \circ_\alpha^\beta \gamma_B$. Therefore $A \circ_\alpha^\beta B$ is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.

Remark 2. *If A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of an LA-ring R, then $A \circ_\alpha^\beta A$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.*

Lemma 5. *Let R be an LA-ring with left identity e. Then every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of an LA-ring R and $x, y \in R$. Thus

$$\begin{aligned} max\{\mu_A(xy), \alpha\} &= max\{\mu_A((ex)y), \alpha\} = max\{\mu_A((yx)e), \alpha\} \\ &\geq min\{\mu_A(yx), \beta\} \geq min\{\mu_A(y), \beta\} \\ \text{and } min\{\gamma_A(xy), (1-\alpha)\} &= min\{\gamma_A((ex)y), (1-\alpha)\} \\ &= min\{\gamma_A((yx)e), (1-\alpha)\} \\ &\leq max\{\gamma_A(yx), (1-\beta)\} \leq max\{\gamma_A(y), (1-\beta)\}. \end{aligned}$$

Therefore A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 6. *If A and B are two intuitionistic fuzzy left (resp. right) ideals with thresholds $(\alpha, \beta]$ of an LA-ring R with left identity e, then $A \circ_\alpha^\beta B$ is also an intuitionistic fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_A, \gamma_A)$ be two intuitionistic fuzzy left ideals with thresholds $(\alpha, \beta]$ of an LA-ring R. We have to show that $A \circ_\alpha^\beta B$ is also an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. Since, $\mu_A \circ_\alpha^\beta \mu_B -_\alpha^\beta \mu_A \circ_\alpha^\beta \mu_B \subseteq \mu_A \circ_\alpha^\beta \mu_B$ and $\gamma_A \circ_\alpha^\beta \gamma_B -_\alpha^\beta \gamma_A \circ_\alpha^\beta \gamma_B \supseteq \gamma_A \circ_\alpha^\beta \gamma_B$ by the Lemma 4. Now

$$\begin{aligned} R \circ_\alpha^\beta (\mu_A \circ_\alpha^\beta \mu_B) &= (R \circ_\alpha^\beta R) \circ_\alpha^\beta (\mu_A \circ_\alpha^\beta \mu_B) \\ &= (R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta (R \circ_\alpha^\beta \mu_B) \subseteq \mu_A \circ_\alpha^\beta \mu_B \\ \text{and } R \circ_\alpha^\beta (\gamma_A \circ_\alpha^\beta \gamma_B) &= (R \circ_\alpha^\beta R) \circ_\alpha^\beta (\gamma_A \circ_\alpha^\beta \gamma_B) \\ &= (R \circ_\alpha^\beta \gamma_A) \circ_\alpha^\beta (R \circ_\alpha^\beta \gamma_B) \supseteq \gamma_A \circ_\alpha^\beta \gamma_B. \end{aligned}$$

Hence $A \circ_\alpha^\beta B$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. Similarly, we can prove for right ideals.

Remark 3. *If A is an intuitionistic fuzzy left (resp. right) ideal with thresholds $(\alpha, \beta]$ of an LA-ring R with left identity e, then $A \circ_\alpha^\beta A$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R.*

Lemma 7. *If A and B are two intuitionistic fuzzy ideals with thresholds $(\alpha, \beta]$ of an LA-ring R, then $A \circ_\alpha^\beta B \subseteq A \wedge_\alpha^\beta B$.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy ideals with thresholds $(\alpha, \beta]$ of an LA-ring R and $x \in R$. If $(A \circ_\alpha^\beta B)(x) = 0$, then obvious $A \circ_\alpha^\beta B \subseteq A \wedge_\alpha^\beta B$, otherwise we have

$$(\mu_A \circ_\alpha^\beta \mu_B)(x) = \{(\mu_A \circ \mu_B)(x) \wedge \beta\} \vee \alpha$$

$$
\begin{aligned}
&= \left\{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{\mu_A(a_i) \wedge \mu_B(b_i)\}\}\right) \wedge \beta\right\} \vee \alpha \\
&\leq \left\{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{\mu_A(a_i b_i) \wedge \mu_B(a_i b_i)\}\}\right) \wedge \beta\right\} \vee \alpha \\
&= \left\{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{(\mu_A \wedge \mu_B)(a_i b_i)\}\}\right) \wedge \beta\right\} \vee \alpha \\
&= \{(\mu_A \wedge \mu_B)(x) \wedge \beta\} \vee \alpha = (\mu_A \wedge_\alpha^\beta \mu_B)(x). \\
&\Rightarrow \mu_A \circ_\alpha^\beta \mu_B \subseteq \mu_A \wedge_\alpha^\beta \mu_B.
\end{aligned}
$$

Similarly, we have $\gamma_A \circ_\alpha^\beta \gamma_B \supseteq \gamma_A \vee_\alpha^\beta \gamma_B$. Hence $A \circ_\alpha^\beta B \subseteq A \wedge_\alpha^\beta B$.

Remark 4. *If A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of an LA-ring R, then $A \circ_\alpha^\beta A \subseteq A_\alpha^\beta$.*

Lemma 8. *Let R be an LA-ring. Then $A \circ_\alpha^\beta B \subseteq A \wedge_\alpha^\beta B$ for every intuitionistic fuzzy right ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal B with thresholds $(\alpha, \beta]$ of R.*

Proof. Same as Lemma 7.

Theorem 4. *Let A be a non-empty subset of an LA-ring R. Then the following conditions are true.*

(1) *A is an interior ideal of R if and only if χ_A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R.*

(2) *A is a quasi-ideal of R if and only if χ_A is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R.*

(3) *A is a bi-ideal of R if and only if χ_A is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.*

(4) *A is a generalized bi-ideal of R if and only if χ_A is an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Let A be an interior ideal of an LA-ring R, this implies that A is an additive LA-subgroup. Then χ_A is an intuitionistic fuzzy additive LA-subgroup with thresholds $(\alpha, \beta]$ of R by the Remark 1. Let $x, y, a \in R$. If $a \notin A$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(a) = 0$ and $\gamma_{\chi_A}(a) = 1$. Thus

$$
\begin{aligned}
\mu_{\chi_A}((xa)y) &\geq \mu_{\chi_A}(a) = min\{\mu_{\chi_A}(a), \beta\} \\
&\Rightarrow \mu_{\chi_A}((xa)y) \geq min\{\mu_{\chi_A}(a), \beta\} \\
&\Rightarrow max\{\mu_{\chi_A}((xa)y), \alpha\} \geq min\{\mu_{\chi_A}(a), \beta\}.
\end{aligned}
$$

Similarly, we have $min\{\gamma_{\chi_A}((xa)y), (1-\alpha)\} \leq max\{\gamma_{\chi_A}(a), (1-\beta)\}$. In same lines, we have

$$
\begin{aligned}
max\{\mu_{\chi_A}((xa)y), \alpha\} &\geq min\{\mu_{\chi_A}(a), \beta\} \\
\text{and } min\{\gamma_{\chi_A}((xa)y), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(a), (1-\beta)\},
\end{aligned}
$$

when $a \in A$. Hence the intuitionistic characteristic function χ_A of A is an i ntuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R.

Conversely, suppose that the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R, this means that χ_A is an intuitionistic fuzzy additive LA-subgroup with thresholds $(\alpha, \beta]$ of R. Then A is an additive LA-subgroup of R by the Remark 1. Let $t \in (RA)R$, so $t = (xa)y$, where $a \in A$ and $x, y \in R$. Then by definition $\mu_{\chi_A}(a) = 1$ and $\gamma_{\chi_A}(a) = 0$. Since

$$\begin{aligned} max\{\mu_{\chi_A}((xa)y), \alpha\} &\geq min\{\mu_{\chi_A}(a), \beta\} = \beta \\ \text{and } min\{\gamma_{\chi_A}((xa)y), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(a), (1-\beta)\} = 1-\beta, \end{aligned}$$

χ_A being an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R. This implies that $\mu_{\chi_A}((xa)y) \geq \beta$ and $\gamma_{\chi_A}((xa)y) \leq 1-\beta$, thus $\mu_{\chi_A}((xa)y) = 1$ and $\mu_{\chi_A}((xa)y) = 0$, i.e., $(xa)y \in A$. Hence A is an interior ideal of R.

(2) Let A be a quasi-ideal of R, this implies that A is an additive LA-subgroup. Then χ_A is an intuitionistic fuzzy additive LA-subgroup with thresholds $(\alpha, \beta]$ of R by the Remark 1. Let $x \in R$ and $x \notin A$, then $x \notin RA$ or $x \notin AR$. If $x \notin RA$, then definition of intuitionistic characteristic function $(R \circ \mu_{\chi_A})(x) = 0$ and $(R \circ \gamma_{\chi_A})(x) = 1$. Thus

$$\begin{aligned} max\{\mu_{\chi_A}(x), \alpha\} &\geq 0 = min\{(\mu_{\chi_A} \circ R)(x), (R \circ \mu_{\chi_A})(x), \beta\} \\ \text{and } min\{\gamma_{\chi_A}(x), (1-\alpha)\} &\leq 1 = max\{(\gamma_{\chi_A} \circ R)(x), (R \circ \gamma_{\chi_A})(x), (1-\beta)\}. \end{aligned}$$

If $x \in A$, then

$$\begin{aligned} max\{\mu_{\chi_A}(x), \alpha\} &= 1 \geq min\{(\mu_{\chi_A} \circ R)(x), R \circ \mu_{\chi_A}(x), \beta\} \\ \text{and } min\{\gamma_{\chi_A}(x), (1-\alpha)\} &= 0 \leq max\{(\gamma_{\chi_A} \circ R)(x), R \circ \gamma_{\chi_A}(x), (1-\beta)\}. \end{aligned}$$

Therefore the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R.

Conversely, assume that the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R, this means that χ_A is an intuitionistic fuzzy additive LA-subgroup with thresholds $(\alpha, \beta]$ of R. Then A is an additive LA-subgroup of R by the Remark 1. Let x be an element of $AR \cap RA$, this means that $x \in AR$ and RA. Since

$$\begin{aligned} max\{\mu_{\chi_A}(x), \alpha\} &\geq min\{(\mu_{\chi_A} \circ R)(x), (R \circ \mu_{\chi_A})(x), \beta\} \\ &= min\{(\mu_{\chi_A} \circ \mu_{\chi_R})(x), (\mu_{\chi_R} \circ \mu_{\chi_A})(x), \beta\} \\ &= min\{\mu_{\chi_{AR}}(x), \mu_{\chi_{RA}}(x), \beta\} = \beta. \\ &\Rightarrow max\{\mu_{\chi_A}(x), \alpha\} \geq \beta. \end{aligned}$$

Similarly, we have $min\{\gamma_{\chi_A}(x), (1-\alpha)\} \leq 1-\beta$, thus $\mu_{\chi_A}(x) = 1$ and $\gamma_{\chi_A}(x) = 0$, i.e., $x \in A$. Therefore A is a quasi-ideal of R.

(3) Let A be a bi-ideal of R, this implies that A is an LA-subring of R. Then χ_A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R by the Remark 1. Let

$x, y, a \in R$. If $x, y \notin A$, then by definition of intuitionistic characteristic function $\mu_{\chi_A}(x) = \mu_{\chi_A}(y) = 0$ and $\gamma_{\chi_A}(x) = \gamma_{\chi_A}(y) = 1$. Thus

$$\begin{aligned} \mu_{\chi_A}((xa)y) &\geq \mu_{\chi_A}(x) \wedge \mu_{\chi_A}(y) = min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ &\Rightarrow \mu_{\chi_A}((xa)y) \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ &\Rightarrow max\{\mu_{\chi_A}((xa)y), \alpha\} \geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\}. \end{aligned}$$

Similarly, we have $min\{\gamma_{\chi_A}((xa)y), (1-\alpha)\} \leq max\{\gamma_{\chi_A}(x), \mu_{\chi_A}(y), (1-\beta)\}$. In same lines we have

$$\begin{aligned} max\{\mu_{\chi_A}((xa)y), \alpha\} &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} \\ \text{and } min\{\gamma_{\chi_A}((xa)y), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1-\beta)\}, \end{aligned}$$

when $x, y \in A$. ence the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.

Conversely, suppose that the intuitionistic characteristic function χ_A of A is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R, this means that χ_A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R. Then A is an LA-subring of R by the Remark 1. Let $t \in (AR)A$, so $t = (xa)y$, where $x, y \in A$ and $a \in R$. Then the definition $\mu_{\chi_A}(x) = \mu_{\chi_A}(y) = 1$ and $\gamma_{\chi_A}(x) = \gamma_{\chi_A}(y) = 0$. As

$$\begin{aligned} max\{\mu_{\chi_A}((xa)y), \alpha\} &\geq min\{\mu_{\chi_A}(x), \mu_{\chi_A}(y), \beta\} = \beta \\ \text{and } min\{\gamma_{\chi_A}((xa)y), (1-\alpha)\} &\leq max\{\gamma_{\chi_A}(x), \gamma_{\chi_A}(y), (1-\beta)\} = 1-\beta, \end{aligned}$$

χ_A being an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R. This implies that $\mu_{\chi_A}((xa)y) \geq \beta$ and $\gamma_{\chi_A}((xa)y) \leq 1-\beta$, thus $\mu_{\chi_A}((xa)y) = 1$ and $\mu_{\chi_A}((xa)y) = 0$, i.e., $(xa)y \in A$. Hence A is bi-ideal of R. Similarly, we can prove (4) .

Theorem 5. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. Then A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R if and only if $(R \circ_\alpha^\beta A) \circ_\alpha^\beta R \subseteq A_\alpha^\beta$ and $A -_\alpha^\beta A \subseteq A_\alpha^\beta$.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of an LA-ring R and $x \in R$. If $((R \circ_\alpha^\beta A) \circ_\alpha^\beta R)(x) = 0$, then obvious $(R \circ_\alpha^\beta A) \circ_\alpha^\beta R \subseteq A_\alpha^\beta$. Otherwise there exist $a_i, b_i, c_i, d_i \in R$ such that $x = \sum_{i=1}^n a_i b_i$ and $a_i = \sum_{i=1}^n c_i d_i$. Since A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R, this implies that $max\{\mu_A((c_i d_i) b_i), \alpha\} \geq min\{\mu_A(d_i), \beta\}$ and $min\{\gamma_A((c_i d_i) b_i), (1-\alpha)\} \leq max\{\gamma_A(d_i), (1-\beta)\}$. Now

$$\begin{aligned} &((R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta R)(x) = \{((R \circ \mu_A) \circ R)(x) \wedge \beta\} \vee \alpha \\ = \;& \{\left(\vee_{x=\sum_{i=1}^n a_i b_i}\{\wedge_{i=1}^n \{(R \circ \mu_A)(a_i) \wedge R(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\ = \;& \{\left(\vee_{x=\sum_{i=1}^n a_i b_i}\{\wedge_{i=1}^n \{(R \circ \mu_A)(a_i) \wedge 1\}\}\right) \wedge \beta\} \vee \alpha \end{aligned}$$

$$
\begin{aligned}
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}(R \circ \mu_A)(a_i)\}\right) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\left(\vee_{a_i=\sum_{i=1}^{n} c_i d_i}\{\wedge_{i=1}^{n}\{R(c_i) \wedge \mu_A(d_i)\}\}\right)\}\right) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\left(\vee_{a_i=\sum_{i=1}^{n} c_i d_i}\{\wedge_{i=1}^{n}\{1 \wedge \mu_A(d_i)\}\}\right)\}\right) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\left(\vee_{a_i=\sum_{i=1}^{n} c_i d_i}\{\wedge_{i=1}^{n}\mu_A(d_i)\}\right)\}\right) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n}(c_i d_i) b_i}\{\wedge_{i=1}^{n}\mu_A(d_i)\}\right) \wedge \beta\} \vee \alpha \\
&\leq \{\left(\vee_{x=\sum_{i=1}^{n}(c_i d_i) b_i}\{\wedge_{i=1}^{n}\mu_A((c_i d_i) b_i)\}\right) \wedge \beta\} \vee \alpha \\
&= \{\mu_A(x) \wedge \beta\} \vee \alpha = (\mu_A)_\alpha^\beta(x). \\
&\Rightarrow (R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta R \subseteq (\mu_A)_\alpha^\beta.
\end{aligned}
$$

Similarly, we have $(R \circ_\alpha^\beta \gamma_A) \circ_\alpha^\beta R \supseteq (\gamma_A)_\alpha^\beta$. Hence $(R \circ_\alpha^\beta A) \circ_\alpha^\beta R \subseteq A_\alpha^\beta$.
Conversely, assume that $(R \circ_\alpha^\beta A) \circ_\alpha^\beta R \subseteq A_\alpha^\beta$ and $x, y, z \in R$ such that $a = (xy)z$. Now

$$
\begin{aligned}
& max\{\mu_A((xy)z), \alpha\} = max\{min\{\mu_A((xy)z), \beta\}, \alpha\} \\
=\ & max\{min\{\mu_A(a), \beta\}, \alpha\} = (\mu_A)_\alpha^\beta(a) \\
\geq\ & ((R \circ_\alpha^\beta \mu_A) \circ_\alpha^\beta R)(a) = \{((R \circ \mu_A) \circ R)(a) \wedge \beta\} \vee \alpha \\
=\ & \{\left(\vee_{a=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{(R \circ \mu_A)(a_i) \wedge R(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
\geq\ & \{((R \circ \mu_A)(xy) \wedge R(z)) \wedge \beta\} \vee \alpha \\
=\ & \{((R \circ \mu_A)(xy) \wedge 1) \wedge \beta\} \vee \alpha = \{(R \circ \mu_A)(xy) \wedge \beta\} \vee \alpha \\
=\ & \{\left(\vee_{xy=\sum_{i=1}^{n} c_i d_i}\{\wedge_{i=1}^{n}\{R(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\
\geq\ & \{(R(x) \wedge \mu_A(y)) \wedge \beta\} \vee \alpha = \{(1 \wedge \mu_A(y)) \wedge \beta\} \vee \alpha \\
=\ & \mu_A(y) \wedge \beta = min\{\mu_A(y), \beta\}. \\
\Rightarrow\ & max\{\mu_A((xy)z), \alpha\} \geq min\{\mu_A(y), \beta\}.
\end{aligned}
$$

Similarly, we have $min\{\gamma_A((xy)z), (1-\alpha)\} \leq max\{\gamma_A(y), (1-\beta)\}$. Therefore A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R.

Theorem 6. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of an LA-ring R. Then A is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R if and only if $(A \circ_\alpha^\beta R) \circ_\alpha^\beta A \subseteq A_\alpha^\beta$.*

Proof. Same as Theorem 5.

Theorem 7. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R. Then A is an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of R if and only if $(A \circ_\alpha^\beta R) \circ_\alpha^\beta A \subseteq A_\alpha^\beta$ and $A -_\alpha^\beta A \subseteq A_\alpha^\beta$.*

Proof. Same as Theorem 5.

Lemma 9. *If A and B are two intuitionistic fuzzy bi- (resp. generalized bi-, quasi-, interior) ideals with thresholds $(\alpha, \beta]$ of an LA-ring R, then $A \wedge_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, quasi-, interior) ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy bi-ideals with thresholds $(\alpha, \beta]$ of an LA-ring R. We have to show that $A \wedge_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R. Since A and B are intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta]$ of R, then $A \wedge_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R by the Lemma 3. We have to show that $max\{(\mu_A \wedge_{\alpha}^{\beta} \mu_B)((xa)y), \alpha\} \geq min\{(\mu_A \wedge_{\alpha}^{\beta} \mu_B)(x), (\mu_A \wedge_{\alpha}^{\beta} \mu_B)(y), \beta\}$ and $min\{(\gamma_A \vee_{\alpha}^{\beta} \gamma_B)((xa)y), (1-\alpha)\} \leq max\{(\gamma_A \vee_{\alpha}^{\beta} \gamma_B)(x), (\gamma_A \vee_{\alpha}^{\beta} \gamma_B)(y), (1-\beta)\}$. Now

$$\begin{aligned}
& max\{(\mu_A \wedge_{\alpha}^{\beta} \mu_B)((xa)y), \alpha\} \\
= & max\{\{\{(\mu_A \wedge \mu_B)((xa)y) \wedge \beta\} \vee \alpha\}, \alpha\} \\
= & \{(\mu_A \wedge \mu_B)((xa)y) \wedge \beta\} \vee \alpha \\
= & \{\mu_A((xa)y) \wedge \mu_B((xa)y) \wedge \beta\} \vee \alpha \\
\geq & \{\mu_A(x) \wedge \mu_A(y) \wedge \mu_B(x) \wedge \mu_B(y) \wedge \beta\} \vee \alpha \\
= & \{\mu_A(x) \wedge \mu_B(x) \wedge \mu_A(y) \wedge \mu_B(y) \wedge \beta\} \vee \alpha \\
= & \{(\mu_A \wedge \mu_B)(x) \wedge (\mu_A \wedge \mu_B)(y) \wedge \beta \wedge \beta \wedge \beta\} \vee \alpha \\
= & \{((\mu_A \wedge \mu_B)(x) \wedge \beta) \wedge ((\mu_A \wedge \mu_B)(y) \wedge \beta) \wedge \beta\} \vee \alpha \\
= & (\{(\mu_A \wedge \mu_B)(x) \wedge \beta\} \vee \alpha) \wedge (\{(\mu_A \wedge \mu_B)(y) \wedge \beta\} \vee \alpha) \wedge (\beta \vee \alpha) \\
= & (\mu_A \wedge_{\alpha}^{\beta} \mu_B)(x) \wedge (\mu_A \wedge_{\alpha}^{\beta} \mu_B)(y) \wedge \beta \\
= & min\{(\mu_A \wedge_{\alpha}^{\beta} \mu_B)(x), (\mu_A \wedge_{\alpha}^{\beta} \mu_B)(y), \beta\}.
\end{aligned}$$

Thus $max\{(\mu_A \wedge_{\alpha}^{\beta} \mu_B)((xa)y), \alpha\} \geq min\{(\mu_A \wedge_{\alpha}^{\beta} \mu_B)(x), (\mu_A \wedge_{\alpha}^{\beta} \mu_B)(y), \beta\}$. Similarly, we have $min\{(\gamma_A \vee_{\alpha}^{\beta} \gamma_B)((xa)y), (1-\alpha)\} \leq max\{(\gamma_A \vee_{\alpha}^{\beta} \gamma_B)(x), (\gamma_A \vee_{\alpha}^{\beta} \gamma_B)(y), (1-\beta)\}$. Hence $A \wedge_{\alpha}^{\beta} B$ is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 10. *If A and B are two intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideals with thresholds $(\alpha, \beta]$ of an LA-ring R with left identity e, then $A \circ_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy bi- (resp. generalized bi-, interior) ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy bi-ideals with thresholds $(\alpha, \beta]$ of an LA-ring R. We have to show that $A \circ_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R. Since A and B are intuitionistic fuzzy LA-subrings with thresholds $(\alpha, \beta]$ of R, then $A \circ_{\alpha}^{\beta} B$ is also an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R by the Lemma 4. Now

$$\begin{aligned}
((\mu_A \circ_{\alpha}^{\beta} \mu_B) \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} (\mu_A \circ_{\alpha}^{\beta} \mu_B) &= ((\mu_A \circ_{\alpha}^{\beta} \mu_B) \circ_{\alpha}^{\beta} (R \circ_{\alpha}^{\beta} R)) \circ_{\alpha}^{\beta} (\mu_A \circ_{\alpha}^{\beta} \mu_B) \\
&= ((\mu_A \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} (\mu_B \circ_{\alpha}^{\beta} R)) \circ_{\alpha}^{\beta} (\mu_A \circ_{\alpha}^{\beta} \mu_B) \\
&= ((\mu_A \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} \mu_A) \circ_{\alpha}^{\beta} ((\mu_B \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} \mu_B)
\end{aligned}$$

$$\subseteq \ (\mu_A)_\alpha^\beta \circ_\alpha^\beta (\mu_B)_\alpha^\beta = \mu_A \circ_\alpha^\beta \mu_B.$$

Similarly, we have $((\gamma_A \circ_\alpha^\beta \gamma_B) \circ_\alpha^\beta R) \circ_\alpha^\beta (\gamma_A \circ_\alpha^\beta \gamma_B) \supseteq \gamma_A \circ_\alpha^\beta \gamma_B$. Therefore $A \circ_\alpha^\beta B$ is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 11. *Every intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of an LA-ring R is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R. The converse is not true in general.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of an LA-ring R and $x, y, z \in R$. Thus

$$\begin{aligned} max\{\mu_A((xy)z), \alpha\} &\geq min\{\mu_A(xy), \beta\} \geq min\{\mu_A(y), \beta\} \\ \text{and } min\{\gamma_A((xy)z), (1-\alpha)\} &\leq max\{\gamma_A(xy), (1-\beta)\} \leq max\{\gamma_A(y), (1-\beta)\}. \end{aligned}$$

Hence A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R.

Proposition 2. *Let $A = (\mu_A, \gamma_A)$ be an IFS of an LA-ring R with left identity e. Then A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R if and only if A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of an LA-ring R and $x, y \in R$. Thus

$$\begin{aligned} max\{\mu_A(xy), \alpha\} &= max\{\mu_A((ex)y), \alpha\} \geq min\{\mu_A(x), \beta\} \\ \text{and } min\{\gamma_A(xy), (1-\alpha)\} &= min\{\gamma_A((ex)y), (1-\alpha)\} \leq max\{\gamma_A(x), (1-\beta)\}. \end{aligned}$$

So A is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R. Therefore A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 5. Converse is true by the Lemma 11.

Lemma 12. *Every intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds $(\alpha, \beta]$ of an LA-ring R is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R. The converse is not true in general.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of an LA-ring R and $x, y, z \in R$. Thus

$$\begin{aligned} max\{\mu_A((xy)z), \alpha\} &\geq min\{\mu_A(xy), \beta\} \geq min\{\mu_A(x), \beta\} \\ \text{and } max\{\mu_A((xy)z), \alpha\} &= max\{\mu_A((zy)x), \alpha\} \\ &\geq min\{\mu_A(zy), \beta\} \geq min\{\mu_A(z), \beta\}. \end{aligned}$$

This implies that $max\{\mu_A((xy)z), \alpha\} \geq min\{\mu_A(x), \mu_A(z), \beta\}$. Similarly, we have

$$min\{\gamma_A((xy)z), (1-\alpha)\} \leq max\{\gamma_A(x), \gamma_A(z), (1-\beta)\}.$$

So A is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 13. *Every intuitionistic fuzzy bi-ideal with thresholds* $(\alpha, \beta]$ *of an LA-ring* R *is an intuitionistic fuzzy generalized bi-ideal with thresholds* $(\alpha, \beta]$ *of* R. *The converse is not true in general.*

Proof. Obvious.

Lemma 14. *Every intuitionistic fuzzy left (resp. right, two-sided) ideal with thresholds* $(\alpha, \beta]$ *of an LA-ring* R *is an intuitionistic fuzzy quasi-ideal with thresholds* $(\alpha, \beta]$ *of* R. *The converse is not true in general.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of an LA-ring R. Thus

$$\begin{aligned} max\{\mu_A(x), \alpha\} &\geq min\{(R \circ \mu_A)(x), \beta\} \\ &\geq min\{(\mu_A \circ R)(x), (R \circ \mu_A)(x), \beta\} \\ \text{and } min\{\gamma_A(x), (1-\alpha)\} &\leq max\{(R \circ \gamma_A)(x), (1-\beta)\} \\ &\leq max\{(\gamma_A \circ R)(x), (R \circ \gamma_A)(x), (1-\beta)\}. \end{aligned}$$

Hence A is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R.

Proposition 3. *Every intuitionistic fuzzy quasi-ideal with thresholds* $(\alpha, \beta]$ *of an LA-ring* R *is an intuitionistic fuzzy LA-subring with thresholds* $(\alpha, \beta]$ *of* R.

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of an LA-ring R. Since $\mu_A \circ_\alpha^\beta \mu_A \subseteq \mu_A \circ_\alpha^\beta R$ and $\mu_A \circ_\alpha^\beta \mu_A \subseteq R \circ_\alpha^\beta \mu_A$, this implies that $\mu_A \circ_\alpha^\beta \mu_A \subseteq \mu_A \circ_\alpha^\beta R \wedge R \circ_\alpha^\beta \mu_A \subseteq (\mu_A)_\alpha^\beta$. Similarly we have, $\gamma_A \circ_\alpha^\beta \gamma_A \supseteq \gamma_A \circ_\alpha^\beta R \vee R \circ_\alpha^\beta \gamma_A \supseteq (\gamma_A)_\alpha^\beta$. Therefore A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R.

Proposition 4. *Let* $A = (\mu_A, \gamma_A)$ *be an intuitionistic fuzzy right ideal with thresholds* $(\alpha, \beta]$ *and* $B = (\mu_B, \gamma_B)$ *be an intuitionistic fuzzy left ideal with thresholds* $(\alpha, \beta]$ *of an LA-ring* R, *respectively. Then* $A \wedge_\alpha^\beta B$ *is an intuitionistic fuzzy quasi-ideal with thresholds* $(\alpha, \beta]$ *of* R.

Proof. We have to show that $A \wedge_\alpha^\beta B$ is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of an LA-ring R. Since

$$\begin{aligned} & max\{(\mu_A \wedge_\alpha^\beta \mu_B)(x-y), \alpha\} \\ \geq\ & min\{(\mu_A \wedge_\alpha^\beta \mu_B)(x), (\mu_A \wedge_\alpha^\beta \mu_B)(y), \beta\} \\ & \text{and } min\{(\gamma_A \vee_\alpha^\beta \gamma_B)(x-y), (1-\alpha)\} \\ \leq\ & max\{(\gamma_A \vee_\alpha^\beta \gamma_B)(x), (\gamma_A \vee_\alpha^\beta \gamma_B)(y), (1-\beta)\}, \end{aligned}$$

by the Lemma 3 and

$$((\mu_A \wedge_\alpha^\beta \mu_B) \circ_\alpha^\beta R) \wedge (R \circ_\alpha^\beta (\mu_A \wedge_\alpha^\beta \mu_B))$$

$$\begin{aligned}
&\subseteq \quad (\mu_A \circ_\alpha^\beta R) \wedge (R \circ_\alpha^\beta \mu_B) \subseteq (\mu_A)_\alpha^\beta \wedge (\mu_B)_\alpha^\beta = \mu_A \wedge_\alpha^\beta \mu_B \\
&\quad\quad \text{and } ((\gamma_A \vee_\alpha^\beta \gamma_B) \circ_\alpha^\beta R) \vee (R \circ_\alpha^\beta (\gamma_A \vee_\alpha^\beta \gamma_B)) \\
&\supseteq \quad (\gamma_A \circ_\alpha^\beta R) \vee (R \circ_\alpha^\beta \gamma_B) \supseteq (\gamma_A)_\alpha^\beta \vee (\gamma_B)_\alpha^\beta = \gamma_A \wedge_\alpha^\beta \gamma_B.
\end{aligned}$$

Thus $A \wedge_\alpha^\beta B$ is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 15. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then every intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of an LA-ring R. This implies that A is an intuitionistic fuzzy LA-subring with thresholds $(\alpha, \beta]$ of R. We have to show that $(\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A \subseteq (\mu_A)_\alpha^\beta$ and $(\gamma_A \circ_\alpha^\beta R) \circ_\alpha^\beta \gamma_A \supseteq (\gamma_A)_\alpha^\beta$. Now

$$\begin{aligned}
(\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A &\subseteq (R \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A \subseteq R \circ_\alpha^\beta \mu_A \\
\text{and } (\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A &\subseteq (\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta R = (\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta (e \circ_\alpha^\beta R) \\
&= (\mu_A \circ_\alpha^\beta e) \circ_\alpha^\beta (R \circ_\alpha^\beta R) \subseteq (\mu_A \circ_\alpha^\beta e) \circ_\alpha^\beta R_\alpha^\beta \\
&= (\mu_A)_\alpha^\beta \circ_\alpha^\beta R_\alpha^\beta = \mu_A \circ_\alpha^\beta R. \\
&\Rightarrow (\mu_A \circ_\alpha^\beta R) \circ_\alpha^\beta \mu_A \subseteq \mu_A \circ_\alpha^\beta R \wedge R \circ_\alpha^\beta \mu_A \subseteq (\mu_A)_\alpha^\beta.
\end{aligned}$$

Similarly, we have $(\gamma_A \circ_\alpha^\beta R) \circ_\alpha^\beta \gamma_A \supseteq (\gamma_A)_\alpha^\beta$. So A is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.

Proposition 5. *If A and B are two intuitionistic fuzzy quasi-ideals with thresholds $(\alpha, \beta]$ of an LA-ring R with left identity e, such that $(xe)R = xR$ for all $x \in R$, then $A \circ_\alpha^\beta B$ is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Let A and B be two intuitionistic fuzzy quasi-ideals with thresholds $(\alpha, \beta]$ of an LA-ring R, this implies that A and B be two intuitionistic fuzzy bi-ideals with thresholds $(\alpha, \beta]$ of R, by the Lemma 15. Then $A \circ_\alpha^\beta B$ is also an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 10.

3. Regular LA-rings

In this section, we characterize regular LA-rings by the properties of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$. An intuitionistic fuzzy ideal $A = (\mu_A, \gamma_A)$ with thresholds $(\alpha, \beta]$ of an LA-ring R is an intuitionistic fuzzy idempotent with thresholds $(\alpha, \beta]$ of R if $A \circ_\alpha^\beta A = A_\alpha^\beta$.

Lemma 16. *Every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of a regular LA-ring R is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R. Let $x, y \in R$, this implies that there exists $a \in R$, such that $x = (xa)x$. Thus

$$\begin{aligned} max\{\mu_A(xy), \alpha\} &= max\{\mu_A(((xa)x)y), \alpha\} = max\{\mu_A((yx)(xa)), \alpha\} \\ &\geq min\{\mu_A(yx), \beta\} \geq min\{\mu_A(y), \beta\} \\ \text{and } min\{\gamma_A(xy), (1-\alpha)\} &= min\{\gamma_A(((xa)x)y), (1-\alpha)\} \\ &= min\{\gamma_A((yx)(xa)), (1-\alpha)\} \\ &\leq max\{\gamma_A(yx), (1-\beta)\} \leq max\{\gamma_A(y), (1-\beta)\}. \end{aligned}$$

Hence A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 17. *Every intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of a regular LA-ring R is an intuitionistic fuzzy idempotent with thresholds $(\alpha, \beta]$.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R and $A \circ_\alpha^\beta A \subseteq A_\alpha^\beta$. We have to show that $A_\alpha^\beta \subseteq A \circ_\alpha^\beta A$. Let $x \in R$, this means that there exists $a \in R$ such that $x = (xa)x$. Thus

$$\begin{aligned} (\mu_A \circ_\alpha^\beta \mu_A)(x) &= \{(\mu_A \circ \mu_A)(x) \wedge \beta\} \vee \alpha \\ &= \{\left(\vee_{x=\sum_{i=1}^n a_i b_i} \{\wedge_{i=1}^n \{\mu_A(a_i) \wedge \mu_A(b_i)\}\}\right) \wedge \beta\} \vee \alpha \\ &\geq \{\{\mu_A(xa) \wedge \mu_A(x)\} \wedge \beta\} \vee \alpha \\ &= (\mu_A(xa) \vee \alpha) \wedge (\mu_A(x) \vee \alpha) \wedge (\beta \vee \alpha) \\ &\geq (\mu_A(x) \wedge \beta) \wedge \mu_A(x) \wedge \beta = \mu_A(x) \wedge \beta \\ &= (\mu_A(x) \wedge \beta) \vee \alpha = (\mu_A)_\alpha^\beta(x). \\ &\Rightarrow (\mu_A)_\alpha^\beta \subseteq \mu_A \circ_\alpha^\beta \mu_A. \end{aligned}$$

Similarly, we have $(\gamma_A)_\alpha^\beta \supseteq \gamma_A \circ_\alpha^\beta \gamma_A$. Therefore $A_\alpha^\beta = A \circ_\alpha^\beta A$.

Remark 5. *Every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of a regular LA-ring R is an intuitionistic fuzzy idempotent with thresholds $(\alpha, \beta]$.*

Proposition 6. *Let $A = (\mu_A, \gamma_A)$ be an IFS of a regular LA-ring R. Then A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R if and only if A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Consider that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R. Let $x, y \in R$, then there exists an element $a \in R$, such that $x = (xa)x$. Thus

$$\begin{aligned} max\{\mu_A(xy), \alpha\} &= max\{\mu_A(((xa)x)y), \alpha\} \\ &= max\{\mu_A((yx)(xa)), \alpha\} \geq min\{\mu_A(x), \beta\} \\ \text{and } min\{\gamma_A(xy), (1-\alpha)\} &= min\{\gamma_A(((xa)x)y), (1-\alpha)\} \end{aligned}$$

$$\begin{aligned} &= min\{\gamma_A((yx)(xa)),(1-\alpha)\} \\ &\leq max\{\gamma_A(x),(1-\beta)\}. \end{aligned}$$

Consequently A is an intuitionistic fuzzy right ideal with thresholds $(\alpha,\beta]$ of R. So A is an intuitionistic fuzzy ideal with thresholds $(\alpha,\beta]$ of R by the Lemma 16. Converse is true by the Lemma 11.

Remark 6. *The concept of intuitionistic fuzzy* (*interior, two-sided*) *ideals with thresholds* $(\alpha,\beta]$ *coincides in regular LA-rings.*

Proposition 7. *Let R be a regular LA-ring. Then* $(A\circ_\alpha^\beta R)\wedge(R\circ_\alpha^\beta A)=A_\alpha^\beta$ *for every intuitionistic fuzzy right ideal A with thresholds* $(\alpha,\beta]$ *of R.*

Proof. Suppose that $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha,\beta]$ of R. This implies that $(A\circ_\alpha^\beta R)\wedge(R\circ_\alpha^\beta A)\subseteq A_\alpha^\beta$, because every intuitionistic fuzzy right ideal with thresholds $(\alpha,\beta]$ of R is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha,\beta]$ of R by the Lemma 14. Let $x\in R$, this implies that there exists $a\in R$, such that $x=(xa)x$. Thus

$$\begin{aligned} (\mu_A\circ_\alpha^\beta R)(x) &= \{(\mu_A\circ R)(x)\wedge\beta\}\vee\alpha \\ &= \{\left(\vee_{x=\sum_{i=1}^n a_ib_i}\{\wedge_{i=1}^n\{\mu_A(a_i)\wedge R(b_i)\}\}\right)\wedge\beta\}\vee\alpha \\ &\geq \{\{\mu_A(xa)\wedge R(x)\}\wedge\beta\}\vee\alpha=\{\mu_A(xa)\wedge\beta\}\vee\alpha \\ &= (\mu_A(xa)\vee\alpha)\wedge(\beta\vee\alpha)\geq(\mu_A(x)\wedge\beta)\wedge\beta \\ &= \mu_A(x)\wedge\beta=(\mu_A(x)\wedge\beta)\vee\alpha=(\mu_A)_\alpha^\beta(x). \\ &\Rightarrow (\mu_A)_\alpha^\beta\subseteq\mu_A\circ_\alpha^\beta R. \end{aligned}$$

Similarly, we have $(\mu_A)_\alpha^\beta\subseteq R\circ_\alpha^\beta\mu_A$, i.e., $(\mu_A)_\alpha^\beta\subseteq(\mu_A\circ_\alpha^\beta R)\wedge(R\circ_\alpha^\beta\mu_A)$. In same lines, we have $(\gamma_A)_\alpha^\beta\supseteq(\gamma_A\circ_\alpha^\beta R)\vee(R\circ_\alpha^\beta\gamma_A)$. Hence $(A\circ_\alpha^\beta R)\wedge(R\circ_\alpha^\beta A)=A_\alpha^\beta$.

Lemma 18. *Let R be a regular LA-ring. Then* $A\circ_\alpha^\beta B=A\wedge_\alpha^\beta B$ *for every intuitionistic fuzzy right ideal* $A=(\mu_A,\gamma_A)$ *with thresholds* $(\alpha,\beta]$ *and every intuitionistic fuzzy left ideal* $B=(\mu_B,\gamma_B)$ *with thresholds* $(\alpha,\beta]$ *of R.*

Proof. Since $A\circ_\alpha^\beta B\subseteq A\wedge_\alpha^\beta B$, for every intuitionistic fuzzy right ideal $A=(\mu_A,\gamma_A)$ with thresholds $(\alpha,\beta]$ and every intuitionistic fuzzy left ideal $B=(\mu_B,\gamma_B)$ with thresholds $(\alpha,\beta]$ of R by the Lemma 8. Let $x\in R$, this means that there exists $a\in R$ such that $x=(xa)x$. Thus

$$\begin{aligned} (\mu_A\circ_\alpha^\beta\mu_B)(x) &= \{(\mu_A\circ\mu_B)(x)\wedge\beta\}\vee\alpha \\ &= \{\left(\vee_{x=\sum_{i=1}^n a_ib_i}\{\wedge_{i=1}^n\{\mu_A(a_i)\wedge\mu_B(b_i)\}\}\right)\wedge\beta\}\vee\alpha \\ &\geq \{\{\mu_A(xa)\wedge\mu_B(x)\}\wedge\beta\}\vee\alpha \\ &= (\mu_A(xa)\vee\alpha)\wedge(\mu_B(x)\vee\alpha)\wedge(\beta\vee\alpha) \end{aligned}$$

$$\begin{aligned}
&\geq (\mu_A(x) \wedge \beta) \wedge \mu_B(x) \wedge \beta \\
&= \mu_A(x) \wedge \mu_B(x) \wedge \beta = (\mu_A \wedge \mu_B)(x) \wedge \beta \\
&= \{(\mu_A \wedge \mu_B)(x) \wedge \beta\} \vee \alpha = (\mu_A \wedge_\alpha^\beta \mu_B)(x). \\
&\Rightarrow \mu_A \wedge_\alpha^\beta \mu_B \subseteq \mu_A \circ_\alpha^\beta \mu_B.
\end{aligned}$$

Similarly, we have $\gamma_A \vee_\alpha^\beta \gamma_B \supseteq \gamma_A \circ_\alpha^\beta \gamma_B$, i.e., $A \wedge_\alpha^\beta B \subseteq A \circ_\alpha^\beta B$. Therefore $A \circ_\alpha^\beta B = A \wedge_\alpha^\beta B$.

Lemma 19. *Let R be an LA-ring with left identity e. Then Ra is the smallest left ideal of R containing a.*

Proof. Let $x, y \in Ra$ and $r \in R$. This implies that $x = r_1 a$ and $y = r_2 a$, where $r_1, r_2 \in R$. Now

$$\begin{aligned}
x - y &= r_1 a - r_2 a = (r_1 - r_2)a \in Ra \\
\text{and } rx &= r(r_1 a) = (er)(r_1 a) = ((r_1 a)r)e = ((r_1 a)(er))e \\
&= ((r_1 e)(ar))e = (e(ar))(r_1 e) = (ar)(r_1 e) \\
&= ((r_1 e)r)a \in Ra.
\end{aligned}$$

Since $a = ea \in Ra$. Thus Ra is a left ideal of R containing a. Let I be another left ideal of R containing a. Since $ra \in I$, where $ra \in Ra$, i.e., $Ra \subseteq I$. Hence Ra is the smallest left ideal of R containing a.

Lemma 20. *Let R be an LA-ring with left identity e. Then aR is a left ideal of R.*

Proof. Straight forward.

Proposition 8. *Let R be an LA-ring with left identity e. Then $aR \cup Ra$ is the smallest right ideal of R containing a.*

Proof. Let $x, y \in aR \cup Ra$, this means that $x, y \in aR$ or Ra. Since aR and Ra both are left ideals of R, so $x - y \in aR$ and Ra, i.e., $x - y \in aR \cup Ra$. We have to show that $(aR \cup Ra)R \subseteq (aR \cup Ra)$. Now

$$\begin{aligned}
(aR \cup Ra)R &= (aR)R \cup (Ra)R = (RR)a \cup (Ra)(eR) \\
&\subseteq Ra \cup (Re)(aR) = Ra \cup R(aR) \\
&= Ra \cup a(RR) \subseteq Ra \cup aR = aR \cup Ra. \\
&\Rightarrow (aR \cup Ra)R \subseteq aR \cup Ra.
\end{aligned}$$

Since $a \in Ra$, i.e., $a \in aR \cup Ra$. Let I be another right ideal of R containing a. Since $aR \in IR \subseteq I$ and $Ra = (RR)a = (aR)R \in (IR)R \subseteq IR \subseteq I$, i.e., $aR \cup Ra \subseteq I$. Therefore $aR \cup Ra$ is the smallest right ideal of R containing a.

Theorem 8. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$A \wedge_{\alpha}^{\beta} B = A \circ_{\alpha}^{\beta} B$ for every intuitionistic fuzzy right ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal B with thresholds $(\alpha, \beta]$ of R.*

(3) *$C_{\alpha}^{\beta} = (C \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} C$ for every intuitionistic fuzzy quasi-ideal C with thresholds $(\alpha, \beta]$ of R.*

Proof. Consider that (1) holds and $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R. This implies that $(C \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} C \subseteq C_{\alpha}^{\beta}$, because every intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 15. Let $x \in R$, then there exists an element $a \in R$ such that $x = (xa)x$. Thus

$$\begin{aligned}
& ((\mu_C \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} \mu_C)(x) \\
= \; & \{((\mu_C \circ R) \circ \mu_C)(x) \wedge \beta\} \vee \alpha \\
= \; & \left\{ \left(\vee_{x=\sum_{i=1}^{n} p_i q_i} \{\wedge_{i=1}^{n} \{(\mu_C \circ R)(p_i) \wedge \mu_C(q_i)\}\} \right) \wedge \beta \right\} \vee \alpha \\
\geq \; & \{\{(\mu_C \circ R)(xa) \wedge \mu_C(x)\} \wedge \beta\} \vee \alpha \\
= \; & ((\mu_C \circ R)(xa) \vee \alpha) \wedge (\mu_C(x) \vee \alpha) \wedge (\beta \vee \alpha) \\
= \; & ((\mu_C \circ R)(xa) \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
= \; & \left(\left(\vee_{xa=\sum_{i=1}^{n} m_i n_i} \{\wedge_{i=1}^{n} \{\mu_C(m_i) \wedge R(n_i)\}\} \right) \vee \alpha \right) \wedge \mu_C(x) \wedge \beta \\
\geq \; & (\{\mu_C(x) \wedge R(a)\} \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
= \; & (\{\mu_C(x) \wedge 1\} \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
= \; & (\mu_C(x) \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
= \; & \mu_C(x) \wedge \beta = (\mu_C(x) \wedge \beta) \vee \alpha = (\mu_C)_{\alpha}^{\beta}(x). \\
\Rightarrow \; & (\mu_C)_{\alpha}^{\beta} \subseteq (\mu_C \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} \mu_C.
\end{aligned}$$

Similarly, we have $(\gamma_C)_{\alpha}^{\beta} \supseteq (\gamma_C \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} \gamma_C$. So $C_{\alpha}^{\beta} = (C \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} C$, i.e., (1) implies (3). Suppose that (3) holds. Let A be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ and B be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. This implies that A and B be intuitionistic fuzzy quasi-ideals with thresholds $(\alpha, \beta]$ of R by the Lemma 14, so $A \wedge_{\alpha}^{\beta} B$ be also an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R. Then by our supposition, $A \wedge_{\alpha}^{\beta} B = ((A \wedge_{\alpha}^{\beta} B) \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} (A \wedge_{\alpha}^{\beta} B) \subseteq (A \circ_{\alpha}^{\beta} R) \circ_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$, i.e., $A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$. Since $A \circ_{\alpha}^{\beta} B \subseteq A \wedge_{\alpha}^{\beta} B$, so $A \circ_{\alpha}^{\beta} B = A \wedge_{\alpha}^{\beta} B$, i.e., (3) $\Rightarrow$ (2). Assume that (2) is true and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 19 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 8. This means that χ_{Ra} is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ and $\chi_{aR \cup Ra}$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R, by the Theorem 2. Then by our assumption $\chi_{aR \cup Ra} \wedge_{\alpha}^{\beta} \chi_{Ra} = \chi_{aR \cup Ra} \circ_{\alpha}^{\beta} \chi_{Ra}$, i.e., $(\chi_{(aR \cup Ra) \cap Ra})_{\alpha}^{\beta} = (\chi_{(aR \cup Ra)Ra})_{\alpha}^{\beta}$ by the Theorem 1. Thus $(aR \cup Ra) \cap Ra = (aR \cup Ra)Ra$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in (aR \cup Ra)Ra$, so $a \in$

$(aR)(Ra) \cup (Ra)(Ra)$. This implies that $a \in (aR)(Ra)$ or $a \in (Ra)(Ra)$. If $a \in (aR)(Ra)$, then $a = (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a = ((xe)(ay))a = (a((xe)y))a$ for any $x, y \in R$. If $a \in (Ra)(Ra)$, then $(Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$, i.e., $a \in (aR)(Ra)$. Therefore a is a regular, i.e., R is a regular. So $(2) \Rightarrow (1)$.

Theorem 9. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$A_\alpha^\beta = (A \circ_\alpha^\beta R) \circ_\alpha^\beta A$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ of R.*

(3) *$B_\alpha^\beta = (B \circ_\alpha^\beta R) \circ_\alpha^\beta B$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(4) *$C_\alpha^\beta = (C \circ_\alpha^\beta R) \circ_\alpha^\beta C$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta]$ of R.*

Proof. $(1) \Rightarrow (4)$, is obvious. $(4) \Rightarrow (3)$, since every intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 13. $(3) \Rightarrow (2)$, since every intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 15. $(2) \Rightarrow (1)$, by the Theorem 8.

Theorem 10. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$A \wedge_\alpha^\beta I = (A \circ_\alpha^\beta I) \circ_\alpha^\beta A$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta]$ of R.*

(3) *$B \wedge_\alpha^\beta I = (B \circ_\alpha^\beta I) \circ_\alpha^\beta B$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta]$ of R.*

(4) *$C \wedge_\alpha^\beta I = (C \circ_\alpha^\beta I) \circ_\alpha^\beta C$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R. Now $(C \circ_\alpha^\beta I) \circ_\alpha^\beta C \subseteq (R \circ_\alpha^\beta I) \circ_\alpha^\beta R \subseteq I \circ_\alpha^\beta R \subseteq I_\alpha^\beta$ and $(C \circ_\alpha^\beta I) \circ_\alpha^\beta C \subseteq (C \circ_\alpha^\beta R) \circ_\alpha^\beta C \subseteq C_\alpha^\beta$, i.e., $(C \circ_\alpha^\beta I) \circ_\alpha^\beta C \subseteq C_\alpha^\beta \wedge I_\alpha^\beta = C \wedge_\alpha^\beta I$. Let $x \in R$, this implies that there exists $a \in R$ such that $x = (xa)x$. Now $xa = ((xa)x)a = (ax)(xa) = x((ax)a)$. Thus

$$
\begin{aligned}
& ((\mu_C \circ_\alpha^\beta \mu_I) \circ_\alpha^\beta \mu_C)(x) \\
= \; & \{((\mu_C \circ \mu_I) \circ \mu_C)(x) \wedge \beta\} \vee \alpha \\
= \; & \left\{ \left(\vee_{x=\sum_{i=1}^n p_i q_i} \{\wedge_{i=1}^n \{(\mu_C \circ \mu_I)(p_i) \wedge \mu_C(q_i)\}\} \right) \wedge \beta \right\} \vee \alpha \\
\geq \; & \{\{(\mu_C \circ \mu_I)(xa) \wedge \mu_C(x)\} \wedge \beta\} \vee \alpha \\
= \; & ((\mu_C \circ \mu_I)(xa) \vee \alpha) \wedge (\mu_C(x) \vee \alpha) \wedge (\beta \vee \alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= ((\mu_C \circ \mu_I)(xa) \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
&= \left(\left(\vee_{xa=\sum_{i=1}^{n} m_i n_i}\{\wedge_{i=1}^{n}\{\mu_C(m_i) \wedge \mu_I(n_i)\}\}\right) \vee \alpha\right) \wedge \mu_C(x) \wedge \beta \\
&\geq (\{\mu_C(x) \wedge \mu_I((ax)a)\} \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
&= (\mu_C(x) \vee \alpha) \wedge (\mu_I((ax)a) \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
&\geq \mu_C(x) \wedge (\mu_I(x) \wedge \beta) \wedge \mu_C(x) \wedge \beta \\
&= \mu_C(x) \wedge \mu_I(x) \wedge \beta = (\mu_C \wedge \mu_I)(x) \wedge \beta \\
&= \{(\mu_C \wedge \mu_I)(x) \wedge \beta\} \vee \alpha = (\mu_C \wedge_\alpha^\beta \mu_I)(x). \\
&\Rightarrow \mu_C \wedge_\alpha^\beta \mu_I \subseteq (\mu_C \circ_\alpha^\beta \mu_I) \circ_\alpha^\beta \mu_C.
\end{aligned}
$$

Similarly, we have $\gamma_C \vee_\alpha^\beta \gamma_I \supseteq (\gamma_C \circ_\alpha^\beta \gamma_I) \circ_\alpha^\beta \gamma_C$. Therefore $C \wedge_\alpha^\beta I = (C \circ_\alpha^\beta I) \circ_\alpha^\beta C$, i.e., $(1) \Rightarrow (4)$. Since $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Assume that (2) holds. Then $A \wedge_\alpha^\beta R = (A \circ_\alpha^\beta R) \circ_\alpha^\beta A$, where R itself is an intuitionistic fuzzy two-sided ideal with thresholds $(\alpha, \beta]$ of R, i.e., $A_\alpha^\beta = (A \circ_\alpha^\beta R) \circ_\alpha^\beta A$. Hence R is a regular by the Theorem 8, i.e., $(2) \Rightarrow (1)$.

Theorem 11. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$A \wedge_\alpha^\beta D \subseteq D \circ_\alpha^\beta A$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ of R.*

(3) *$B \wedge_\alpha^\beta D \subseteq D \circ_\alpha^\beta B$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ of R.*

(4) *$C \wedge_\alpha^\beta D \subseteq D \circ_\alpha^\beta C$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ of R.*

Proof. $(1) \Rightarrow (4)$, is obvious. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Assume that (2) holds, this means that $D \wedge_\alpha^\beta A = A \wedge_\alpha^\beta D \subseteq D \circ_\alpha^\beta A$, where A is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. Since $D \circ_\alpha^\beta A \subseteq D \wedge_\alpha^\beta A$, so $D \wedge_\alpha^\beta A = D \circ_\alpha^\beta A$. Therefore R is a regular by the Theorem 8, i.e., $(2) \Rightarrow (1)$.

Theorem 12. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$A \wedge_\alpha^\beta D \wedge_\alpha^\beta L \subseteq (A \circ_\alpha^\beta D) \circ_\alpha^\beta L$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$, every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ of R.*

(3) *$B \wedge_\alpha^\beta D \wedge_\alpha^\beta L \subseteq (B \circ_\alpha^\beta D) \circ_\alpha^\beta L$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$, every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ of R.*

(4) *$C \wedge_\alpha^\beta D \wedge_\alpha^\beta L \subseteq (C \circ_\alpha^\beta D) \circ_\alpha^\beta L$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta]$, every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ of R.*

Proof. Consider that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$, $L = (\mu_L, \gamma_L)$ be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ and $D = (\mu_D, \gamma_D)$ be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R. Let $x \in R$, then there exists an element $a \in R$ such that $x = (xa)x$. Now

$$\begin{aligned}
x &= (xa)x. \\
xa &= ((xa)x)a = (ax)(xa) = x((ax)a). \\
(ax)a &= (a((xa)x))a = ((xa)(ax))a = (a(ax))(xa) \\
&= x((a(ax))a) = x(((ea)(ax))a) = x(((xa)(ae))a) \\
&= x((((ae)a)x)a) = x((nx)a) = x((nx)(ea)) = x((ae)(xn)) \\
&= x(x((ae)n)) = x(xm). \\
&\Rightarrow xa = x((ax)a) = x(x(xm)) = (ex)(x(xm)) = ((xm)x)(xe).
\end{aligned}$$

Thus

$$\begin{aligned}
& ((\mu_C \circ_\alpha^\beta \mu_D) \circ_\alpha^\beta \mu_L)(x) \\
= {} & \{((\mu_C \circ \mu_D) \circ \mu_L)(x) \wedge \beta\} \vee \alpha \\
= {} & \left\{\left(\vee_{x=\sum_{i=1}^n p_i q_i} \{\wedge_{i=1}^n \{(\mu_C \circ \mu_D)(p_i) \wedge \mu_L(q_i)\}\}\right) \wedge \beta\right\} \vee \alpha \\
\geq {} & \{\{(\mu_C \circ \mu_D)(xa) \wedge \mu_L(x)\} \wedge \beta\} \vee \alpha \\
= {} & ((\mu_C \circ \mu_D)(xa) \vee \alpha) \wedge (\mu_L(x) \vee \alpha) \wedge (\beta \vee \alpha) \\
= {} & ((\mu_C \circ \mu_D)(xa) \vee \alpha) \wedge \mu_L(x) \wedge \beta \\
= {} & \left(\left(\vee_{xa=\sum_{i=1}^n m_i n_i} \{\wedge_{i=1}^n \{\mu_C(m_i) \wedge \mu_D(n_i)\}\}\right) \vee \alpha\right) \wedge \mu_L(x) \wedge \beta \\
\geq {} & (\{\mu_C((xm)x) \wedge \mu_D(xe)\} \vee \alpha) \wedge \mu_L(x) \wedge \beta \\
= {} & (\mu_C((xm)x) \vee \alpha) \wedge (\mu_D(xe) \vee \alpha) \wedge \mu_L(x) \wedge \beta \\
\geq {} & (\mu_C(x) \wedge \mu_C(x) \wedge \beta) \wedge (\mu_D(x) \wedge \beta) \wedge \mu_L(x) \wedge \beta \\
= {} & \mu_C(x) \wedge \mu_D(x) \wedge \mu_L(x) \wedge \beta \\
= {} & (\mu_C(x) \wedge \mu_D(x) \wedge \mu_L(x) \wedge \beta) \vee \alpha = (\mu_C \wedge_\alpha^\beta \mu_D \wedge_\alpha^\beta \mu_L)(x). \\
\Rightarrow {} & \mu_C \wedge_\alpha^\beta \mu_D \wedge_\alpha^\beta \mu_L \subseteq (\mu_C \circ_\alpha^\beta \mu_D) \circ_\alpha^\beta \mu_L.
\end{aligned}$$

Similarly, we have $\gamma_C \vee_\alpha^\beta \gamma_D \vee_\alpha^\beta \gamma_L \supseteq (\gamma_C \circ_\alpha^\beta \gamma_D) \circ_\alpha^\beta \gamma_L$. Hence $C \wedge_\alpha^\beta D \wedge_\alpha^\beta L \subseteq (C \circ_\alpha^\beta D) \circ_\alpha^\beta L$, i.e., $(1) \Rightarrow (4)$. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Assume that (2) holds. Then $A \wedge_\alpha^\beta R \wedge_\alpha^\beta L \subseteq (A \circ_\alpha^\beta R) \circ_\alpha^\beta L$, where A is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R, i.e., $A \wedge_\alpha^\beta L \subseteq A \circ_\alpha^\beta L$. Since $A \circ_\alpha^\beta L \subseteq A \wedge_\alpha^\beta L$, thus $A \circ_\alpha^\beta L = A \wedge_\alpha^\beta L$. So R is regular by the Theorem 8, i.e., $(2) \Rightarrow (1)$.

4. Intra-regular LA-rings

In this section, we characterize intra-regular LA-rings in terms of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

Lemma 21. *Every intuitionistic fuzzy left (right) ideal with thresholds $(\alpha, \beta]$ of an intra-regular LA-ring R is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. Let $x, y \in R$, this implies that there exist $a_i, b_i \in R$, such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Thus

$$
\begin{aligned}
max\{\mu_A(xy), \alpha\} &= max\{\mu_A(((a_i x^2)b_i)y), \alpha\} \\
&= max\{\mu_A((yb_i)(a_i x^2)), \alpha\} \\
&\geq min\{\mu_A(a_i(xx)), \beta\} \geq min\{\mu_A(xx), \beta\} \\
&\geq min\{\mu_A(x), \beta\} \\
\text{and } min\{\gamma_A(xy), (1-\alpha)\} &= min\{\gamma_A(((a_i x^2)b_i)y), (1-\alpha)\} \\
&= min\{\gamma_A((yb_i)(a_i x^2)), (1-\alpha)\} \\
&\leq max\{\gamma_A(a_i(xx)), (1-\beta)\} \\
&\leq max\{\gamma_A(xx), (1-\beta)\} \leq max\{\gamma_A(x), (1-\beta)\}.
\end{aligned}
$$

Hence A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R.

Lemma 22. *Let R be an intra-regular LA-ring with left identity e. Then every intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy idempotent with thresholds $(\alpha, \beta]$.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R and $A \circ_{\alpha}^{\beta} A \subseteq A_{\alpha}^{\beta}$. Let $x \in R$, this means that there exist $a_i, b_i \in R$, such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$
\begin{aligned}
x &= (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i x))b_i \\
&= (x(a_i x))(eb_i) = (xe)((a_i x)b_i) = (a_i x)((xe)b_i).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(\mu_A \circ_{\alpha}^{\beta} \mu_A)(x) &= \{(\mu_A \circ \mu_A)(x) \wedge \beta\} \vee \alpha \\
&= \left\{\left(\vee_{x=\sum_{i=1}^{n} p_i q_i} \{\wedge_{i=1}^{n} \{\mu_A(p_i) \wedge \mu_A(q_i)\}\}\right) \wedge \beta\right\} \vee \alpha \\
&\geq \{\{\mu_A(a_i x) \wedge \mu_A((xe)b_i)\} \wedge \beta\} \vee \alpha \\
&= (\mu_A(a_i x) \vee \alpha) \wedge (\mu_A((xe)b_i) \vee \alpha) \wedge (\beta \vee \alpha) \\
&\geq (\mu_A(x) \wedge \beta) \wedge (\mu_A(x) \wedge \beta) \wedge \beta \\
&= \mu_A(x) \wedge \beta = (\mu_A(x) \wedge \beta) \vee \alpha = (\mu_A)_{\alpha}^{\beta}(x). \\
&\Rightarrow (\mu_A)_{\alpha}^{\beta} \subseteq \mu_A \circ_{\alpha}^{\beta} \mu_A.
\end{aligned}
$$

Similarly, we have $(\gamma_A)_{\alpha}^{\beta} \supseteq \gamma_A \circ_{\alpha}^{\beta} \gamma_A$. Therefore $A_{\alpha}^{\beta} = A \circ_{\alpha}^{\beta} A$.

Proposition 9. *Let A be an IFS of an intra-regular LA-ring R with left identity e. Then A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R if and only if A is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R.*

Proof. Consider that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy interior ideal with thresholds $(\alpha, \beta]$ of R. Let $x, y \in R$, then there exist elements $a_i, b_i \in R$, such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Thus

$$\begin{aligned} max\{\mu_A(xy), \alpha\} &= max\{\mu_A(((a_i x^2)b_i)y), \alpha\} = max\{\mu_A((yb_i)(a_i x^2)), \alpha\} \\ &= max\{\mu_A((yb_i)(a_i(xx))), \alpha\} = max\{\mu_A((yb_i)(x(a_i x))), \alpha\} \\ &= max\{\mu_A((yx)(b_i(a_i x))), \alpha\} \geq min\{\mu_A(x), \beta\}. \\ &\Rightarrow max\{\mu_A(xy), \alpha\} \geq min\{\mu_A(x), \beta\}. \end{aligned}$$

Similarly, we have $min\{\gamma_A(xy), (1-\alpha)\} \leq max\{\gamma_A(x), (1-\beta)\}$, i.e., A is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R. Therefore A is an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 21. Converse is true by the Lemma 11.

Remark 7. *The concept of intuitionistic fuzzy (interior, two-sided) ideals with thresholds $(\alpha, \beta]$ coincides in intra-regular LA-rings with left identity.*

Lemma 23. *Let R be an intra-regular LA-ring with left identity e. Then $B \wedge_\alpha^\beta A \subseteq A \circ_\alpha^\beta B$ for every intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal $B = (\mu_B, \gamma_B)$ with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ and $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R. Let $x \in R$, this implies that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$\begin{aligned} x &= (a_i x^2)b_i = (a_i(xx))b_i = (x(a_i x))b_i \\ &= (x(a_i x))(eb_i) = (xe)((a_i x)b_i) = (a_i x)((xe)b_i). \end{aligned}$$

Thus

$$\begin{aligned} (\mu_A \circ_\alpha^\beta \mu_B)(x) &= \{(\mu_A \circ \mu_B)(x) \wedge \beta\} \vee \alpha \\ &= \{\left(\vee_{x=\sum_{i=1}^{n} p_i q_i}\{\wedge_{i=1}^{n}\{\mu_A(p_i) \wedge \mu_B(q_i)\}\}\right) \wedge \beta\} \vee \alpha \\ &\geq \{\{\mu_A(a_i x) \wedge \mu_B((xe)b_i)\} \wedge \beta\} \vee \alpha \\ &= (\mu_A(a_i x) \vee \alpha) \wedge (\mu_B((xe)b_i) \vee \alpha) \wedge (\beta \vee \alpha) \\ &\geq (\mu_A(x) \wedge \beta) \wedge (\mu_B(x) \wedge \beta) \wedge \beta \\ &= \mu_A(x) \wedge \mu_B(x) \wedge \beta = \mu_B(x) \wedge \mu_A(x) \wedge \beta \\ &= \{(\mu_B \wedge \mu_A)(x) \wedge \beta \\ &= \{(\mu_B \wedge \mu_A)(x) \wedge \beta\} \vee \alpha = (\mu_B \wedge_\alpha^\beta \mu_A)(x). \\ &\Rightarrow \mu_B \wedge_\alpha^\beta \mu_A \subseteq \mu_A \circ_\alpha^\beta \mu_B. \end{aligned}$$

Similarly, we have $\gamma_B \vee_\alpha^\beta \gamma_A \supseteq \gamma_A \circ_\alpha^\beta \gamma_B$. Hence $B \wedge_\alpha^\beta A \subseteq A \circ_\alpha^\beta B$.

Theorem 13. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$B \wedge_\alpha^\beta A \subseteq A \circ_\alpha^\beta B$ for every intuitionistic fuzzy left ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal B with thresholds $(\alpha, \beta]$ of R.*

Proof. $(1) \Rightarrow (2)$, is true by the Lemma 23. Assume that (2) holds and $a \in R$. Then Ra is a left ideal of R containing a by the Lemma 19 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 8. This means that χ_{Ra} is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ and $\chi_{aR\cup Ra}$ is an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R, by the Theorem 2. By our assumption $\chi_{aR\cup Ra} \wedge_\alpha^\beta \chi_{Ra} \subseteq \chi_{Ra} \circ_\alpha^\beta \chi_{aR\cup Ra}$, i.e., $(\chi_{(aR\cup Ra)\cap Ra})_\alpha^\beta \subseteq (\chi_{(Ra)(aR\cup Ra)})_\alpha^\beta$ by the Theorem 1. Thus $(aR \cup Ra) \cap Ra \subseteq Ra(aR \cup Ra)$. Since $a \in (aR \cup Ra) \cap Ra$, i.e., $a \in Ra(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra)$. This implies that $a \in (Ra)(aR)$ or $a \in (Ra)(Ra)$. If $a \in (Ra)(aR)$, then

$$
\begin{aligned}
(Ra)(aR) &= (Ra)((ea)(RR)) = (Ra)((RR)(ae)) \\
&= (Ra)(((ae)R)R) = (Ra)((aR)R) \\
&= (Ra)((RR)a) = (Ra)(Ra) = ((Ra)a)R \\
&= ((Ra)(ea))R = ((Re)(aa))R = (Ra^2)R.
\end{aligned}
$$

Thus $a \in (Ra^2)R$. If $a \in (Ra)(Ra)$, then obvious $a \in (Ra^2)R$. So a is an intra regular. Therefore R is an intra-regular, i.e., $(2) \Rightarrow (1)$.

Theorem 14. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$A \wedge_\alpha^\beta I = (A \circ_\alpha^\beta I) \circ_\alpha^\beta A$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta]$ of R.*

(3) *$B \wedge_\alpha^\beta I = (B \circ_\alpha^\beta I) \circ_\alpha^\beta B$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta]$ of R.*

(4) *$C \wedge_\alpha^\beta I = (C \circ_\alpha^\beta I) \circ_\alpha^\beta C$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy ideal I with thresholds $(\alpha, \beta]$ of R.*

Proof. Consider that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ and $I = (\mu_I, \gamma_I)$ be an intuitionistic fuzzy ideal with thresholds $(\alpha, \beta]$ of R. Now $(C \circ_\alpha^\beta I) \circ_\alpha^\beta C \subseteq (R \circ_\alpha^\beta I) \circ_\alpha^\beta R \subseteq I \circ_\alpha^\beta R \subseteq I_\alpha^\beta$ and $(C \circ_\alpha^\beta I) \circ_\alpha^\beta C \subseteq (C \circ_\alpha^\beta R) \circ_\alpha^\beta C \subseteq C_\alpha^\beta$, thus $(C \circ_\alpha^\beta I) \circ_\alpha^\beta C \subseteq C_\alpha^\beta \wedge I_\alpha^\beta = C \wedge_\alpha^\beta I$. Let $x \in R$, then there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^n (a_i x^2) b_i$. Now

$$
\begin{aligned}
x &= (a_i x^2) b_i = (a_i(xx)) b_i = (x(a_i x)) b = (b_i(a_i x))x. \\
b_i(a_i x) &= b_i(a_i((a_i x^2) b_i)) = b_i((a_i x^2)(a_i b_i)) = b_i((a_i x^2) c_i) \\
&= (a_i x^2)(b_i c_i) = (a_i x^2) d_i = (a_i x^2)(e d_i) = (d_i e)(x^2 a_i) \\
&= m_i(x^2 a_i) = x^2(m_i a_i) = (xx) l_i = (l_i x)x = (l_i x)(ex)
\end{aligned}
$$

$$= \quad (xe)(xl_i) = x((xe)l_i).$$

Thus

$$\begin{aligned}
& ((\mu_C \circ_\alpha^\beta \mu_I) \circ_\alpha^\beta \mu_C)(x) \\
= \ & \{((\mu_C \circ \mu_I) \circ \mu_C)(x) \wedge \beta\} \vee \alpha \\
= \ & \left\{\left(\vee_{x=\sum_{i=1}^n p_i q_i} \{\wedge_{i=1}^n \{(\mu_C \circ \mu_I)(p_i) \wedge \mu_C(q_i)\}\}\right) \wedge \beta\right\} \vee \alpha \\
\geq \ & \{\{(\mu_C \circ \mu_I)(b_i(a_i x)) \wedge \mu_C(x)\} \wedge \beta\} \vee \alpha \\
= \ & ((\mu_C \circ \mu_I)(b_i(a_i x)) \vee \alpha) \wedge (\mu_C(x) \vee \alpha) \wedge (\beta \vee \alpha) \\
= \ & ((\mu_C \circ \mu_I)(b_i(a_i x)) \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
= \ & \left(\left(\vee_{b_i(a_i x)=\sum_{i=1}^n m_i n_i} \{\wedge_{i=1}^n \{\mu_C(m_i) \wedge \mu_I(n_i)\}\}\right) \vee \alpha\right) \wedge \mu_C(x) \wedge \beta \\
\geq \ & (\{\mu_C(x) \wedge \mu_I((xe)l_i)\} \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
= \ & (\mu_C(x) \vee \alpha) \wedge (\mu_I((xe)l_i) \vee \alpha) \wedge \mu_C(x) \wedge \beta \\
\geq \ & \mu_C(x) \wedge (\mu_I(x) \wedge \beta) \wedge \mu_C(x) \wedge \beta \\
= \ & \mu_C(x) \wedge \mu_I(x) \wedge \beta = (\mu_C \wedge \mu_I)(x) \wedge \beta \\
= \ & \{(\mu_C \wedge \mu_I)(x) \wedge \beta\} \vee \alpha \\
= \ & (\mu_C \wedge_\alpha^\beta \mu_I)(x). \\
\Rightarrow \ & \mu_C \wedge_\alpha^\beta \mu_I \subseteq (\mu_C \circ_\alpha^\beta \mu_I) \circ_\alpha^\beta \mu_C.
\end{aligned}$$

Similarly, we have $\gamma_C \vee_\alpha^\beta \gamma_I \supseteq (\gamma_C \circ_\alpha^\beta \gamma_I) \circ_\alpha^\beta \gamma_C$. Hence $C \wedge_\alpha^\beta I = (C \circ_\alpha^\beta I) \circ_\alpha^\beta C$, i.e., (1) implies (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) holds. Let A be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ and I be an intuitionistic fuzzy two-sided ideal with thresholds $(\alpha, \beta]$ of R. Since every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 14, This implies that A is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R. By our supposition $A \wedge_\alpha^\beta I = (A \circ_\alpha^\beta I) \circ_\alpha^\beta A \subseteq (R \circ_\alpha^\beta I) \circ_\alpha^\beta A \subseteq I \circ_\alpha^\beta A$, i.e., $A \wedge_\alpha^\beta I \subseteq I \circ_\alpha^\beta A$. So R is an intra-regular by the Theorem 13, i.e., (2) $\Rightarrow$ (1).

Theorem 15. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$A \wedge_\alpha^\beta L \subseteq L \circ_\alpha^\beta A$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ of R.*

(3) *$B \wedge_\alpha^\beta L \subseteq L \circ_\alpha^\beta B$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ of R.*

(4) *$C \wedge_\alpha^\beta L \subseteq L \circ_\alpha^\beta C$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ of R.*

Proof. Suppose that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$ and $L = (\mu_L, \gamma_L)$ be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. Let $x \in R$, this implies that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now $x = (a_i(xx))b_i = (x(a_i x))b_i = (b_i(a_i x))x$. Thus

$$\begin{aligned}
(\mu_L \circ_\alpha^\beta \mu_C)(x) &= \{(\mu_L \circ \mu_C)(x) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} p_i q_i}\{\wedge_{i=1}^{n}\{\mu_L(p_i) \wedge \mu_C(q_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\geq \{\{\mu_L(b_i(a_i x)) \wedge \mu_C(x)\} \wedge \beta\} \vee \alpha \\
&= (\mu_L(b_i(a_i x)) \vee \alpha) \wedge (\mu_C(x) \vee \alpha) \wedge (\beta \vee \alpha) \\
&\geq (\mu_L(x) \wedge \beta) \wedge \mu_C(x) \wedge \beta \\
&= \mu_L(x) \wedge \mu_C(x) \wedge \beta = \mu_C(x) \wedge \mu_L(x) \wedge \beta \\
&= (\mu_C \wedge \mu_L)(x) \wedge \beta \\
&= \{(\mu_C \wedge \mu_L)(x) \wedge \beta\} \vee \alpha = (\mu_C \wedge_\alpha^\beta \mu_L)(x). \\
&\Rightarrow \mu_C \wedge_\alpha^\beta \mu_L \subseteq \mu_L \circ_\alpha^\beta \mu_C.
\end{aligned}$$

Similarly, we have $\gamma_C \vee_\alpha^\beta \gamma_L \supseteq \gamma_L \circ_\alpha^\beta \gamma_C$. Hence $C \wedge_\alpha^\beta L \subseteq L \circ_\alpha^\beta C$, i.e., (1) implies (4). It is clear that (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Assume that (2) holds. Let A be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ and L be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. Since every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R, this means that A is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R. By our assumption, $A \wedge_\alpha^\beta L \subseteq L \circ_\alpha^\beta A$. Hence R is an intra-regular by the Theorem 13, i.e., (2) $\Rightarrow$ (1).

Theorem 16. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$A \wedge_\alpha^\beta L \wedge_\alpha^\beta D \subseteq (L \circ_\alpha^\beta A) \circ_\alpha^\beta D$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$, every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ of R.*

(3) *$B \wedge_\alpha^\beta L \wedge_\alpha^\beta D \subseteq (L \circ_\alpha^\beta B) \circ_\alpha^\beta D$ for every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$, every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ of R.*

(4) *$C \wedge_\alpha^\beta L \wedge_\alpha^\beta D \subseteq (L \circ_\alpha^\beta C) \circ_\alpha^\beta D$ for every intuitionistic fuzzy generalized bi-ideal C with thresholds $(\alpha, \beta]$, every intuitionistic fuzzy left ideal L with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy right ideal D with thresholds $(\alpha, \beta]$ of R.*

Proof. Assume that (1) holds. Let $C = (\mu_C, \gamma_C)$ be an intuitionistic fuzzy generalized bi-ideal with thresholds $(\alpha, \beta]$, $L = (\mu_L, \gamma_L)$ be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ and $D = (\mu_D, \gamma_D)$ be an intuitionistic

fuzzy right ideal with thresholds $(\alpha, \beta]$ of R. Let $x \in R$, this means that there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$\begin{aligned}
x &= (a_i(xx))b_i = (x(a_i x))b_i = (b_i(a_i x))x \\
\text{and } b_i(a_i x) &= b_i(a_i((a_i x^2)b_i)) = b_i((a_i x^2)(a_i b_i))
\end{aligned}$$

$$
\begin{aligned}
&= b_i((a_ix^2)c_i) = (a_ix^2)(b_ic_i) = (a_ix^2)d_i \\
&= (a_i(xx))d_i = (x(a_ix))d_i = (d_i(a_ix))x.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& ((\mu_L \circ_\alpha^\beta \mu_C) \circ_\alpha^\beta \mu_D)(x) \\
= & \{((\mu_L \circ \mu_C) \circ \mu_D)(x) \wedge \beta\} \vee \alpha \\
= & \left\{\left(\vee_{x=\sum_{i=1}^n p_iq_i} \{\wedge_{i=1}^n \{(\mu_L \circ \mu_C)(p_i) \wedge \mu_D(q_i)\}\}\right) \wedge \beta\right\} \vee \alpha \\
\geq & \{\{(\mu_L \circ \mu_C)(b_i(a_ix)) \wedge \mu_D(x)\} \wedge \beta\} \vee \alpha \\
= & ((\mu_L \circ \mu_C)(b_i(a_ix)) \vee \alpha) \wedge (\mu_D(x) \vee \alpha) \wedge (\beta \vee \alpha) \\
\geq & ((\mu_L \circ \mu_C)(b_i(a_ix)) \vee \alpha) \wedge \mu_D(x) \wedge \beta \\
= & \left(\left(\vee_{b_i(a_ix)=\sum_{i=1}^n m_in_i} \{\wedge_{i=1}^n \{\mu_L(m_i) \wedge \mu_C(n_i)\}\}\right) \vee \alpha\right) \wedge \mu_D(x) \wedge \beta \\
\geq & (\{\mu_L(d_i(a_ix)) \wedge \mu_C(x)\} \vee \alpha) \wedge \mu_D(x) \wedge \beta \\
= & (\mu_L(d_i(a_ix)) \vee \alpha) \wedge (\mu_C(x) \vee \alpha) \wedge \mu_D(x) \wedge \beta \\
\geq & (\mu_L(x) \wedge \beta) \wedge \mu_C(x) \wedge \mu_D(x) \wedge \beta \\
= & \mu_L(x) \wedge \mu_C(x) \wedge \mu_D(x) \wedge \beta = (\mu_L \wedge \mu_C \wedge \mu_D)(x) \wedge \beta \\
= & \{(\mu_C \wedge \mu_L \wedge \mu_D)(x) \wedge \beta\} \vee \alpha \\
= & (\mu_C \wedge_\alpha^\beta \mu_L \wedge_\alpha^\beta \mu_D)(x). \\
\Rightarrow & \mu_C \wedge_\alpha^\beta \mu_L \wedge_\alpha^\beta \mu_D \subseteq (\mu_L \circ_\alpha^\beta \mu_C) \circ_\alpha^\beta \mu_D.
\end{aligned}
$$

Similarly, we have $\mu_C \vee_\alpha^\beta \mu_L \vee_\alpha^\beta \mu_D \supseteq (\mu_L \circ_\alpha^\beta \mu_C) \circ_\alpha^\beta \mu_D$. Hence $C \wedge_\alpha^\beta L \wedge_\alpha^\beta D \subseteq (L \circ_\alpha^\beta C) \circ_\alpha^\beta D$, i.e., (1) implies (4). Since (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2). Suppose that (2) holds. Then $A \wedge_\alpha^\beta R \wedge_\alpha^\beta D \subseteq (R \circ_\alpha^\beta A) \circ_\alpha^\beta D$, where A is an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R, i.e., $A \wedge_\alpha^\beta D \subseteq A \circ_\alpha^\beta D$. Therefore R is an intra-regular, i.e., (2) $\Rightarrow$ (1).

5. Regular and Intra-regular LA-rings

In this section, we characterize both regular and intra-regular LA-rings in terms of intuitionistic fuzzy left (right, quasi-, bi-, generalized bi-) ideals with thresholds $(\alpha, \beta]$.

Theorem 17. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is both a regular and an intra-regular.*

(2) *Every intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy idempotent with thresholds $(\alpha, \beta]$.*

Proof. Suppose that R is both a regular and an intra-regular. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R.Then A be an intuitionistic fuzzy bi-ideal with thresholds $(\alpha, \beta]$ of R and $A \circ_\alpha^\beta A \subseteq A_\alpha^\beta$. Let $x \in R$, this implies that there exists $a \in R$ such that $x = (xa)x$, and also there exist $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Now

$$\begin{aligned}
x &= (xa)x \\
xa &= ((a_i x^2)b_i)a = (ab_i)(a_i x^2) = c_i(a_i(xx)) \\
&= c_i(x(a_i x)) = x(c_i(a_i x)) = x((ec_i)(a_i x)) \\
&= x((xa_i)(c_i e)) = x((xa_i)d_i) = x((d_i a_i)x) \\
&= x(l_i x) = l_i(xx) = (el_i)(xx) = (xx)(l_i e) \\
&= (xx)m_i = (m_i x)x. \\
m_i x &= m_i((a_i x^2)b_i) = (a_i x^2)(m_i b_i) = (a_i(xx))n_i \\
&= (x(a_i x))n_i = (x(a_i x))(en_i) = (xe)((a_i x)n_i) \\
&= (xe)((a_i x)(en_i)) = (xe)((a_i e)(xn_i)) \\
&= (xe)(x((a_i e)n_i)) = (xe)(xu_i) = x((xe)u_i) = xw_i. \\
&\Rightarrow xa = (m_i x)x = (xw_i)x.
\end{aligned}$$

Thus

$$\begin{aligned}
(\mu_A \circ_\alpha^\beta \mu_A)(x) &= \{(\mu_A \circ \mu_A)(x) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} p_i q_i}\{\wedge_{i=1}^{n}\{\mu_A(p_i) \wedge \mu_A(q_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\geq \{\{\mu_A((xw_i)x) \wedge \mu_A(x)\} \wedge \beta\} \vee \alpha \\
&= (\mu_A((xw_i)x) \vee \alpha) \wedge (\mu_A(x) \vee \alpha) \wedge (\beta \vee \alpha) \\
&\geq (\mu_A(x) \wedge \mu_A(x) \wedge \beta) \wedge \mu_A(x) \wedge \beta \\
&= \mu_A(x) \wedge \beta = (\mu_A(x) \wedge \beta) \vee \alpha = (\mu_A)_\alpha^\beta(x). \\
&\Rightarrow (\mu_A)_\alpha^\beta \subseteq \mu_A \circ_\alpha^\beta \mu_A.
\end{aligned}$$

Similarly, we have $(\gamma_A)_\alpha^\beta \supseteq \gamma_A \circ_\alpha^\beta \gamma_A$. Hence $A_\alpha^\beta = A \circ_\alpha^\beta A$. Conversely, assume that every intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy idempotent with thresholds $(\alpha, \beta]$. Let $a \in R$, then Ra is a left ideal of R containing a by the Lemma 19. This implies that Ra is a quasi-ideal of R, so χ_{Ra} is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R by the Theorem 4. By our assumption $(\chi_{Ra})_\alpha^\beta = \chi_{Ra} \circ_\alpha^\beta \chi_{Ra} = (\chi_{(Ra)(Ra)})_\alpha^\beta$, i.e., $Ra = (Ra)(Ra)$. Since $a \in Ra$, i.e., $a \in (Ra)(Ra)$. Thus a is both a regular and an intra-regular by the Theorems 8 and 13, respectively. Hence R is both a regular and an intra-regular, i.e., (2) $\Rightarrow$ (1) .

Theorem 18. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is both a regular and an intra-regular.*

(2) *$A \wedge_\alpha^\beta B \subseteq A \circ_\alpha^\beta B$ for all intuitionistic fuzzy quasi-ideals A and B with thresholds $(\alpha, \beta]$ of R.*

(3) *$A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(4) *$A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ for every intuitionistic fuzzy bi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy quasi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(5) *$A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ for all intuitionistic fuzzy bi-ideals A and B with thresholds $(\alpha, \beta]$ of R.*

(6) *$A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ for every intuitionistic fuzzy bi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy generalized bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(7) *$A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ for every intuitionistic fuzzy generalized bi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy quasi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(8) *$A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ for every intuitionistic fuzzy generalized bi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(9) *$A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ for all intuitionistic fuzzy generalized bi-ideals A and B with thresholds $(\alpha, \beta]$ of R.*

Proof. Assume that (1) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy generalized bi-ideals with thresholds $(\alpha, \beta]$ of R. Let $x \in R$, then means that there exists an element $a \in R$ such that $x = (xa)x$, and also there exist elements $a_i, b_i \in R$ such that $x = \sum_{i=1}^{n}(a_i x^2)b_i$. Since $x = (xa)x = ((xw_i)x)x$ by the Theorem 17. Thus

$$\begin{aligned}
(\mu_A \circ_{\alpha}^{\beta} \mu_B)(x) &= \{(\mu_A \circ \mu_B)(x) \wedge \beta\} \vee \alpha \\
&= \{\left(\vee_{x=\sum_{i=1}^{n} p_i q_i} \{\wedge_{i=1}^{n} \{\mu_A(p_i) \wedge \mu_B(q_i)\}\}\right) \wedge \beta\} \vee \alpha \\
&\geq \{\{\mu_A((xw_i)x) \wedge \mu_B(x)\} \wedge \beta\} \vee \alpha \\
&= (\mu_A((xw_i)x) \vee \alpha) \wedge (\mu_B(x) \vee \alpha) \wedge (\beta \vee \alpha) \\
&\geq (\mu_A(x) \wedge \mu_A(x) \wedge \beta) \wedge \mu_B(x) \wedge \beta \\
&= \mu_A(x) \wedge \mu_B(x) \wedge \beta = (\mu_A \wedge \mu_B)(x) \wedge \beta \\
&= \{(\mu_A \wedge \mu_B)(x) \wedge \beta\} \vee \alpha \\
&= (\mu_A \wedge_{\alpha}^{\beta} \mu_B)(x). \\
&\Rightarrow \mu_A \wedge_{\alpha}^{\beta} \mu_B \subseteq \mu_A \circ_{\alpha}^{\beta} \mu_B.
\end{aligned}$$

Similarly, we have $\gamma_A \vee_{\alpha}^{\beta} \gamma_B \supseteq \gamma_A \circ_{\alpha}^{\beta} \gamma_B$. Therefore $A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$, i.e., (1) implies (9). It is clear that (9) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (4) $\Rightarrow$ (2) and (9) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (3). Suppose that (2) holds. Let A be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ and B be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. Since every intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ and intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R is an intuitionistic fuzzy quasi-ideal with thresholds $(\alpha, \beta]$ of R by the Lemma 14. By our supposition, $A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$. Since $A \circ_{\alpha}^{\beta} B \subseteq A \wedge_{\alpha}^{\beta} B$, so $A \wedge_{\alpha}^{\beta} B = A \circ_{\alpha}^{\beta} B$, i.e., R is a regular. Again by our supposition, $A \wedge_{\alpha}^{\beta} B = B \wedge_{\alpha}^{\beta} A \subseteq B \circ_{\alpha}^{\beta} A$, i.e., R is an intra-regular. Therefore R is both a regular and an intra-regular, i.e., (2) $\Rightarrow$ (1). In similar way, we can prove that (3) $\Rightarrow$ (1).

Theorem 19. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is both a regular and an intra-regular.*

(2) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy right ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy left ideal B with thresholds $(\alpha, \beta]$ of R.*

(3) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy right ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy quasi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(4) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy right ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(5) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy right ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy generalized bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(6) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy left ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy quasi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(7) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy left ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(8) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy left ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy generalized bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(9) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for all intuitionistic fuzzy quasi-ideals A and B with thresholds $(\alpha, \beta]$ of R.*

(10) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(11) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy quasi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy generalized bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(12) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for all intuitionistic fuzzy bi-ideals A with thresholds $(\alpha, \beta]$ and B with thresholds $(\alpha, \beta]$ of R.*

(13) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for every intuitionistic fuzzy bi-ideal A with thresholds $(\alpha, \beta]$ and every intuitionistic fuzzy generalized bi-ideal B with thresholds $(\alpha, \beta]$ of R.*

(14) *$A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$ for all intuitionistic fuzzy generalized bi-ideals A and B with thresholds $(\alpha, \beta]$ of R.*

Proof. Consider that (1) holds. Since $A \wedge_{\alpha}^{\beta} B \subseteq A \circ_{\alpha}^{\beta} B$ and $A \wedge_{\alpha}^{\beta} B \subseteq B \circ_{\alpha}^{\beta} A$ for all intuitionistic fuzzy generalized bi-ideals A and B with thresholds $(\alpha, \beta]$ of R by the Theorem 18. Hence $A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A)$, i.e., $(1) \Rightarrow (14)$. It is clear that $(14) \Rightarrow (13) \Rightarrow (12) \Rightarrow (9) \Rightarrow (6) \Rightarrow (2)$, $(14) \Rightarrow (11) \Rightarrow (10) \Rightarrow (9)$, $(14) \Rightarrow (8) \Rightarrow (7) \Rightarrow (6)$ and $(14) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2)$. Suppose that (2) holds. Let A be an intuitionistic fuzzy right ideal with thresholds $(\alpha, \beta]$ and B be an intuitionistic fuzzy left ideal with thresholds $(\alpha, \beta]$ of R. By our supposition $A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A) \subseteq B \circ_{\alpha}^{\beta} A$, i.e., R is an intra-regular. Again $A \wedge_{\alpha}^{\beta} B \subseteq (A \circ_{\alpha}^{\beta} B) \wedge (B \circ_{\alpha}^{\beta} A) \subseteq A \circ_{\alpha}^{\beta} B$. Since $A \circ_{\alpha}^{\beta} B \subseteq A \wedge_{\alpha}^{\beta} B$, so $A \wedge_{\alpha}^{\beta} B = A \circ_{\alpha}^{\beta} B$, i.e., R is a regular. Hence R is both a regular and an inta-regular, i.e., $(2) \Rightarrow (1)$.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems, 20(1986) 87-96.

[2] K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy sets and systems, 61(1994) 137-142.

[3] B. Banerjee and D. K. Basnet, Intuitionistic fuzzy subrings and ideals, J. Fuzzy Math., 11(2003) 139-155.

[4] S. K. Bhakat and P. Das, On the definition of a fuzzy subgroup, Fuzzy sets and systems, 51(1992) 235-241.

[5] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroups, Fuzzy sets and systems, 80(1996) 359-368.

[6] S. K. Bhakat and P. Das, Fuzzy subrings and ideals redefined, Fuzzy sets and systems, 81(1996) 383-393.

[7] B. Davvaz, $(\in, \in \vee q)$-fuzzy subnear-rings and ideals, Soft Comput., 10(2006) 206-211.

[8] J. R. Cho, J. Jezek and T. Kepka, Paramedial groupoids, Czechoslovak Math. J., 49(1999) 277-290.

[9] K. Hur, S. Y. Jang and H. W. Kang, Intuitionistic fuzzy ideals of a ring, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 12(2005) 193-209.

[10] K. Hur, H. W. Kang and H. k. Song, Intuitionistic fuzzy subgroups and subrings, Honam Math. J., 25(2003) 19-41.

[11] K. C. Gupta and M. K. Kantroo, The intrinsic product of fuzzy subsets of a ring, Fuzzy sets and systems, 57(1993) 103-110.

[12] J. Jezek and T. Kepka, Medial groupoids, Rozpravy CSAV Rada mat. a prir. ved 93/2, 1983, 93 pp.

[13] Y. B. Jun and S. Z. Song, Generalized fuzzy interior ideals in semigroups, Inf. Sci., 176(2006) 3079-3093.

[14] M. S. Kamran, Conditions for LA-semigroups to resemble associative structures, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, 1993.

[15] N. Kausar, A. Waqar, Characterizations of non-associative rings by their intuitionistic fuzzy bi-ideals, European Journal of Pure and Applied Mathematics, 12(2019) 226-250.

[16] N. Kausar, Characterizations of non-associative ordered semigroups by the properties of their fuzzy ideals with thresholds, Prikladnaya Diskretnaya Matematika, 43(2019) 37-59.

[17] N. Kausar, Direct product of finite intuitionistic fuzzy normal subrings over non-associative rings, European Journal of Pure and Applied Mathematics, 12(2019) 622-648.

[18] M. A. Kazim and M. Naseerudin, On almost semigroups, Alig. Bull. Math., 2(1972) 1-7.

[19] N. Kuroki, Regular fuzzy duo rings, Inform. Sci., 94(1996) 119-139.

[20] W. J. Liu, Fuzzy invariant subgroups and ideals, Fuzzy sets and systems, 8(1982) 133-139.

[21] T. K. Mukherjee and M. K. Sen, On fuzzy ideals of a ring 1, Fuzzy sets and systems, 21(1987) 99-104.

[22] T. K. Mukherjee and M. K. Sen, Prime fuzzy ideals in rings, Fuzzy sets and systems, 32(1989) 337-341.

[23] A. L. Narayanan and T. Manikantan, $(\in, \in \vee q)$-fuzzy subnear-rings and $(\in, \in \vee q)$-fuzzy ideals of near-rings, Fuzzy J. Appl. Math. Comput., 18(2005) 419-430.

[24] P. V. Protic and N. Stevanovic, AG-test and some general properties of Abel-Grassmann's groupoids, Pure Math. Appl., 6(1995) 371-383.

[25] T. Shah and I. Rehman, On LA-rings of finitely non-zero functions, Int. J. Contempt. Math. Sci., 5(2010) 209-222.

[26] T. Shah, N. Kausar and I. Rehman, Intuitionistic fuzzy normal subrings over a non-associative ring, An. St. Univ. Ovidius Constanta, 1(2012) 369-386.

[27] T. Shah, N. Kausar, Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals, Theoretical Computer Science, 529(2014), 96-110.

[28] M. Shabir, Y. Nawaz and M. Aslam, Semigroups characterized by the properties of their fuzzy ideals with thresholds, World Appl. Sci. J., 14(2011) 1851-1865.

[29] U. M. Swamy and K. L. N. Swamy, Fuzzy prime ideals of rings, J. Math. Anal. Appl., 134(1988) 94-103.

[30] X. Yuan, C. Zhang and Y. Ren, Generalized fuzzy groups and many-valued implications, Fuzzy sets and systems, 138(2003) 205-211.

[31] L. A. Zadeh, Fuzzy sets, Information and control, 8(1965) 338-363.