



## Cost Effective Domination in the Join, Corona and Composition of Graphs

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**Abstract.** Let  $G$  be a connected graph. A cost effective dominating set in a graph  $G$  is any set  $S$  of vertices of  $G$  satisfying the condition that each vertex in  $S$  is adjacent to at least as many vertices outside  $S$  as inside  $S$  and every vertex outside  $S$  is adjacent to at least one vertex in  $S$ . The minimum cardinality of a cost effective dominating set is the cost effective domination number of  $G$ . The maximum cardinality of a cost effective dominating set is the upper cost effective domination number of  $G$ . A cost effective dominating set is said to be minimal if it does not contain a proper subset which is itself a cost effective dominating in  $G$ . The maximum cardinality of a minimal cost effective dominating set in a graph  $G$  is the minimal cost effective domination number of  $G$ .

In this paper, we characterized the cost effective dominating sets in the join, corona and composition of graphs. As direct consequences, the bounds or the exact cost effective domination numbers, minimal cost effective domination numbers and upper cost effective domination numbers of these graphs were obtained.

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### 1. Introduction

Throughout this paper, we consider simple, finite and undirected connected graphs  $G = (V(G), E(G))$ . All basic graph theoretic concepts used here are adapted from [1]. The symbols  $V(G)$  and  $E(G)$  are the *vertex set* and *edge set*, respectively, of  $G$ . For  $S \subseteq V(G)$ ,  $|S|$  is the cardinality of  $S$ . In particular,  $|V(G)|$  is called the *order* of  $G$ .

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Given graphs  $G$  and  $H$  with disjoint vertex sets, the *join* of  $G$  and  $H$  is the graph  $G+H$  with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$ . The *corona* of  $G$  and  $H$  is the graph  $G \circ H$  obtained by taking one copy of  $G$  and  $|V(G)|$  copies of  $H$ , and then joining the  $i^{\text{th}}$  vertex of  $G$  to every vertex in the  $i^{\text{th}}$  copy of  $H$ . The *composition* (or *lexicographic product*)  $G[H]$  of  $G$  and  $H$  is the graph with  $V(G[H]) = V(G) \times V(H)$  and  $(u, v)(u', v') \in E(G[H])$  if and only if either  $uu' \in E(G)$  or  $u = u'$  and  $vv' \in E(H)$ .

For  $v \in V(G)$ , the *neighborhood* of  $v$  is the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The *degree* of a vertex  $v \in V(G)$ , denoted by  $\deg_G(v)$ , is equal to the cardinality of  $N_G(v)$  and the *maximum degree* of  $G$  is  $\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}$ . A vertex is *isolated* if its degree is zero, and a graph is *isolate-free* if it has no isolated vertices.

For  $S \subseteq V(G)$ ,  $N_G(S) = \cup_{v \in S} N_G(v)$  and  $N_G[S] = S \cup N_G(S)$ . A *dominating set* of  $G$  is any  $S \subseteq V(G)$  for which  $N_G[S] = V(G)$ . The *domination number* of  $G$ , denoted by,  $\gamma(G)$  is the smallest cardinality of a dominating set of  $G$ . A dominating set  $S$  of  $G$  is said to be a *minimal dominating set* if it has no proper subset which is itself a dominating set in  $G$ . The maximum cardinality of a minimal domination set in  $G$  is denoted by  $\gamma_m(G)$ . A dominating set  $S$  is said to be an *independent dominating set* of  $G$  if for every two vertices  $u, v \in S$ ,  $uv \notin E(G)$ . The minimum cardinality of an independent dominating set is called an *independent domination number* and is denoted by  $i(G)$ . We refer to [2–7] for the fundamental concepts and history of the theory of domination in graphs as well as for some of its relevant applications. Investigation of the concept in the join, corona or composition of graphs can be found in [8, 9, 13].

A subset  $S \subseteq V(G)$  is said to be a *cost effective set* of  $G$  if for every  $v \in S$ ,  $|N_G(v) \cap S| \leq |N_G(v) \setminus S|$ . A subset  $S \subseteq V(G)$  is said to be a *very cost effective set* of  $G$  if for every  $v \in S$ ,  $|N_G(v) \cap S| < |N_G(v) \setminus S|$ . A subset  $S \subseteq V(G)$  is said to be a *(very) cost effective dominating set* of  $G$  if  $S$  is both a (very) cost effective set and a dominating set of  $G$ . The minimum (resp. maximum) cardinality of a cost effective dominating set of a graph  $G$  is called the *cost effective domination number* (resp. *upper cost effective domination number*) of  $G$ , and is denoted by  $\gamma_{ce}(G)$  (resp.  $\gamma_{ce}^+(G)$ ). The minimum cardinality of a very cost effective dominating set of a graph  $G$  is called the *very cost effective domination number* of  $G$ , and is denoted by  $\gamma_{vce}(G)$ . An excellent introduction and exposition on cost effective domination in graphs can be found in [11, 12].

A cost effective dominating set  $S \subseteq V(G)$  is a *minimal cost effective set* if  $S$  does not contain a proper subset which is itself a cost effective dominating set. We use the symbol  $\gamma_{mce}(G)$  to denote the maximum cardinality of a minimal cost effective dominating set of  $G$ . It is worth noting that, in particular, an independent dominating set is a minimal cost effective dominating set. Clearly  $\gamma(G) \leq \gamma_{ce}(G) \leq \gamma_{mce}(G) \leq \gamma_{ce}^+(G)$  for all graphs  $G$ .

For simplicity, we use the terms *ced-set*,  $\gamma_{ce}$ -set,  $\gamma_{ce}^+$ -set and  $\gamma_{mce}$ -set to refer to the cost effective dominating set, the cost effective dominating sets with cardinality  $\gamma_{ce}(G)$ ,  $\gamma_{ce}^+(G)$  and  $\gamma_{mce}(G)$ , respectively.

In this paper we characterized the cost effective dominating sets and minimal cost effective dominating sets in the join, corona and composition of graphs. As consequences, we determined the cost effective domination number, minimal cost effective domination

number and upper cost effective domination number of the aforementioned graphs.

### 2. Cost Effective Domination in the Join of Graphs

**Remark 1.** Given two connected graphs  $G$  and  $H$ , a ced-set  $S$  of  $G+H$ , where  $S \subseteq V(G)$ , need not be a ced-set of  $G$  as shown in Example 1.

**Example 1.** Let  $G = K_5 \circ \overline{K_3}$ . Consider the graph  $G + K_1$  as shown in Figure 1.

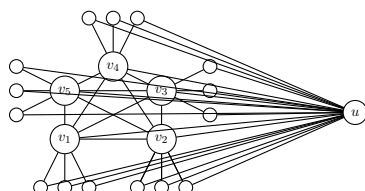


Figure 1: The graph  $G + K_1$

Observe that the set  $\{v_1, v_2, v_3, v_4, v_5\} \subseteq V(G)$  is a cost effective dominating set of  $G + K_1$  but not a cost effective dominating set of  $G$ .

**Proposition 1.** Let  $G$  be a nontrivial connected graph and  $H$  be any graph. If  $S \subseteq V(G)$  is a cost effective dominating set of  $G$ , then  $S$  is a very cost effective dominating set of  $G + H$ .

*Proof.* Let  $S \subseteq V(G)$  be a cost effective dominating set of  $G$ . Then  $S$  is a dominating set of  $G + H$ . For each  $v \in S$ ,

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |N_G(v) \cap S| \\ &\leq |N_G(v) \setminus S| \\ &< |N_G(v) \setminus S| + |V(H)| \\ &= |N_{G+H}(v) \setminus S|. \end{aligned}$$

Thus,  $S$  is a very cost effective dominating set of  $G + H$ . □

**Theorem 1.** Let  $G$  and  $H$  be nontrivial graphs of orders  $m$  and  $n$ , respectively, and let  $S \subseteq V(G + H)$ . Then  $S$  is a cost effective dominating set of  $G + H$  if and only if one of the following holds:

- (i)  $S \subseteq V(G)$  is a dominating set of  $G$  and  $|N_G(v) \cap S| \leq n + |N_G(v) \setminus S|$  for all  $v \in S$
- (ii)  $S \subseteq V(H)$  is a dominating set of  $H$  and  $|N_H(v) \cap S| \leq m + |N_H(v) \setminus S|$  for all  $v \in S$
- (iii)  $S \cap V(G) \neq \emptyset$  and  $S \cap V(H) \neq \emptyset$  satisfying
  - (a)  $|N_G(v) \cap S| + 2|S \cap V(H)| \leq n + |N_G(v) \setminus S|$  for all  $v \in S \cap V(G)$
  - (b)  $|N_H(v) \cap S| + 2|S \cap V(G)| \leq m + |N_H(v) \setminus S|$  for all  $v \in S \cap V(H)$ .

*Proof.* Let  $S \subseteq V(G + H)$ . Suppose that  $S$  is a cost effective dominating set of  $G + H$ . If  $S \subseteq V(G)$ , then  $S$  is a dominating set of  $G$ , and for each  $v \in S$ ,

$$\begin{aligned} |N_G(v) \cap S| &= |N_{G+H}(v) \cap S| \leq |N_{G+H}(v) \setminus S| \\ &= n + |N_G(v) \setminus S|. \end{aligned}$$

Similarly, if  $S \subseteq V(H)$ , then  $S$  is dominating of  $H$  and for each  $v \in S$ ,  $|N_H(v) \cap S| \leq m + |N_H(v) \setminus S|$ . Suppose that  $S_1 = S \cap V(G) \neq \emptyset$  and  $S_2 = S \cap V(H) \neq \emptyset$ . For each  $v \in S_1$ ,

$$\begin{aligned} |N_G(v) \cap S| + |S_2| &= |N_{G+H}(v) \cap S| \\ &\leq |N_{G+H}(v) \setminus S| \\ &= |N_G(v) \setminus S| + |V(H) \setminus S_2| \\ &= |N_G(v) \setminus S| + n - |S_2| \end{aligned}$$

Thus,  $|N_G(v) \cap S| + 2|S_2| \leq n + |N_G(v) \setminus S|$ . Similarly, for each  $v \in S_2$ ,

$$|N_H(v) \cap S| + 2|S_1| \leq m + |N_H(v) \setminus S|.$$

Conversely, suppose that  $S$  satisfies Property (i). Then  $S$  is a dominating set of  $G + H$ . Let  $v \in S$ . Then

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |N_G(v) \cap S| \leq n + |N_G(v) \setminus S| \\ &= |N_{G+H}(v) \setminus S|. \end{aligned}$$

Thus,  $S$  is a cost effective set of  $G + H$ , and the conclusion follows. Similarly, if  $S$  satisfies (ii), then  $S$  is a cost effective dominating set of  $G + H$ .

Finally, suppose that  $S$  satisfies (iii). Then  $S$  is a dominating set of  $G + H$ . For each  $v \in S \cap V(G)$ , we have from Property (iii)(a),

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |N_G(v) \cap S| + |N_H(v) \cap S| \\ &= |N_G(v) \cap S| + |S \cap V(H)| \\ &\leq n + |N_G(v) \setminus S| - 2|S \cap V(H)| + |S \cap V(H)| \\ &= n + |N_G(v) \setminus S| - |S \cap V(H)| \\ &= |N_G(v) \setminus S| + |V(H) \setminus S| \\ &= |N_{G+H}(v) \setminus S|. \end{aligned}$$

Similarly, for each  $v \in S \cap V(H)$ ,  $|N_H(v) \cap S| + 2|S \cap V(G)| \leq m + |N_H(v) \setminus S|$ . Therefore,  $S$  is a cost effective dominating set of  $G + H$ . □

In view of Theorem 1, all independent dominating sets of  $G$  and all independent dominating sets of  $H$  are cost effective dominating sets of  $G + H$ . Moreover, if  $m$  and  $n$  are the orders of  $G$  and  $H$ , respectively, and if  $m \leq n$ , then all dominating sets of  $G$  are cost effective dominating sets of  $G + H$  so that  $\gamma_{ce}(G + H) \leq \gamma(G)$ .

**Corollary 1.** *For any graphs  $G$  and  $H$ ,*

$$\gamma_{ce}(G + H) = \begin{cases} 1, & \text{if } \gamma(G) = 1 \text{ or } \gamma(H) = 1 \\ 2, & \text{otherwise.} \end{cases}$$

*Proof.* Suppose that  $\gamma(G) = 1$ , and let  $S = \{v\}$  be a  $\gamma$ -set of  $G$ . Then,  $|N_{G+H}(v) \cap S| = |N_G(v) \cap S| = 0 < |N_{G+H}(v) \setminus S|$ . Thus,  $S$  is a cost effective dominating set of  $G + H$ , showing that  $\gamma_{ce}(G + H) = 1$ . Similarly, if  $\gamma(H) = 1$ , then  $\gamma_{ce}(G + H) = 1$ . Suppose that  $\gamma(G) \geq 2$  and  $\gamma(H) \geq 2$ . Let  $S = \{u, v\}$ , where  $u \in V(G)$  and  $v \in V(H)$ . Then  $S$  satisfies Theorem 1(iii) so that  $S$  is a cost effective dominating set of  $G + H$ . In this case,  $\gamma_{ce}(G + H) = 2$ . □

**Theorem 2.** *Let  $G$  and  $H$  be nontrivial graphs of orders  $m$  and  $n$ , respectively, and let  $S \subseteq V(G + H)$ . Then  $S$  is a minimal cost effective dominating set of  $G + H$  if and only if one of the following holds:*

- (i)  $S \subseteq V(G)$  is a minimal dominating set of  $G$  and  $|N_G(v) \cap S| \leq n + |N_G(v) \setminus S|$  for all  $v \in S$
- (ii)  $S \subseteq V(H)$  is a minimal dominating set of  $H$  and  $|N_H(v) \cap S| \leq m + |N_H(v) \setminus S|$  for all  $v \in S$
- (iii)  $S = \{u, v\}$ , where  $u \in V(G)$  and  $v \in V(H)$  do not dominate  $V(G)$  and  $V(H)$ , respectively.

*Proof.* Suppose that  $S$  is a minimal cost effective dominating set of  $G + H$ . Suppose that  $S \subseteq V(G)$ . By Theorem 1,  $S$  is a dominating set of  $G$  satisfying

$$|N_G(v) \cap S| \leq n + |N_G(v) \setminus S| \tag{1}$$

for all  $v \in S$ . Suppose that  $S$  is not a minimal dominating set of  $G$ . Then, there exists a dominating set  $S^* \subseteq S$  of  $G$  with  $|S^*| < |S|$ . Since  $|N_{G+H}(v) \cap S^*| = |N_G(v) \cap S^*| \leq |N_G(v) \cap S| = |N_{G+H}(v) \setminus S| \leq |N_{G+H}(v) \setminus S^*|$  for all  $v \in S^*$ ,  $S^*$  is a cost effective dominating set of  $G$ , contrary to the minimality of  $S$ . Thus,  $S$  is a minimal dominating set of  $G$ . Similarly, if  $S \subseteq V(H)$ , then Property (ii) holds. Suppose that  $S \cap V(G) \neq \emptyset$  and  $S \cap V(H) \neq \emptyset$ . Pick any  $u \in S \cap V(G)$  and  $v \in S \cap V(H)$ . Then  $\{u, v\}$  satisfies Theorem 1(iii), and is thus a cost effective dominating set of  $G + H$ . Therefore, in this case, if  $S$  is a minimal cost effective set of  $G + H$ , then  $S = \{u, v\}$  for some  $u \in V(G)$  and  $v \in V(H)$ . Moreover,  $u$  and  $v$  do not dominate  $V(G)$  and  $V(H)$ , respectively.

Conversely, following similar arguments, if Property (i) or Property (ii) holds, then  $S$  is a minimal cost effective dominating set of  $G + H$ . Suppose that  $S = \{u, v\}$ , where  $u \in V(G)$  and  $v \in V(H)$  and  $u$  and  $v$  do not dominate  $V(G)$  and  $V(H)$ , respectively. By Theorem 1,  $S$  is a cost effective dominating set of  $G + H$ . Since  $u$  and  $v$  each does not dominate  $V(G + H)$ ,  $S$  is a minimal cost effective dominating set of  $G + H$ . □

**Corollary 2.** *Let  $G$  and  $H$  be isolate-free graphs with  $G$  noncomplete. Then,*

$$\max\{\gamma_{mce}(G), \gamma_{mce}(H)\} \leq \gamma_{mce}(G + H) \leq \max\{\gamma_m(G), \gamma_m(H)\}.$$

*Proof.* Let  $S \subseteq V(G + H)$  be a  $\gamma_{mce}$ -set of  $G + H$ . In view of Theorem 2, since  $G$  is noncomplete,  $\gamma_{mce}(G + H) \geq 2$ . By the same theorem, if  $S \subseteq V(G)$ , then  $S$  is a minimal dominating set of  $G$  so that  $|S| \leq \gamma_m(G)$ . Similarly, if  $S \subseteq V(H)$ , then  $|S| \leq \gamma_m(H)$ . Hence,  $\gamma_{mce}(G + H) \leq \max\{\gamma_m(G), \gamma_m(H)\}$ .

On the other hand, by Proposition 1 and Theorem 1, every minimal cost effective dominating set of  $G$  is a minimal cost effective dominating set of  $G + H$ . Thus,  $\gamma_{mce}(G) \leq \gamma_{mce}(G + H)$ . Similarly,  $\gamma_{mce}(H) \leq \gamma_{mce}(G + H)$ . Thus,  $\max\{\gamma_{mce}(G), \gamma_{mce}(H)\} \leq \gamma_{mce}(G + H)$ . Therefore,  $\max\{\gamma_{mce}(G), \gamma_{mce}(H)\} \leq \gamma_{mce}(G + H) \leq \max\{\gamma_m(G), \gamma_m(H)\}$ . □

**Corollary 3.** *Let  $G$  be any isolate-free graph and  $m \geq 1$ . Then*

$$\gamma_{mce}(G) \leq \gamma_{mce}(G + K_m) \leq \gamma_m(G).$$

*In particular, if  $G$  is any of the following:  $K_n, K_{r,s}, P_n, C_n$ , then*

$$\gamma_{mce}(G + K_m) = \gamma_{mce}(G).$$

In view of Corollary 3 and results on the minimal dominating sets by [10], the lower and upper bounds in Corollary 2 are sharp.

The following is directly from Proposition 1.

**Proposition 2.** *For any connected graph  $G$ ,*

$$\gamma_{ce}^+(G) \leq \gamma_{ce}^+(G + K_1),$$

*and this bound is sharp.*

Consider the graph  $G = K_n$  for  $n = 2k + 1, k \geq 0$ . Note that

$$\gamma_{ce}^+(K_n) = \left\lfloor \frac{n + 1}{2} \right\rfloor = \left\lfloor \frac{2k + 1 + 1}{2} \right\rfloor = k + 1.$$

Now,  $G + K_1 = K_{n+1}$  where  $n + 1$  is even. Now,

$$\gamma_{ce}^+(G + K_1) = \gamma_{ce}^+(K_{n+1}) = \left\lfloor \frac{(2k + 1 + 1) + 1}{2} \right\rfloor = \left\lfloor \frac{2k + 3}{2} \right\rfloor = k + 1 = \gamma_{ce}^+(G).$$

Thus, the bound in Proposition 2 is sharp.

However, strict inequality in Proposition 2 may be attained as illustrated by the following example.

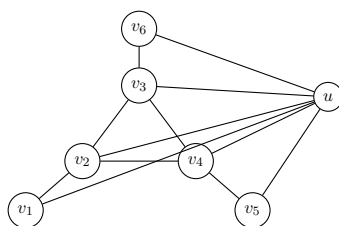


Figure 2:  $G + K_1$

**Example 2.** Let  $G = C_3 \circ K_1$  and  $V(G) = \{v_1, v_2, \dots, v_6\}$ . Consider the join  $G + K_1$  as shown in Figure 2.

Observe that the set  $\{v_2, v_5, v_6\}$  is a  $\gamma_{ce}^+$ -set of  $G$ . On the other hand, the set  $\{v_2, v_5, v_6, u\}$  is a  $\gamma_{ce}^+$ -set in  $G + K_1$ . Thus,

$$\gamma_{ce}^+(G) = 3 < 4 = \gamma_{ce}^+(G + K_1).$$

**Remark 2.** [11] For any connected graph  $G$  of order  $n \geq 2$ ,  $\gamma_{ce}(G) \leq \lfloor \frac{n}{2} \rfloor$ .

**Theorem 3.** Let  $G$  and  $H$  be any connected nontrivial graphs. Then,

$$\max \{ \gamma_{ce}^+(G), \gamma_{ce}^+(H), \alpha + \beta \} \leq \gamma_{ce}^+(G + H),$$

where

$$\alpha = \max \left\{ |S| : S \subseteq V(G) \text{ is a ced-set of } G \text{ with } |S| \leq \left\lfloor \frac{m}{2} \right\rfloor \right\},$$

and

$$\beta = \max \left\{ |S| : S \subseteq V(H) \text{ is a ced-set of } H \text{ with } |S| \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

*Proof.* First, note that the existence of  $\alpha$  and  $\beta$  is guaranteed by Remark 2. By Proposition 1 and Theorem 1, every  $\gamma_{ce}^+$ -set of  $G$  is a cost effective dominating set of  $G + H$ . Hence,

$$\gamma_{ce}^+(G + H) \geq \gamma_{ce}^+(G).$$

Similarly,

$$\gamma_{ce}^+(G + H) \geq \gamma_{ce}^+(H).$$

Now, let  $S_1 \subseteq V(G)$  be a cost effective dominating set of  $G$  with  $|S_1| \leq \lfloor \frac{m}{2} \rfloor$ , and  $S_2 \subseteq V(H)$  a cost effective dominating set of  $H$  with  $|S_2| \leq \lfloor \frac{n}{2} \rfloor$ , and put  $S = S_1 \cup S_2$ . For each  $v \in S_1$ ,

$$|N_G(v) \cap S| + 2|S_2| \leq |N_G(v) \setminus S| + 2 \left\lfloor \frac{n}{2} \right\rfloor \leq |N_G(v) \setminus S| + n.$$

Similarly, for each  $v \in S_2$ ,

$$|N_G(v) \cap S| + 2|S_1| \leq |N_G(v) \setminus S| + m.$$

By Theorem 1,  $S$  is a cost effective dominating set of  $G + H$ . Thus,  $|S| \leq \gamma_{ce}^+(G + H)$ . Since  $S_1$  and  $S_2$  are arbitrary,  $\alpha + \beta \leq \gamma_{ce}^+(G + H)$ . The inequality follows immediately.  $\square$

**Theorem 4.** *Let  $G$  be a connected graph of order  $m$ . Then for  $n \geq m$ ,*

$$\gamma_{ce}^+(G + K_n) \leq \frac{m + n + 1}{2}.$$

*Proof.* Let  $S \subseteq V(G + K_n)$  be a  $\gamma_{ce}^+$ -set of  $G + K_n$ . Since  $S$  is a  $\gamma_{ce}^+$ -set and  $V(G)$  satisfies Theorem 1(i), if  $S \subseteq V(G)$ , then  $S = V(G)$ . Suppose that  $S \subseteq V(K_n)$ . Then  $S$  is dominating set of  $K_n$  and for a given  $v \in S$ ,  $|S| - 1 = |N_{K_n}(v) \cap S| \leq m + |N_{K_n}(v) \setminus S| = m + n - |S|$  so that  $|S| \leq \frac{m+n+1}{2}$ . Now suppose that  $S_1 = S \cap V(G) \neq \emptyset$  and  $S_2 = S \cap V(K_n) \neq \emptyset$ . By Theorem 1, in particular, for each  $u \in S_2$ ,

$$\begin{aligned} 2|S_1| &\leq m + |N_{K_n}(u) \setminus S| - |N_{K_n}(u) \cap S| \\ &= m + n - 2|S_2| + 1, \end{aligned}$$

or equivalently,

$$|S| \leq \frac{m + n + 1}{2}.$$

□

### 3. Cost Effective Domination in the Corona of Graphs

**Theorem 5.** [8] *Let  $G$  be a connected graph of order  $n$  and let  $H$  be any graph of order  $m$ . Then  $S \subseteq V(G \circ H)$  is a dominating set of  $G \circ H$  if and only if  $S \cap V(H^v + v)$  is a dominating set of  $H^v + v$  for each  $v \in V(G)$ .*

**Corollary 4.** *Let  $G$  be a connected graph and  $H$  be any graph, and  $S \subseteq V(G \circ H)$ . If  $S$  is a cost effective dominating set of  $G \circ H$ , then  $S \cap V(H^v + v)$  is a dominating set of  $H^v + v$  for each  $v \in V(G)$ .*

Corollary 4 guarantees that if  $S$  is a cost effective dominating set of  $G \circ H$ , then  $S \cap V(H^v + v) \neq \emptyset$  for each  $v \in V(G)$ .

**Proposition 3.** *Let  $G$  be a connected graph and  $H$  be any isolate-free graph. If for each  $v \in V(G)$ ,  $S_v \subseteq V(H^v)$  is a cost effective dominating set of  $H^v$ , then  $\cup_{v \in V(G)} S_v$  is a cost effective dominating set of  $G \circ H$ .*

*Proof.* For each  $v \in V(G)$ , let  $S_v \subseteq V(H^v)$  be a cost effective dominating set of  $H^v$ . Then  $S_v$  is a very cost effective dominating set of  $H^v + v$ , by Proposition 1. Let  $S = \cup_{v \in V(G)} S_v$ . Then  $S$  is a dominating set of  $G \circ H$ . Let  $u \in S_v$ . Then,

$$\begin{aligned} |N_{G \circ H}(u) \cap S| &= |N_{H^v + v}(u) \cap S_v| \\ &\leq |N_{H^v + v}(u) \setminus S_v| \\ &= |N_{G \circ H}(u) \setminus S|, \end{aligned}$$

showing that  $S$  is a cost effective dominating set of  $G \circ H$ . □



**Proposition 4.** *Let  $G$  be a connected graph and  $H$  be any graph of order  $n \geq 2$ . For each  $v \in V(G)$  with  $\deg_G(v) \leq n$ , let  $S_v = \{v\}$ , and for each  $v \in V(G)$  with  $\deg_G(v) > n$ , let  $S_v \subseteq V(H^v)$  be a cost effective dominating set of  $H^v$ . Then  $\cup_{v \in V(G)} S_v$  is a cost effective dominating set of  $G \circ H$ . Consequently,*

$$\gamma_{ce}(G \circ H) \leq |V(G)| + (\gamma_{ce}(H) - 1)|L|,$$

where  $L = \{v \in V(G) : \deg_G(v) > n\}$ .

*Proof.* Let  $S = \cup_{v \in V(G)} S_v$ . Then  $S$  is a dominating set of  $G \circ H$ . Let  $v \in V(G)$  with  $\deg_G(v) \leq n$ . Then  $|N_{G \circ H}(v) \cap S| \leq \deg_G(v) \leq n \leq |N_{G \circ H}(v) \setminus S|$ . Suppose that  $\deg_G(v) > n$ . Then  $S_v$  is a cost effective dominating set of  $H^v$ . By Proposition 1,  $S_v$  is a very cost effective dominating set of  $H^v + v$ . Thus, for each  $u \in S_v$ ,

$$\begin{aligned} |N_{G \circ H}(u) \cap S| &= |N_{H^v+v}(u) \cap S_v| \\ &< |N_{H^v+v}(u) \setminus S_v| \\ &= |N_{G \circ H}(u) \setminus S|. \end{aligned}$$

Therefore,  $S$  is a cost effective dominating set of  $G \circ H$ . □

**Corollary 5.** *For any nontrivial connected graph  $G$  of order  $m$  and any graph  $H$ , if  $\Delta(G) \leq |V(H)|$ , then*

$$\gamma_{ce}(G \circ H) = m.$$

**Definition 1.** *Let  $G$  be any graph. A subset  $S \subseteq V(G)$  is called a  $K_n$ -cost effective set of  $G$  if  $S$  is a cost effective set of  $K_n + G$ . A  $K_n$ -cost effective set which is dominating in  $G$  is called a  $K_n$ -cost effective dominating set of  $G$ . A  $K_n$ -cost effective dominating set is called minimal  $K_n$ -cost effective dominating set if it does not contain a proper subset that is itself  $K_n$ -cost effective dominating set.*

The symbols  $\gamma_{K_n ce}(G)$ ,  $\gamma_{K_n mce}(G)$  and  $\gamma_{K_n ce}^+(G)$  denote the minimum cardinality of a  $K_n$ -cost effective dominating set, the maximum cardinality of a minimal  $K_n$ -cost effective dominating set and maximum cardinality of a  $K_n$ -cost effective dominating set, respectively, in  $G$ .

**Remark 3.** *Every cost effective (dominating) set of  $G$  is a  $K_n$ -cost effective (dominating) set of  $G$ . Consequently,  $\gamma_{K_n ce}(G) \leq \gamma_{ce}(G)$  and  $\gamma_{ce}^+(G) \leq \gamma_{K_n ce}^+(G)$ .*

**Remark 4.** *If  $C_1, C_2, \dots, C_n$  are the components of a graph  $G$  and  $S \subseteq V(G)$ , then  $S$  is a  $K_n$ -cost effective dominating set of  $G$  if and only if  $S \cap V(C_k)$  is a  $K_n$ -cost effective dominating set of  $C_k$  for all  $k = 1, 2, \dots, n$ .*

**Lemma 1.** *Let  $G$  be a connected graph and  $H$  any graph, and let  $S \subseteq V(G \circ H)$  and  $v \in V(G) \setminus S$ . If  $S$  is a cost effective dominating set of  $G \circ H$ , then  $S \cap V(H^v)$  is a  $K_1$ -cost effective dominating set of  $H^v$ .*

*Proof.* Suppose that  $S$  is a cost effective dominating set of  $G \circ H$  and  $v \in V(G) \setminus S$ . Let  $S_v = S \cap V(H^v)$  and let  $u \in S_v$ . Then,  $|N_{H^v+v}(u) \cap S_v| = |N_{G \circ H}(u) \cap S| \leq |N_{G \circ H}(u) \setminus S| = |N_{H^v+v}(u) \setminus S_v|$ .  $\square$

**Theorem 6.** *Let  $G$  be a connected graph and  $H$  an isolate-free graph of order  $n$ , and let  $S \subseteq V(G \circ H)$ . Then  $S$  is a cost effective dominating set of  $G \circ H$  if and only if the following hold:*

(i) *For each  $v \in S \cap V(G)$ ,  $S \cap V(H^v)$  is a cost effective set of  $H^v$  satisfying*

$$|S \cap V(H^v)| \leq \frac{1}{2} (n + |N_G(v) \setminus S| - |N_G(v) \cap S|).$$

(ii) *For each  $v \in V(G) \setminus S$ ,  $S \cap V(H^v)$  is a  $K_1$ -cost effective dominating set of  $H^v$*

*Proof.* Suppose that  $S$  is a cost effective dominating set of  $G \circ H$ . By Corollary 4,  $S \cap V(H^v + v)$  is a dominating set of  $H^v + v$  for each  $v \in V(G)$ . Let  $v \in S \cap V(G)$ , and put  $S_v = S \cap V(H^v)$ . We claim that  $S_v$  is a cost effective set of  $H^v$ . Let  $u \in S_v$ . Then

$$\begin{aligned} 1 + |N_{H^v}(u) \cap S_v| &= |N_{H^v+v}(u) \cap S| \\ &= |N_{G \circ H}(u) \cap S| \\ &\leq |N_{G \circ H}(u) \setminus S| \\ &= |N_{H^v}(u) \setminus S_v|. \end{aligned}$$

Necessarily,  $|N_{H^v}(u) \cap S_v| < |N_{H^v}(u) \setminus S_v|$ . Since  $u$  is arbitrary,  $S_v$  is a cost effective set of  $H^v$ . Further, since  $v \in S$ ,

$$\begin{aligned} |N_G(v) \cap S| + |S_v| &= |N_{G \circ H}(v) \cap S| \\ &\leq |N_{G \circ H}(v) \setminus S| \\ &= |N_G(v) \setminus S| + n - |S_v|, \end{aligned}$$

or equivalently,

$$|S_v| \leq \frac{1}{2} (n + |N_G(v) \setminus S| - |N_G(v) \cap S|).$$

This establishes Property (i). Property (ii) follows immediately from Lemma 1.

Conversely, suppose that  $S$  satisfies all the above prescribed properties. In any case, for  $v \in V(G)$ ,  $S \cap V(H^v + v)$  is a dominating set of  $H^v + v$ . Thus,  $S$  is a dominating set of  $G \circ H$  by Theorem 5. Let  $u \in S$ . In view of Corollary 4,  $u \in S \cap V(H^v + v)$  for some  $v \in V(G)$ . Suppose that  $v \in S$ . By Property (i),  $S_v = S \cap V(H^v)$  is a cost effective set of  $H^v$  satisfying  $|S_v| \leq \frac{1}{2} (n + |N_G(v) \setminus S| - |N_G(v) \cap S|)$ . If  $u = v$ , then

$$\begin{aligned} |N_{G \circ H}(u) \cap S| &= |N_G(u) \cap S| + |S_v| \\ &\leq |N_G(u) \cap S| + n + |N_G(v) \setminus S| - |N_G(u) \cap S| - |S_v| \\ &= |N_G(v) \setminus S| + |V(H^v) \setminus S_v| \end{aligned}$$

$$= |N_{G \circ H}(u) \setminus S|.$$

If  $u \in S_v$ , then

$$\begin{aligned} |N_{G \circ H}(u) \cap S| &= |N_{H^v}(u) \cap S_v| \\ &\leq |N_{H^v}(u) \setminus S_v| \\ &= |N_{G \circ H}(u) \setminus S|. \end{aligned}$$

Now, suppose that  $v \notin S$ . Then, by Property (ii),  $S_v = S \cap V(H^v)$  is a  $K_1$ -cost effective dominating set of  $H^v$ . Since  $N_{G \circ H}(u) \cap S = N_{H^v+v}(u) \cap S_v$  and  $N_{G \circ H}(u) \setminus S = N_{H^v+v}(u) \setminus S_v$ , the desired inequality follows. Therefore,  $S$  is a cost effective set of  $G \circ H$ .  $\square$

**Remark 5.** *The conclusion in Theorem 6 still holds even if the graph  $H$  contains an isolated vertex. Suppose that  $u$  is an isolated vertex of  $H$ . For  $v \in S \cap V(G)$ ,  $u \notin S \cap V(H^v)$ . In fact, if  $H$  is an empty graph, then  $S \cap V(H^v) = \emptyset$  which is a cost effective set, and the claim in the necessity part holds.*

**Lemma 2.** *Let  $G$  be an isolate-free graph. Every cost effective set of maximum cardinality is a dominating set of  $G$ .*

*Proof.* Let  $S \subseteq V(G)$  be a cost effective set of  $G$  of maximum cardinality. Suppose that  $S$  is not a dominating set of  $G$ , and let  $v \in V(G) \setminus N_G[S]$ . Define  $S^* = S \cup \{v\}$ . For each  $u \in S$ ,  $N_G(u) \cap S^* = N_G(u) \cap S$  and  $N_G(u) \setminus S^* = N_G(u) \setminus S$  so that  $|N_G(u) \cap S^*| \leq |N_G(u) \setminus S^*|$ . We also have

$$|N_G(v) \cap S^*| = 0 \leq |N_G(v) \setminus S^*|.$$

Thus,  $S^*$  is a cost effective set of  $G$ , contradicting the assumption on  $S$  being a cost effective set of maximum cardinality. Therefore,  $S$  is a dominating set of  $G$ .  $\square$

**Corollary 6.** *Let  $G$  and  $H$  be nontrivial connected graphs of orders  $m$  and  $n$ , respectively. Then*

- (i)  $\gamma_{ce}(G \circ H) = m\gamma_{K_1ce}(H) - (\gamma_{K_1ce}(H) - 1)\gamma_{K_nce}^+(G)$  whenever  $\Delta(G) > n$ ;
- (ii)  $\gamma_{mce}(G \circ H) = m\gamma_{K_1mce}(H)$ ; and
- (iii)  $m\gamma_{K_1ce}^+(H) \leq \gamma_{ce}^+(G \circ H) \leq |L| + m\gamma_{K_1ce}^+(H)$ , where  $L = \{v \in V(G) : deg_G(v) \geq 2\gamma_{K_1ce}^+(H) - n\}$ .

*Proof.* Let  $D \subseteq V(G)$  be a  $\gamma_{K_nce}^+$  set of  $G$ . Note that if  $\Delta(G) > n$ , then  $D \neq V(G)$ . For each  $v \in V(G) \setminus D$ , let  $S_v \subseteq V(H^v)$  be a  $\gamma_{K_1ce}$ -set of  $H^v$ . Put  $S = D \cup \left( \bigcup_{v \in V(G) \setminus D} S_v \right)$ .

For each  $v \in S \cap V(G) = D$ ,  $S \cap V(H^v) = \emptyset$  so that

$$|S \cap V(H^v)| = 0 \leq |V(H)| + |N_G(v) \setminus S| - |N_G(v) \cap S|.$$

For each  $v \in V(G) \setminus S = V(G) \setminus D$ ,  $S_v = S \cap V(H^v)$  is a  $K_1$ -cost effective dominating set of  $H^v$ . By Theorem 6,  $S$  is a cost effective dominating set of  $G \circ H$ , and

$$\begin{aligned} \gamma_{ce}(G \circ H) \leq |S| &= |D| + \sum_{v \in V(G) \setminus D} |S_v| \\ &= \gamma_{K_n ce}^+(G) + (|V(G)| - \gamma_{K_n ce}^+(G)) \gamma_{K_1 ce}(H) \\ &= |V(G)| \gamma_{K_1 ce}(H) - (\gamma_{K_1 ce}(H) - 1) \gamma_{K_n ce}^+(G). \end{aligned}$$

Conversely, let  $S \subseteq V(G \circ H)$  be a  $\gamma_{ce}$ -set of  $G \circ H$ . In view of Theorem 6 we can write

$$|S| = \sum_{v \in S \cap V(G)} (1 + |S \cap V(H^v)|) + \sum_{v \in V(G) \setminus S} \gamma_{K_1 ce}(H).$$

Now  $S$  can be made as small as desired if  $S \cap V(H^v)$  can be made  $\emptyset$  for all  $v \in S \cap V(G)$ . This is attained when  $S \cap V(G)$  is a  $K_n$ -cost effective set of  $G$  so that  $|S \cap V(G)| \leq \gamma_{K_n ce}^+(G)$  and  $|V(G) \setminus S| \geq |V(G)| - \gamma_{K_n ce}^+(G)$  by Lemma 2. Therefore,

$$\begin{aligned} \gamma_{ce}(G \circ H) = |S| &= |S \cap V(G)| + \sum_{v \in V(G) \setminus S} \gamma_{K_1 ce}(H) \\ &\geq \gamma_{K_n ce}^+(G) + (|V(G)| - \gamma_{ce}^+(G)) \gamma_{K_1 ce}(H) \end{aligned}$$

This proves Statement (i).

To prove Statement (ii), let  $S = \bigcup_{v \in V(G)} S_v$ , where  $S_v \subseteq V(H^v)$  is a  $\gamma_{K_1 mce}$ -set of  $H^v$

for each  $v \in V(G)$ . By Theorem 6,  $S$  is a cost effective dominating set of  $G \circ H$ . Since  $S_v$  is a minimal cost effective dominating set of  $H^v + v$  for each  $v \in V(G)$ ,  $S$  is a minimal cost effective dominating set of  $G \circ H$ . Thus,

$$\gamma_{mce}(G \circ H) \geq |S| = |V(G)| \gamma_{K_1 mce}(H).$$

Conversely, let  $S \subseteq V(G \circ H)$  be a  $\gamma_{mce}$ -set of  $G \circ H$ . Let  $v \in V(G) \setminus S$ . By Theorem 6(ii) and the minimality of  $S$ ,  $S \cap V(H^v)$  is a minimal  $K_1$ -cost effective dominating set of  $H^v$ . Let  $v \in S \cap V(G)$ . By Theorem 6(i),  $S_v = S \cap V(H^v)$  is a cost effective set of  $H^v$  satisfying

$$|S_v| \leq \frac{1}{2} (n + |N_G(v) \setminus S| - |N_G(v) \cap S|) < \frac{n}{2}.$$

Thus,

$$\begin{aligned} |N_G(v) \cap S| \leq |N_G(v) \cap S| + 2|S_v| &= |N_{G \circ H}(v) \cap S| + |S_v| \\ &\leq |N_{G \circ H}(v) \setminus S| + |S_v| \\ &= |N_G(v) \setminus S| + n. \end{aligned}$$

Let  $S^* = S \setminus S_v$ . Since  $v$  dominates  $V(H^v + v)$ ,  $S^*$  is a dominating set of  $G \circ H$ . Note that each  $u \in S^* \setminus \{v\}$  is cost effective relative to  $S^*$  as it is relative to  $S$  in  $G \circ H$ . Now,

$$|N_{G \circ H}(v) \cap S^*| = |N_G(v) \cap S|$$

$$\begin{aligned} &\leq |N_G(v) \setminus S| + n \\ &= |N_{G \circ H}(v) \setminus S^*|. \end{aligned}$$

Since  $S$  is minimal,  $S = S^*$ , and  $S_v = \emptyset$ . Since  $v$  is arbitrary,

$$\begin{aligned} \gamma_{mce}(G \circ H) = |S| &= \sum_{v \in S \cap V(G)} |S \cap V(H^v + v)| + \sum_{v \in V(G) \setminus S} |S \cap V(H^v)| \\ &\leq \sum_{v \in V(G)} \gamma_{K_1mce}(H). \end{aligned}$$

Finally, we prove Statement (iii). For each  $v \in V(G)$ , let  $S_v \subseteq V(H^v)$  be a  $\gamma_{K_1ce}^+$ -set of  $H^v$ . By Proposition 3,  $S = \cup_{v \in V(G)} S_v$  is a cost effective dominating set of  $G \circ H$ . Thus

$$\gamma_{ce}^+(G \circ H) \geq m\gamma_{K_1ce}^+(H).$$

Now, suppose that  $\gamma_{ce}^+(G \circ H) > m\gamma_{K_1ce}^+(H)$ , and let  $S \subseteq V(G \circ H)$  be a  $\gamma_{ce}^+$ -set of  $G \circ H$ . Then there exists  $v \in S \cap V(G)$  such that  $|S \cap V(H^v + v)| = 1 + \gamma_{K_1ce}^+(H)$ . Thus,  $S_v = S \cap V(H^v)$  is a very cost effective set of  $H^v$  of cardinality  $\gamma_{K_1ce}^+(H)$  and satisfying

$$\gamma_{K_1ce}^+(H) \leq \frac{1}{2} (n + |N_G(v) \setminus S| - |N_G(v) \cap S|) \leq \frac{1}{2} (n + deg_G(v)).$$

In other words,  $deg_G(v) \geq 2\gamma_{K_1ce}^+(H) - n$ . Thus,  $v \in L$ . Therefore,

$$\begin{aligned} \gamma_{ce}^+(G \circ H) &\leq |L| (1 + \gamma_{K_1ce}^+(H)) + (m - |L|) \gamma_{K_1ce}^+(H) \\ &= |L| + m\gamma_{K_1ce}^+(H). \end{aligned}$$

□

**Remark 6.** The bounds in Corollary 4.2.10 are sharp. Note, for example that

$$\gamma_{ce}^+(P_3 \circ K_4) = 9 = 3\gamma_{K_1ce}^+(K_4).$$

Verify also that  $\gamma_{ce}^+(P_3 \circ K_{1,3}) = 10 = 1 + 3\gamma_{K_1ce}^+(K_{1,3})$ .

**Corollary 7.** Let  $G$  be a connected graph and  $m \geq 2$ . Then

- (i)  $\gamma_{ce}(G \circ K_m) = |V(G)|$ ;
- (ii)  $\gamma_{mce}(G \circ K_m) = |V(G)|$ ; and
- (iii)  $\gamma_{ce}^+(G \circ K_m) = |V(G)| \lfloor \frac{m+2}{2} \rfloor$ .

*Proof.* Statement (i) follows from Corollary 5 and Corollary 6. Corollary 6 also yields Statement (ii) and Statement (iii) and the fact that  $\gamma_{K_1ce}(K_m) = 1$ ,  $\gamma_{K_1mce}(K_m) = 1$  and  $\gamma_{K_1ce}^+(K_m) = \lfloor \frac{m+2}{2} \rfloor$ . □

**Example 3.** If  $G$  is either the path  $P_n$  or the cycle  $C_n$  of order  $n$  and  $m \geq 2$ , then

- (i)  $\gamma_{ce}(G \circ K_m) = n$ ;
- (ii)  $\gamma_{mce}(G \circ K_m) = n$ ; and
- (iii)  $\gamma_{ce}^+(G \circ K_m) = n \lfloor \frac{m+2}{2} \rfloor$ .

**Example 4.** For the complete graph  $K_n$  of order  $n \geq 2$  and  $m \geq 2$ ,

- (i)  $\gamma_{ce}(K_n \circ K_m) = n$ ;
- (ii)  $\gamma_{mce}(K_n \circ K_m) = n$ ; and
- (iii)  $\gamma_{ce}^+(K_n \circ K_m) = n \lfloor \frac{m+2}{2} \rfloor$ .

**Proposition 5.** Let  $G$  be a connected graph and  $H$  the union of  $k$  isolated vertices and  $K$  isolate-free subgraph. Then

- (i)  $\gamma_{ce}(G \circ H) = |V(G)|(k + \gamma_{K_1ce}(K)) - \gamma_{K_nce}^+(G)(\gamma_{K_1ce}(K) + k - 1)$ ;
- (ii)  $\gamma_{mce}(G \circ H) = |V(G)|(k + \gamma_{K_1mce}(K))$ ; and
- (iii)  $\gamma_{ce}^+(G \circ H) = |V(G)|(k + \gamma_{K_1ce}^+(K))$ .

*Proof.* Let  $D \subseteq V(G)$  be a  $K_n$ -cost effective set of  $G$  of maximum cardinality. For each  $v \in D$ , define  $S_v = \{v\}$ , and for each  $v \in V(G) \setminus D$ , let  $S_v = (L_v \cup P_v) \subseteq V(H^v)$  where  $L_v = \cup_{i=1}^k \{u_i\}$  be the union of  $k$  isolated vertices and  $P_v \subseteq V(K^v)$  be a  $\gamma_{K_1ce}$ -set of  $K^v$ . Put  $S = \cup_{v \in V(G)} S_v$ . For each  $v \in S \cap V(G) = D$ ,  $S \cap V(H^v) = \emptyset$  so that

$$|S \cap V(H^v)| = 0 \leq |V(H)| + |N_G(v) \setminus S| - |N_G(v) \cap S|.$$

For each  $v \in V(G) \setminus S = V(G) \setminus D$ ,  $S \cap V(H^v) = S_v$  is a  $K_1$ -cost effective dominating set of  $H^v$ . By Remark 6,  $S$  is a cost effective dominating set of  $G \circ H$ , and

$$\begin{aligned} \gamma_{ce}(G \circ H) \leq |S| &= |D| + \sum_{v \in V(G) \setminus D} |S_v| \\ &= \gamma_{K_nce}^+(G) + (|V(G)| - \gamma_{K_nce}^+(G))(k + \gamma_{K_1ce}(K)) \\ &= |V(G)|(k + \gamma_{K_1ce}(K)) - (\gamma_{K_1ce}(K) + k - 1)\gamma_{K_nce}^+(G). \end{aligned}$$

Conversely, let  $S \subseteq V(G \circ H)$  be a  $\gamma_{ce}$ -set of  $G \circ H$ . In view of Theorem 6 we can write

$$|S| = \sum_{v \in S \cap V(G)} (1 + |S \cap V(H^v)|) + \sum_{v \in V(G) \setminus S} \gamma_{K_1ce}(K) + k.$$

Now  $S$  can be made as small as desired if  $S \cap V(H^v)$  can be made  $\emptyset$  for all  $v \in S \cap V(G)$ . This is attained when  $S \cap V(G)$  is a  $K_n$ -cost effective set of  $G$  so that  $|S \cap V(G)| \leq \gamma_{K_nce}^+(G)$  and  $|V(G) \setminus S| \geq |V(G)| - \gamma_{K_nce}^+(G)$ . Therefore,

$$\gamma_{ce}(G \circ H) = |S| = |S \cap V(G)| + \sum_{v \in V(G) \setminus S} (\gamma_{K_1ce}(K) + k)$$

$$\begin{aligned} &\geq \gamma_{K_n^{+ce}}^+(G) + (|V(G)| - \gamma_{K_n^{+ce}}^+(G)) (k + \gamma_{K_1^{ce}}(K)) \\ &= |V(G)| (k + \gamma_{K_1^{ce}}(K)) - \gamma_{K_n^{+ce}}^+(G) (\gamma_{K_1^{ce}}(K) + k - 1) \end{aligned}$$

This proves Statement (i).

To prove Statement (ii), let  $H$  be of order  $n$  and  $S = \cup_{v \in V(G)} S_v$ , where  $S_v = L_v \cup P_v \subseteq V(H^v)$  is a  $\gamma_{K_1^{mce}}$ -set of  $H^v$  for each  $v \in V(G)$ , with  $L_v = \cup_{i=1}^k \{u_i\}$  a union of  $k$  isolated vertices and  $P_v \subseteq V(K^v)$  be a  $\gamma_{K_1^{mce}}$ -set of  $K^v$ . By Theorem 6,  $S$  is a cost effective dominating set of  $G \circ H$ . Since  $S_v$  is a minimal cost effective dominating set of  $H^v + v$  for each  $v \in V(G)$ ,  $S$  is a minimal cost effective dominating set of  $G \circ H$ . Thus,

$$\gamma_{mce}(G \circ H) \geq |S| = |V(G)| (k + \gamma_{K_1^{mce}}(K)).$$

Conversely, let  $S \subseteq V(G \circ H)$  be a  $\gamma_{mce}$ -set of  $G \circ H$ . Let  $v \in V(G) \setminus S$ . By Theorem 6(ii) and the minimality of  $S$ ,  $S \cap V(H^v)$  is a minimal  $K_1$ -cost effective dominating set of  $H^v$ . Let  $v \in S \cap V(G)$ . If  $v$  is cost effective relative to  $S \cap V(G)$ , then the minimality of  $S$  implies that  $S \cap V(H^v + v) = \{v\}$ . Suppose that  $v$  is not cost effective relative to  $S \cap V(G)$ . By Theorem 6(i),  $S_v = S \cap V(H^v)$  is a cost effective set of  $H^v$  satisfying

$$|S_v| \leq \frac{1}{2} (n + |N_G(v) \setminus S| - |N_G(v) \cap S|) < \frac{n}{2}.$$

Thus,

$$\begin{aligned} |N_G(v) \cap S| &\leq |N_G(v) \cap S| + 2|S_v| = |N_{G \circ H}(v) \cap S| + |S_v| \\ &\leq |N_{G \circ H}(v) \setminus S| + |S_v| \\ &= |N_G(v) \setminus S| + n. \end{aligned}$$

Let  $S^* = S \setminus S_v$ . Since  $v$  dominates  $V(H^v + v)$ ,  $S^*$  is a dominating set of  $G \circ H$ . Clearly, each  $u \in S^* \setminus \{v\}$  is cost effective relative to  $S^*$  as it is relative to  $S$  in  $G \circ H$ . Now,

$$\begin{aligned} |N_{G \circ H}(v) \cap S^*| &= |N_G(v) \cap S| \\ &\leq |N_G(v) \setminus S| + n \\ &= |N_{G \circ H}(v) \setminus S^*|. \end{aligned}$$

Since  $S$  is minimal,  $S = S^*$ , and  $S_v = \emptyset$ . Thus,

$$\begin{aligned} \gamma_{mce}(G \circ H) = |S| &= \sum_{v \in S \cap V(G)} |S \cap V(H^v + v)| + \sum_{v \in V(G) \setminus S} |S \cap V(H^v)| \\ &\leq \sum_{v \in V(G)} (k + \gamma_{K_1^{mce}}(K)) \\ &= |V(G)| (k + \gamma_{K_1^{mce}}(K)). \end{aligned}$$

Finally, we prove Statement (iii). For each  $v \in V(G)$ , let  $S_v \subseteq V(H^v)$  be a  $\gamma_{K_1^{+ce}}^+$ -set of  $H^v$ , with  $S_v = L_v \cup P_v \subseteq V(H^v)$  where  $L_v = \cup_{i=1}^k \{u_i\}$  is a union of  $k$  isolated vertices

and  $P_v \subseteq V(K^v)$  is a  $\gamma_{K_1ce}^+$ -set of  $K^v$ . Then  $S = \cup_{v \in V(G)} S_v$  satisfies the properties of Theorem 6, and thus is a cost effective dominating set of  $G \circ H$ . It follows that

$$\begin{aligned} \gamma_{ce}^+(G \circ H) &\geq |S| = \sum_{v \in V(G)} |S_v| \\ &= |V(G)| \left( k + \gamma_{K_1ce}^+(K) \right). \end{aligned}$$

Conversely, suppose that  $S \subseteq V(G \circ H)$  is a  $\gamma_{ce}^+$ -set of  $G \circ H$ . Then,

$$\begin{aligned} \gamma_{ce}^+(G \circ H) = |S| &= \sum_{v \in S \cap V(G)} |S \cap V(H^v + v)| + \sum_{v \in V(G) \setminus S} |S \cap V(H^v)| \\ &\leq \gamma_{ce}^+(G) \left( \gamma_{K_1ce}^+(K) + k \right) + (|V(G)| - \gamma_{ce}^+(G)) \left( \gamma_{K_1ce}^+(K) + k \right) \\ &\leq \gamma_{K_nce}^+(G) \left( \gamma_{K_1ce}^+(K) + k \right) + (|V(G)| - \gamma_{K_nce}^+(G)) \left( \gamma_{K_1ce}^+(K) + k \right) \\ &= |V(G)| \left( k + \gamma_{K_1ce}^+(K) \right). \end{aligned}$$

□

#### 4. Composition of Graphs

**Theorem 7.** [9] Let  $G$  and  $H$  be connected graphs. Then  $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ , where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for every  $x \in S$ , is a dominating set of  $G[H]$  if and only if either

- (i)  $S$  is a total dominating set of  $G$ , or
- (ii)  $S$  is a dominating set of  $G$  and  $T_x$  is a dominating set of  $H$  for every  $x \in S \setminus N_G(S)$ .

**Theorem 8.** [9] Let  $G$  and  $H$  be nontrivial connected graphs with  $\gamma(H) = 1$ . Then  $\gamma(G[H]) = 1$ .

**Remark 7.** Let  $G$  and  $H$  be connected graphs. For  $C = \cup_{u \in S} (\{u\} \times T_u) \subseteq V(G[H])$  and  $(u, v) \in C$ ,

$$|N_{G[H]}((u, v)) \cap C| = \sum_{x \in S \cap N_G(u)} |T_x| + |N_H(v) \cap T_u| \tag{2}$$

and

$$|N_{G[H]}((u, v)) \setminus C| = |N_G(u) \setminus S| |V(H)| + |N_H(v) \setminus T_u| + \sum_{x \in N_G(u) \cap S} |V(H) \setminus T_x| \tag{3}$$

It is worth noting that a graph may not have a total dominating set that is cost effective. A good example is the path  $P_3$ .



**Theorem 9.** Let  $G$  and  $H$  be nontrivial connected graphs, and  $C = \cup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ , where  $S \subseteq V(G)$  and  $T_x \subseteq V(H)$  for each  $x \in S$ . If  $S$  is an independent dominating set of  $G$  and  $T_x$  is a dominating set of  $H$  for each  $x \in S$ , then  $C$  is a cost effective dominating set of  $G[H]$ .

*Proof.* Suppose that  $S$  is an independent dominating set of  $G$  and  $T_x$  is a dominating set of  $H$  for each  $x \in S$ . By Theorem 7,  $C$  is a dominating set of  $G[H]$ . Let  $u \in S$  and  $v \in T_u$ . Since  $G$  is nontrivial and  $S$  is independent,  $N_G(u) \setminus S \neq \emptyset$ . Using Equations 2 and 3,

$$\begin{aligned} |N_{G[H]}(u, v) \cap C| &= |N_H(v) \cap T_u| \\ &< |N_H(v) \setminus T_u| + |N_G(u) \setminus S| |V(H)| \\ &= |N_{G[H]}((u, v)) \setminus C|. \end{aligned}$$

Since  $u$  and  $v$  are arbitrary,  $C$  is a cost effective dominating set of  $G[H]$ . □

**Corollary 8.** For any nontrivial connected graphs  $G$  and  $H$ ,  $\gamma_{ce}(G[H]) \leq \gamma_i(G)\gamma(H)$ .

**Corollary 9.** For any nontrivial connected graphs  $G$  and  $H$  with  $G$  claw-free,  $\gamma_{ce}(G[H]) = \gamma(G)\gamma(H)$ .

**Theorem 10.** Let  $G$  and  $H$  be nontrivial connected graphs, and let  $C = \cup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ . If  $C$  has the following properties:

- (i)  $S$  is a cost effective dominating set of  $G$ ;
- (ii) For each  $x \in S \setminus N_G(S)$ ,  $T_x$  is a dominating set of  $H$ ; and
- (iii) For each  $x \in S \cap N_G(S)$ ,  $T_x$  is a cost effective set of  $H$ ,

then  $C$  is a cost effective dominating set of  $G[H]$ .

*Proof.* By Theorem 7, properties (i) and (ii) imply that  $C$  is a dominating set of  $G[H]$ . Let  $(x, y) \in C$ . Suppose that  $x \in S \setminus N_G(S)$ . Then  $N_G(x) \setminus S \neq \emptyset$ . Following Equations 2 and 3,

$$\begin{aligned} |N_{G[H]}((x, y)) \cap C| &= |N_H(y) \cap T_x| \\ &< |N_G(x) \setminus S| |V(H)| + |N_H(y) \setminus T_x| \\ &= |N_{G[H]}((x, y)) \setminus C|. \end{aligned}$$

Suppose that  $x \in S \cap N_G(S)$ . Properties (i) and (iii) and Equations 2 and 3 yield

$$\begin{aligned} |N_{G[H]}((x, y)) \cap C| &= \sum_{x \in S \cap N_G(x)} |T_x| + |N_H(y) \cap T_x| \\ &\leq |S \cap N_G(x)| |V(H)| + |N_H(y) \cap T_x| \end{aligned}$$

$$\begin{aligned} &\leq |N_G(x) \setminus S| |V(H)| + |N_H(y) \setminus T_x| + \sum_{z \in N_G(x) \cap S} |V(H) \setminus T_z| \\ &= |N_{G[H]}((x, y)) \setminus C|. \end{aligned}$$

Accordingly,  $C$  is a cost effective dominating set of  $G[H]$ . □

Removing the condition that  $S$  is a cost effective set of  $G$  in Theorem 10(i) may not yield a cost effective dominating set  $C$  of  $G[H]$ . Consider, for example, the set  $C = V(K_3) \times U$  in the composition  $G = K_3[K_{1,4}]$ , where  $U$  is the partite set of  $K_{1,4}$  with  $|U| = 4$ . For each  $x \in V(K_3)$  and  $u \in U$ ,

$$|N_G((x, u)) \cap C| = 8,$$

and

$$|N_G((x, u)) \setminus C| = 3.$$

Thus,  $C$  is not a cost effective set, hence not a cost effective dominating set, of  $G$ .

Next, let  $S^\circ = S \cap N_G(S)$ .

**Theorem 11.** *For all nontrivial connected graphs  $G$  and  $p \geq 2$ ,*

- (i)  $\gamma_{ce}(G[K_p]) = \gamma(G)$ ;
- (ii)  $\gamma_{mce}(G[K_p]) = \gamma_{mce}(G)$ ; and
- (iii)  $\gamma_{ce}^+(G[K_p]) \geq \max\{p|S| - \left(p - \left\lfloor \frac{p+1}{2} \right\rfloor\right) |S^\circ| : S \text{ is a } \gamma_{ce}^+ \text{- set of } G\}$ .

*Proof.* Let  $S \subseteq V(G)$  be a  $\gamma$ -set of  $G$ , and let  $v \in V(K_p)$ . Define  $C = S \times \{v\}$ . By Theorem 7,  $C$  is a dominating set of  $G[K_m]$ . For each  $u \in S$ ,

$$\begin{aligned} |N_{G[K_p]}((u, v)) \cap C| &= |N_G(u) \cap S| \\ &\leq p|N_G(u) \setminus S| + (p - 1) + (p - 1)|N_G(u) \cap S| \\ &= |N_{G[K_p]}((u, v)) \setminus C|. \end{aligned}$$

Thus,  $C$  is a cost effective dominating set of  $G[K_p]$  so that  $\gamma_{ce}(G[K_p]) \leq |S| = \gamma(G)$ . By Theorem 8,  $\gamma_{ce}(G[K_p]) = \gamma(G)$ .

Suppose that  $S$  is a  $\gamma_{mce}$ -set of  $G$ . By Theorem 10,  $C = S \times \{v\}$  is a cost effective dominating set of  $G[K_p]$ . Let  $(u, v) \in C$ , and put  $C^* = C \setminus \{(u, v)\}$ . Then  $C^* = S^* \times \{v\}$ , where  $S^* = S \setminus \{u\}$ . Let  $w \in S^*$ . If  $u \notin N_G(w)$ , then

$$|N_G(w) \cap S^*| = |N_G(w) \cap S| \leq |N_G(w) \setminus S| = |N_G(w) \setminus S^*|.$$

If  $u \in N_G(w)$ , then

$$|N_G(w) \cap S^*| = |N_G(w) \cap S| - 1 < |N_G(w) \setminus S| + 1 = |N_G(w) \setminus S^*|.$$

That is,  $S^*$  is a cost effective set of  $G$ . Since  $S$  is a minimal cost effective dominating set of  $G$ ,  $S^*$  is not a dominating set of  $G$ . By Theorem 7,  $C^*$  is not a dominating set, hence not a cost effective dominating set, of  $G[K_p]$ . This shows that  $C$  is a minimal cost effective dominating set of  $G \circ H$ . Thus,

$$\gamma_{mce}(G[K_p]) \geq |C| = |S| = \gamma_{mce}(G).$$

Conversely, suppose that  $C = \cup_{u \in S} (\{u\} \times T_u) \subseteq V(G[K_p])$  is a minimal cost effective dominating set of  $G[K_p]$ . We claim that  $|T_u| = 1$  for each  $u \in S$ . Suppose that  $|T_u| \geq 2$  for some  $u \in S$ . Let  $C^* = C \setminus \{(u, x)\}$ , where  $x \in T_u$ , and let  $v \in T_u \setminus \{x\}$ . Since the  $G$ -projection  $C_G^* = S$  and every subset of  $V(K_p)$  is dominating of  $K_p$ ,  $C^*$  is dominating of  $G[K_p]$  by Theorem 7. In view of Equations (1) and (2),

$$|N_{G[K_p]}((u, v)) \cap C^*| \leq |N_{G[K_p]}((u, v)) \cap C|,$$

and

$$|N_{G[K_p]}((u, v)) \setminus C^*| \geq |N_{G[K_p]}((u, v)) \setminus C|.$$

Thus,  $C^*$  is a cost effective set of  $G[K_p]$ . Consequently,  $C^*$  is a cost effective dominating set of  $G[K_p]$ , a contradiction. Thus  $|T_u| = 1$  for all  $u \in S$ . In view of the proof of the necessity part of the statement, we may assume that for some  $v \in V(K_p)$ ,  $T_u = \{v\}$  for all  $u \in S$ . Clearly, the minimality of  $C$  implies that  $S$  is a minimal cost effective dominating set of  $G$ . Thus,  $\gamma_{mce}(G) \geq |S| = |C|$ . Since  $C$  is arbitrary,  $\gamma_{mce}(G) \geq \gamma_{mce}(G[K_p])$ .

Now, let  $S \subseteq V(G)$  be a  $\gamma_{ce}^+$ -set of  $G$ . For each  $u \in S \cap N_G(S)$ , let  $T_u \subseteq V(K_p)$  be a  $\gamma_{ce}^+$ -set of  $K_p$ , and for each  $u \in S \setminus N_G(S)$ , let  $T_u = V(K_p)$ . By Theorem 10,  $C = \cup_{u \in S} (\{u\} \times T_u)$  is a cost effective dominating set of  $G[H]$ . Thus,

$$\gamma_{ce}^+(G[K_m]) \geq |C| = |S^\circ| \gamma_{ce}^+(K_p) + (|S| - |S^\circ|) |V(K_p)|.$$

□

Following a similar proof shows that if  $G$  and  $H$  are nontrivial connected graphs with  $\gamma(H) = 1$ ,  $\gamma_{mce}(G[H]) = \gamma_{ce}(G[H]) = \gamma(G)$ .

**Theorem 12.** *Let  $G$  be a noncomplete connected graph and  $p \geq 3$ . Then*

$$(i) \quad \gamma_{ce}(K_p[G]) = \begin{cases} 1, & \text{if } \gamma(G) = 1 \\ 2, & \text{otherwise.} \end{cases}$$

$$(ii) \quad \gamma_{mce}(K_p[G]) = \gamma_m(G); \text{ and}$$

$$(iii) \quad \gamma_{ce}^+(K_p[G]) = \left\lceil \frac{p+1}{2} \right\rceil |V(G)|.$$

*Proof.* Let  $x, y \in V(K_p)$  be distinct and  $u \in V(G)$ . By Theorem 10,  $C = \{x, y\} \times \{v\}$  is a cost effective dominating set of  $K_p[G]$ . Thus,

$$\gamma_{ce}(K_p[G]) \leq |C| = 2.$$

Also, by Theorem 10, for any  $x \in V(K_p)$  and any dominating set  $S \subseteq V(G)$  of  $G$ ,  $C = \{x\} \times S$  is a cost effective dominating set of  $K_p[G]$ . Thus,  $\gamma_{ce}(K_p[G]) = \min\{\gamma(G), 2\}$ , and the conclusion follows.

To prove Statement (ii), note first that since  $G$  is not complete,  $\gamma_m(G) \geq 2$ . Let  $S \subseteq V(G)$  be a minimal dominating set of  $G$  and  $x \in V(K_p)$ . In view of Theorem 7,  $C = \{x\} \times S$  is a minimal cost effective dominating set of  $K_p[G]$ . Consequently,

$$\gamma_{mce}(K_p[G]) \geq |C| = |S|.$$

Since  $S$  is arbitrary,  $\gamma_{mce}(K_p[G]) \geq \gamma_m(G)$ . Conversely, let  $C = \cup_{u \in S} (\{u\} \times T_u) \subseteq V(K_p[G])$  be a  $\gamma_{mce}$ -set of  $K_p[G]$ . Since  $C$  is a dominating set of  $K_p[G]$ , either  $S$  is a total dominating set of  $K_p$  or  $S$  is a dominating set of  $K_p$ , in which case  $T_u$  is a dominating set for each  $u \in S \setminus N_{K_p}(S)$ . Since  $C$  is minimal, if  $S$  is a total dominating set in  $K_p$ , then  $C = \{(x, y), (u, v)\}$  for some  $x, u \in V(K_p)$ . On the other hand, if  $S$  is not a total dominating set, then  $C = \{x\} \times S$  for some  $x \in V(K_p)$  and  $S$  is a minimal dominating set of  $G$ . Thus,

$$\gamma_{mce}(K_p[G]) \leq \max\{2, \gamma_m(G)\} = \gamma_m(G).$$

This proves Statement (ii).

To prove Statement (iii), let  $C = \cup_{u \in S} (\{u\} \times V(G))$ , where  $S \subseteq V(K_p)$  is a  $\gamma_{ce}^+$ -set in  $K_p$ . By Theorem 10,  $C$  is a cost effective dominating set of  $K_p[G]$ . Consequently,

$$\gamma_{ce}^+(K_p[G]) \geq |S| \times |V(G)|.$$

Conversely, let  $C = \cup_{u \in S} (\{u\} \times T_u)$  be a  $\gamma_{ce}^+$ -set of  $K_p[G]$ . In view of Theorem 10, since we want the largest possible cardinality of  $C$ , we assume that  $T_u = V(G)$ , which is a dominating set of  $G$ . Let  $u \in S$ . Equation 2 and Equation 3, respectively, yield

$$\begin{aligned} |N_{K_p[G]}((u, v)) \cap C| &= |N_{K_p}(u) \cap S| |V(G)| + |N_G(v) \cap V(G)| \\ &= (|S| - 1) |V(G)| + |N_G(v)|. \end{aligned}$$

and

$$\begin{aligned} |N_{K_p[G]}((u, v)) \setminus C| &= |N_{K_p}(u) \setminus S| |V(G)| \\ &= (p - |S|) |V(G)|. \end{aligned}$$

for each  $v \in V(G)$ . Since  $C$  is a cost effective set of  $K_p[G]$ ,

$$(|S| - 1) |V(G)| + |N_G(v)| \leq (p - |S|) |V(G)|,$$

or equivalently,

$$2|S| |V(G)| \leq (p + 1) |V(G)| - |N_G(v)|$$

for each  $v \in V(G)$ . Thus,

$$|S| < \frac{1}{2}(p + 1).$$

Therefore,

$$\gamma_{ce}^+(K_p[G]) = |C| = |S| |V(G)| \leq \left\lfloor \frac{p + 1}{2} \right\rfloor |V(G)|,$$

and the desired conclusion follows. □

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