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Operation on Fine Topology

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Abstract. This paper introduces the concept of an operation γ on τ_f. Using this operation, we define the concept of f_γ-open sets, and study some of their related notions. Also, we introduce the concept of $f_\gamma g$.closed sets and then study some of its properties. Moreover, we introduce and investigate some types of f_γ-separation axioms and $f_{\gamma\beta}$-continuous functions by utilizing the operation γ on τ_f. Finally, some basic properties of functions with f_β-closed graphs have been obtained.

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1. Introduction

Kasahara [11] introduced the notion of an α operation approaches on a class τ of sets and studied the concept of α-continuous functions with α-closed graphs and α-compact spaces. After this, Jankovic [10] introduced the concept of α-closure of a set in X via α-operation and investigated further characterizations of function with α-closed graph. Later, Ogata [12] defined and studied the concept of γ-open sets, and applied it to investigate operation-functions and operation-separation axioms. Asaad et al. [7] introduced the notion of γ-extremally disconnected spaces. Asaad et al. [5] studied further characterizations of γ-extremally disconnected spaces and investigated some relations of functions of γ-extremally disconnected spaces. Asaad [4] defined a γ operation on generalized open sets in X and studied its applications. In 2017-2018, Ahmad and Asaad ([1], [6]) introduced an operation γ on semi generalized open subsets of X and discussed some types of separation axioms, functions and closed spaces with respect to γ. Recently, Asaad and Ameen [8] introduced an operation on $g\alpha$-open sets and studied some of its properties. On

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the other hand, Powar and Rajak [13] defined the concept of fine-open sets. They studied fine-irresolute homeomorphism and fine-quotient function.

The aim of this paper is to introduce the concept of an operation γ on τ_f and to define the notion of f_γ-open sets of (X, τ, τ_f) by using the operation γ on τ_f. Also, some notions of f_γ-open sets with their relationships are studied. In Section 4, we introduce the concept of $f_\gamma g$.closed sets and then investigate some of its properties. In Section 5, some types of f_γ-separation axioms by utilizing the operation γ on τ_f are introduced and investigated. In the last two sections, some basic properties of $f_{\gamma\beta}$-continuous functions with f_β-closed graphs have been obtained.

2. Preliminaries

Throughout this paper, the space (X, τ) (or simply X) always mean topological space on which no separation axioms are assumed unless explicitly stated.

Definition 2.1. [13] Let (X, τ) be a topological space, we define $\tau(A_\alpha) = \tau_\alpha$(say)= $\{G_\alpha(\neq X) : G_\alpha \cap A_\alpha \neq \phi$, for $A_\alpha \in \tau$ and $A_\alpha \neq \phi, X$. for some $\alpha \in J$, where J is the index set$\}$. Now, define $\tau_f = \{\phi, X\} \cup_\alpha \{\tau_\alpha\}$. The above collection τ_f of subsets of X is called the fine collection of subsets of X and (X, τ, τ_f) is said to be the fine space X generated by the topology τ on X.

Definition 2.2. [13] A subset U of a fine space X is said to be fine-open in X, if U belongs to the collection τ_f. It is clear that every open set of X is fine-open in X. The complement of a fine-open set of X is called the fine-closed in X.

Remark 2.3. *[13] Let (X, τ, τ_f) be a fine space. Then the following are holds.*

(i) The arbitrary union of any fine-open sets in X is fine-open of X.

(ii) The intersection of two fine-open sets need not be fine-open.

Definition 2.4. [13] Let A be the subset of a fine space X, the fine interior of A is defined as the union of all fine-open sets contained in A. That means, the fine interior of A is the largest fine-open set contained in A and it is denoted by $f_{int}(A)$.

Definition 2.5. [13] Let A be the subset of a fine space X, the fine closure of A is defined as the intersection of all fine-closed sets containing the set A. That means, the fine closure of A is the smallest fine-closed set containing A and it is denoted by $f_{cl}(A)$.

Definition 2.6. [12] An operation γ on the topology τ on X is a mapping $\gamma\colon \tau \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U. A nonempty subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-open if for each $x \in A$, there exists an open set U containing x such that $\gamma(U) \subseteq A$. The complement of a γ-open subset of a space X as γ-closed. The family of all γ-open subsets of a space (X, τ) is denoted by τ_γ.

Definition 2.7. [10] A point $x \in X$ is in the γ-closure of a set $A \subseteq X$ if $\gamma(U) \cap A \neq \phi$ for each open set U containing x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $Cl_\gamma(A)$.

Definition 2.8. [12] A subset A of (X, τ) with an operation γ on τ is said to be γ-g-closed if $Cl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is γ-open in (X, τ).

Definition 2.9. [12] A topological space (X, τ) with an operation γ on τ is said to be

(i) γ-T_0 if for any two distinct points x, y in X, there exists an open set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.

(ii) γ-T_1 if for any two distinct points x, y in X, there exist two open sets U and V containing x and y respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.

(iii) γ-T_2 if for any two distinct points x, y in X, there exist two open sets U and V containing x and y respectively such that $\gamma(U) \cap \gamma(V) = \phi$.

(iv) γ-$T_{\frac{1}{2}}$ if every γ-g-closed set in X is γ-closed.

3. f_γ-Open Sets

An operation γ on τ_f is a mapping $\gamma\colon \tau_f \to P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in \tau_f$, where $P(X)$ is the power set of X and $\gamma(U)$ is the value of γ at U. From this, we can easily to find $\gamma(X) = X$ for any operation $\gamma\colon \tau_f \to P(X)$. The operators defined by $\gamma(U) = U$, $\gamma(U) = X$, $\gamma(U) = f_{cl}(U)$ and $\gamma(U) = f_{int}(f_{cl}(U))$ are all examples of the operation γ.

Definition 3.1. Let (X, τ, τ_f) be a fine space and $\gamma\colon \tau_f \to P(X)$ be an operation on τ_f. A nonempty set A of X is said to be f_γ-open if for each $x \in A$, there exists a fine-open set U such that $x \in U$ and $\gamma(U) \subseteq A$. The complement of a f_γ-open set of X is f_γ-closed. Suppose that the empty set ϕ is also f_γ-open set for any operation $\gamma\colon \tau_f \to P(X)$. The family of all f_γ-open subsets of a fine space (X, τ, τ_f) is denoted by $\tau_{f\gamma}$.

Theorem 3.2. *The union of any collection of f_γ-open sets in a fine space (X, τ, τ_f) is a f_γ-open set in (X, τ, τ_f).*

Proof. Let $x \in \bigcup_{\alpha\in\Delta}\{A_\alpha\}$, where $\{A_\alpha\}_{\alpha\in\Delta}$ be a class of f_γ-open sets in X. Then $x \in A_\alpha$ for some $\alpha \in \Delta$. Since A_α is f_γ-open set in X, then there exists a fine-open set V such that $x \in V \subseteq \gamma(V) \subseteq A_\alpha \subseteq \bigcup_{\alpha\in\Delta}\{A_\alpha\}$. Therefore, $\bigcup_{\alpha\in\Delta}\{A_\alpha\}$ is a f_γ-open set in X.

Example 3.3. *The intersection of any two f_γ-open sets in (X, τ, τ_f) is generally not a f_γ-open sets. To see this, let $X = \{a, b, c\}$ and $\tau = P(X) = \tau_f$. Let $\gamma\colon \tau_f \to P(X)$ be an*

operation on τ_f *defined as follows:*
For every $A \in \tau_f$

$$\gamma(A) = \begin{cases} A & \text{if } A \neq \{c\} \\ \{b, c\} & \text{if } A = \{c\} \end{cases}$$

Thus, $\tau_{f_\gamma} = P(X) \backslash \{c\}$. *Then* $\{a, c\} \in \tau_{f_\gamma}$ *and* $\{b, c\} \in \tau_{f_\gamma}$, *but* $\{a, c\} \cap \{b, c\} = \{c\} \notin \tau_{f_\gamma}$. *Therefore,* τ_{f_γ} *does not form a topology on* X.

It is clear from Definition 3.1 that every f_γ-open set is fine-open in (X, τ, τ_f) (That is, $\tau_{f_\gamma} \subseteq \tau_f$). But the converse need not be true as shown by the following example.

Example 3.4. *In Example* 3.3*, the set* $\{c\}$ *is fine-open, but it is not* f_γ*-open.*

Definition 3.5. A fine space (X, τ, τ_f) with an operation γ on τ_f is said to be f_γ-regular if for each $x \in X$ and for each fine-open set U containing x, there exists a fine-open set W such that $x \in W$ and $\gamma(W) \subseteq U$.

Theorem 3.6. *Let* (X, τ, τ_f) *be a fine space and* $\gamma \colon \tau_f \to P(X)$ *be an operation on* τ_f*. Then the following conditions are equivalent:*

(i) $\tau_f = \tau_{f_\gamma}$.

(ii) (X, τ, τ_f) *is a* f_γ*-regular space.*

(iii) For every $x \in X$ *and for every fine-open set* U *of* (X, τ, τ_f) *containing* x*, there exists a* f_γ*-open set* W *of* (X, τ, τ_f) *containing* x *such that* $W \subseteq U$.

Proof. (1) ⇒ (2) Let $x \in X$ and U be a fine-open set in X such that $x \in U$. It follows from assumption that U is a f_γ-open set. This implies that there exists a fine-open set W such that $x \in W$ and $\gamma(W) \subseteq U$. Therefore, the fine space (X, τ, τ_f) is f_γ-regular.

(2) ⇒ (3) Let $x \in X$ and U be a fine-open set in (X, τ, τ_f) containing x. Then by (2), there is a fine-open set W such that $x \in W \subseteq \gamma(W) \subseteq U$. Again, by using (2) for the set W, it is shown that W is f_γ-open. Hence W is a f_γ-open set containing x such that $W \subseteq U$.

(3) ⇒ (1) By applying the part (3) and Theorem 3.2, it follows that every fine-open set of X is f_γ-open in X. That is, $\tau_f \subseteq \tau_{f_\gamma}$. But in general, we have $\tau_{f_\gamma} \subseteq \tau_f$. Therefore, $\tau_f = \tau_{f_\gamma}$.

Definition 3.7. Let (X, τ, τ_f) be any fine space. An operation γ on τ_f is said to be

(i) fine-open if for each $x \in X$ and for every fine-open set U containing x, there exists a f_γ-open set W containing x such that $W \subseteq \gamma(U)$.

(ii) fine-regular if for each $x \in X$ and for every pair of fine-open sets U_1 and U_2 such that both containing x, there exists a fine-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$.

Lemma 3.8. *Let a mapping γ be fine-regular operation on τ_f. If A and B are f_γ-open sets in a fine space (X, τ, τ_f), then $A \cap B$ is also f_γ-open set in (X, τ, τ_f).*

Proof. Suppose $x \in A \cap B$ for any f_γ-open sets A and B in (X, τ, τ_f) both containing x. Then there exist fine-open sets U_1 and U_2 such that $x \in U_1 \subseteq A$ and $x \in U_2 \subseteq B$. Since γ is a fine-regular operation on τ_f, then there exists a fine-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2) \subseteq A \cap B$. Therefore, $A \cap B$ is f_γ-open set in (X, τ, τ_f).

Remark 3.9. *By applying Lemma* 3.8*, it is easy to show that $\tau_{f\gamma}$ forms a topology on X for any fine-regular operation γ on τ_f.*

Definition 3.10. Let A be any subset of a fine space (X, τ, τ_f) and γ be an operation on τ_f. The point $x \in X$ is said to be f_γ-closure of A if $\gamma(U) \cap A \neq \phi$ for each $U \in \tau_f$ such that $x \in U$. We denote $fcl_\gamma(A)$ by the f_γ-closure of A which is the set of all f_γ-closure points of A.

Definition 3.11. Let A be any subset of a fine space (X, τ, τ_f) and γ be an operation on τ_f. We define $\tau_{f\gamma}$-$cl(A)$ as the intersection of all f_γ-closed sets of X containing A.

$$\text{i.e. } \tau_{f\gamma}\text{-}cl(A) = \bigcap\{F : A \subseteq F,\ X \backslash F \in \tau_{f\gamma}\}.$$

Theorem 3.12. *Let A be any subset of a fine space (X, τ, τ_f) and γ be an operation on τ_f. Then $x \in \tau_{f\gamma}$-$cl(A)$ if and only if $A \cap U \neq \phi$ for every f_γ-open set U of X containing x.*

Proof. Let $x \in \tau_{f\gamma}$-$cl(A)$ and let $A \cap U = \phi$ for some f_γ-open set U of X containing x. Then $A \subseteq X \backslash U$ and $X \backslash U$ is f_γ-closed set in X. So $\tau_{f\gamma}$-$cl(A) \subseteq X \backslash U$. Thus, $x \in X \backslash U$. This is a contradiction. Hence $A \cap U \neq \phi$ for every f_γ-open set U of X containing x.

Conversely, suppose that $x \notin \tau_{f\gamma}$-$cl(A)$. So there exists a f_γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then $X \backslash F$ is a f_γ-open set such that $x \in X \backslash F$ and $A \cap (X \backslash F) = \phi$. Contradiction of hypothesis. Therefore, $x \in \tau_{f\gamma}$-$cl(A)$.

Lemma 3.13. *The following statements are true for any subsets A and B of a fine space (X, τ, τ_f) with an operation γ on τ_f.*

(i) $fcl_\gamma(A)$ *is fine-closed set in* X *and* $\tau_{f\gamma}$-$cl(A)$ *is* f_γ*-closed set in* X.

(ii) $A \subseteq fcl_\gamma(A) \subseteq \tau_{f\gamma}$-$cl(A)$.

(iii) $\tau_{f\gamma}$-$cl(\phi) = fcl_\gamma(\phi) = \phi$ *and* $\tau_{f\gamma}$-$cl(X) = fcl_\gamma(X) = X$.

(iv) *(a)* A *is* f_γ*-closed if and only if* $\tau_{f\gamma}$-$cl(A) = A$ *and,*

(b) A *is* f_γ*-closed if and only if* $fcl_\gamma(A) = A$.

(v) *If* $A \subseteq B$, *then* $\tau_{f\gamma}$-$cl(A) \subseteq \tau_{f\gamma}$-$cl(B)$ *and* $fcl_\gamma(A) \subseteq fcl_\gamma(B)$.

(vi) *(a)* $\tau_{f\gamma}$-$cl(A \cap B) \subseteq \tau_{f\gamma}$-$cl(A) \cap \tau_{f\gamma}$-$cl(B)$ *and,*

(b) $fcl_\gamma(A \cap B) \subseteq fcl_\gamma(A) \cap fcl_\gamma(B)$.

(vii) *(a)* $\tau_{f\gamma}$-$cl(A) \cup \tau_{f\gamma}$-$cl(B) \subseteq \tau_{f\gamma}$-$cl(A \cup B)$ *and,*
(b) $fcl_\gamma(A) \cup fcl_\gamma(B) \subseteq fcl_\gamma(A \cup B)$.

(viii) $\tau_{f\gamma}$-$cl(\tau_{f\gamma}$-$cl(A)) = \tau_{f\gamma}$-$cl(A)$.

Proof. Straightforward.

Theorem 3.14. *For any subsets A, B of a fine space (X, τ, τ_f). If γ is a fine-regular operation on τ_f, then*

(i) $\tau_{f\gamma}$-$cl(A) \cup \tau_{f\gamma}$-$cl(B) = \tau_{f\gamma}$-$cl(A \cup B)$.

(ii) $fcl_\gamma(A) \cup fcl_\gamma(B) = fcl_\gamma(A \cup B)$.

Proof. (1) It is enough to prove that $\tau_{f\gamma}$-$cl(A \cup B) \subseteq \tau_{f\gamma}$-$cl(A) \cup \tau_{f\gamma}$-$cl(B)$ since the other part follows directly from Lemma 3.13 (7). Let $x \notin \tau_{f\gamma}$-$cl(A) \cup \tau_{f\gamma}$-$cl(B)$. Then by using Theorem 3.12, there exist two f_γ-open sets U and V containing x such that $A \cap U = \phi$ and $B \cap V = \phi$. Since γ is a fine-regular operation on τ_f, then by Lemma 3.8, $U \cap V$ is f_γ-open in X such that

$$(U \cap V) \cap (A \cup B) = \phi.$$

Therefore, we have $x \notin \tau_{f\gamma}$-$cl(A \cup B)$ and hence

$$\tau_{f\gamma}\text{-}cl(A \cup B) \subseteq \tau_{f\gamma}\text{-}cl(A) \cup \tau_{f\gamma}\text{-}cl(B).$$

(2) Let $x \notin fcl_\gamma(A) \cup fcl_\gamma(B)$. Then there exist fine-open sets U_1 and U_2 such that $x \in U_1$, $x \in U_2$, $A \cap \gamma(U_1) = \phi$ and $A \cap \gamma(U_2) = \phi$. Since γ is a fine-regular operation on τ_f, then there exists a fine-open set W containing x such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$. Thus, we have

$$(A \cup B) \cap \gamma(W) \subseteq (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)).$$

This implies that $(A \cup B) \cap \gamma(W) = \phi$ since $(A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)) = \phi$. This means that $x \notin fcl_\gamma(A \cup B)$ and hence $fcl_\gamma(A \cup B) \subseteq fcl_\gamma(A) \cup fcl_\gamma(B)$. Using Lemma 3.13 (7), we have the equality.

Theorem 3.15. *Let A be any subset of a fine space (X, τ, τ_f). If γ is a fine-open operation on τ_f, then $fcl_\gamma(A) = \tau_{f\gamma}$-$cl(A)$, $fcl_\gamma(fcl_\gamma(A)) = fcl_\gamma(A)$ and $fcl_\gamma(A)$ is f_γ-closed set in X.*

Proof. By Lemma 3.13 (2), we have $fcl_\gamma(A) \subseteq \tau_{f\gamma}$-$cl(A)$. Now, we need to show that $\tau_{f\gamma}$-$cl(A) \subseteq fcl_\gamma(A)$. Let $x \notin fcl_\gamma(A)$, then there exists a fine-open set U containing x such that $A \cap \gamma(U) = \phi$. Since γ is a fine-open on τ_f, then there exists a f_γ-open set W containing x such that $W \subseteq \gamma(U)$. So $A \cap W = \phi$ and hence by Theorem 3.12, $x \notin \tau_{f\gamma}$-$cl(A)$. Therefore, $\tau_{f\gamma}$-$cl(A) \subseteq fcl_\gamma(A)$. Hence $fcl_\gamma(A) = \tau_{f\gamma}$-$cl(A)$. Moreover, using the above result and by Lemma 3.13 (8), we get $fcl_\gamma(fcl_\gamma(A)) = fcl_\gamma(A)$ and by Lemma 3.13 (4b), we obtain $fcl_\gamma(A)$ is f_γ-closed set in X.

Theorem 3.16. *Let A be any subset of a fine space (X, τ, τ_f) and γ be an operation on τ_f. Then the following statements are equivalent:*

(i) A *is* f_γ*-open set.*

(ii) $fcl_\gamma(X\backslash A) = X\backslash A$.

(iii) $\tau_{f\gamma}$*-cl*$(X\backslash A) = X\backslash A$.

(iv) $X\backslash A$ *is* f_γ*-closed set.*

Proof. Clear.

Lemma 3.17. *Let (X, τ, τ_f) be a fine space and γ be a fine-regular operation on τ_f. Then $\tau_{f\gamma}$-cl$(A) \cap U \subseteq \tau_{f\gamma}$-cl$(A \cap U)$ holds for every f_γ-open set U and every subset A of X.*

Proof. Suppose that $x \in \tau_{f\gamma}$-$cl(A) \cap U$ for every f_γ-open set U, then $x \in \tau_{f\gamma}$-$cl(A)$ and $x \in U$. Let V be any f_γ-open set of X containing x. Since γ is fine-regular on τ_f. So by Lemma 3.8, $U \cap V$ is f_γ-open set containing x. Since $x \in \tau_{f\gamma}$-$cl(A)$, then by Theorem 3.12, we have $A \cap (U \cap V) \neq \phi$. This means that $(A \cap U) \cap V \neq \phi$. Therefore, again by Theorem 3.12, we obtain that $x \in \tau_{f\gamma}$-$cl(A \cap U)$. Thus, $\tau_{f\gamma}$-$cl(A) \cap U \subseteq \tau_{f\gamma}$-$cl(A \cap U)$.

4. $f_\gamma g$.Closed Sets

Definition 4.1. A subset A of a fine space (X, τ, τ_f) with an operation γ on τ_f is said to be f_γ-generalized closed (briefly $f_\gamma g$.closed) if $fcl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and U is a f_γ-open set in X.

Lemma 4.2. *Let (X, τ, τ_f) be a fine space and γ be an operation on τ_f. A set A in (X, τ, τ_f) is $f_\gamma g$.closed if and only if $A \cap \tau_{f\gamma}$-cl$(\{x\}) \neq \phi$ for every $x \in fcl_\gamma(A)$.*

Proof. Suppose A is $f_\gamma g$.closed set in X and suppose (if possible) that there exists an element $x \in fcl_\gamma(A)$ such that $A \cap \tau_{f\gamma}$-$cl(\{x\}) = \phi$. This follows that $A \subseteq X\backslash\tau_{f\gamma}$-$cl(\{x\})$. Since $\tau_{f\gamma}$-$cl(\{x\})$ is f_γ-closed implies $X\backslash\tau_{f\gamma}$-$cl(\{x\})$ is f_γ-open and A is $f_\gamma g$.closed set in X. Then, we have that $fcl_\gamma(A) \subseteq X\backslash\tau_{f\gamma}$-$cl(\{x\})$. This means that $x \notin fcl_\gamma(A)$. This is a contradiction. Hence $A \cap \tau_{f\gamma}$-$cl(\{x\}) \neq \phi$.

Conversely, let $U \in \tau_{f\gamma}$ such that $A \subseteq U$. To show that $fcl_\gamma(A) \subseteq U$. Let $x \in fcl_\gamma(A)$. Then by hypothesis, $A \cap \tau_{f\gamma}$-$cl(\{x\}) \neq \phi$. So there exists an element $y \in A \cap \tau_{f\gamma}$-$cl(\{x\})$. Thus $y \in A \subseteq U$ and $y \in \tau_{f\gamma}$-$cl(\{x\})$. By Theorem 3.12, $\{x\} \cap U \neq \phi$. Hence $x \in U$ and so $fcl_\gamma(A) \subseteq U$. Therefore, A is $f_\gamma g$.closed set in (X, τ, τ_f).

Theorem 4.3. *Let A be a subset of fine space (X, τ, τ_f) and γ be an operation on τ_f. If A is $f_\gamma g$.closed, then $fcl_\gamma(A)\backslash A$ does not contain any non-empty f_γ-closed set.*

Proof. Let F be a non-empty f_γ-closed set in X such that $F \subseteq fcl_\gamma(A)\backslash A$. Then $F \subseteq X\backslash A$ implies $A \subseteq X\backslash F$. Since $X\backslash F$ is f_γ-open set and A is $f_\gamma g$.closed set, then $fcl_\gamma(A) \subseteq X\backslash F$. That is $F \subseteq X\backslash fcl_\gamma(A)$. Hence $F \subseteq X\backslash fcl_\gamma(A) \cap fcl_\gamma(A)\backslash A \subseteq X\backslash fcl_\gamma(A) \cap fcl_\gamma(A) = \phi$. This shows that $F = \phi$. This is a contradiction. Therefore, $F \not\subseteq fcl_\gamma(A)\backslash A$.

Theorem 4.4. *If $\gamma\colon \tau_f \to P(X)$ is a fine-open operation, then the converse of the Theorem 4.3 is true.*

Proof. Let U be a f_γ-open set in (X, τ, τ_f) such that $A \subseteq U$. Since $\gamma\colon \tau_f \to P(X)$ is a fine-open operation, then by Theorem 3.15, $fcl_\gamma(A)$ is f_γ-closed set in X. Thus, using Theorem 3.2, we have $fcl_\gamma(A) \cap X\backslash U$ is a f_γ-closed set in (X, τ, τ_f). Since $X\backslash U \subseteq X\backslash A$, $fcl_\gamma(A) \cap X\backslash U \subseteq fcl_\gamma(A)\backslash A$. Using the assumption of the converse of the Theorem 4.3, $fcl_\gamma(A) \subseteq U$. Therefore, A is $f_\gamma g$.closed set in (X, τ, τ_f).

Corollary 4.5. *Let A be a $f_\gamma g$.closed subset of fine space (X, τ, τ_f) and let γ be an operation on τ_f. Then A is f_γ-closed if and only if $fcl_\gamma(A)\backslash A$ is f_γ-closed set.*

Proof. Let A be a f_γ-closed set in (X, τ, τ_f). Then by Lemma 3.13 (4b), $fcl_\gamma(A) = A$ and hence $fcl_\gamma(A)\backslash A = \phi$ which is f_γ-closed set.

Conversely, suppose $fcl_\gamma(A)\backslash A$ is f_γ-closed and A is $f_\gamma g$.closed. Then by Theorem 4.3, $fcl_\gamma(A)\backslash A$ does not contain any non-empty f_γ-closed set and since $fcl_\gamma(A)\backslash A$ is f_γ-closed subset of itself, then $fcl_\gamma(A)\backslash A = \phi$ implies $fcl_\gamma(A) \cap X\backslash A = \phi$. Hence $fcl_\gamma(A) = A$. This follows from Lemma 3.13 (4b) that A is f_γ-closed set in (X, τ, τ_f).

Theorem 4.6. *Let (X, τ) be a fine space and γ be an operation on τ_f. If a subset A of X is $f_\gamma g$.closed and f_γ-open, then A is f_γ-closed.*

Proof. Since A is $f_\gamma g$.closed and f_γ-open set in X, then $fcl_\gamma(A) \subseteq A$ and hence by Lemma 3.13 (4b), A is f_γ-closed.

Theorem 4.7. *In any fine space (X, τ, τ_f) with an operation γ on τ_f. For an element $x \in X$, the set $X\backslash\{x\}$ is $f_\gamma g$.closed or f_γ-open.*

Proof. Suppose that $X\backslash\{x\}$ is not f_γ-open. Then X is the only f_γ-open set containing $X\backslash\{x\}$. This implies that $fcl_\gamma(X\backslash\{x\}) \subseteq X$. Thus $X\backslash\{x\}$ is a $f_\gamma g$.closed set in X.

Corollary 4.8. *In any fine space (X, τ, τ_f) with an operation γ on τ_f. For an element $x \in X$, either the set $\{x\}$ is f_γ-closed or the set $X\backslash\{x\}$ is $f_\gamma g$.closed.*

Proof. Suppose $\{x\}$ is not f_γ-closed, then $X\backslash\{x\}$ is not f_γ-open. Hence by Theorem 4.7, $X\backslash\{x\}$ is $f_\gamma g$.closed set in X.

Definition 4.9. Let A be any subset of a fine space (X, τ, τ_f) and γ be an operation on τ_f. Then the $\tau_{f\gamma}$-kernel of A is denoted by $\tau_{f\gamma}$-ker(A) and is defined as follows:

$$\tau_{f\gamma}\text{-ker}(A) = \cap\{U : A \subseteq U \text{ and } U \in \tau_{f\gamma}\}$$

In other words, $\tau_{f\gamma}$-ker(A) is the intersection of all f_γ-open sets of (X, τ, τ_f) containing A.

Theorem 4.10. *Let $A \subseteq (X, \tau, \tau_f)$ and γ be an operation on τ_f. Then A is $f_\gamma g$.closed if and only if $fcl_\gamma(A) \subseteq \tau_{f\gamma}$-ker(A).*

Proof. Suppose that A is $f_\gamma g$.closed. Then $fcl_\gamma(A) \subseteq U$, whenever $A \subseteq U$ and U is f_γ-open. Let $x \in fcl_\gamma(A)$. Then by Lemma 4.2, $A \cap \tau_{f\gamma}\text{-}cl(\{x\}) \neq \phi$. So there exists a point z in X such that $z \in A \cap \tau_{f\gamma}\text{-}cl(\{x\})$ implies that $z \in A \subseteq U$ and $z \in \tau_{f\gamma}$-$cl(\{x\})$. By Theorem 3.12, $\{x\} \cap U \neq \phi$. Hence we show that $x \in \tau_{f\gamma}$-ker(A). Therefore, $fcl_\gamma(A) \subseteq \tau_{f\gamma}$-ker$(A)$.

Conversely, let $fcl_\gamma(A) \subseteq \tau_{f\gamma}$-ker$(A)$. Let U be any f_γ-open set containing A. Let x be a point in X such that $x \in fcl_\gamma(A)$. Then $x \in \tau_{f\gamma}$-ker(A). Namely, we have $x \in U$, because $A \subseteq U$ and $U \in \tau_{f\gamma}\}$. That is $fcl_\gamma(A) \subseteq \tau_{f\gamma}$-ker$(A) \subseteq U$. Therefore, A is $f_\gamma g$.closed set in X.

5. On f_γ-Separation Axioms

Definition 5.1. A fine space (X, τ, τ_f) with an operation γ on τ_f is said to be
(i) f_γ-T_0 if for any two distinct points x, y in X, there exists a fine-open set U such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.
(ii) f_γ-T_0^* if for each pair of distinct points x, y in X, there exists a f_γ-open set U containing one of the points but not the other.

Definition 5.2. A fine space (X, τ, τ_f) with an operation γ on τ_f is said to be
(i) f_γ-T_1 if for any two distinct points x, y in X, there exist two fine-open sets U and V such that $x \in U$, $y \notin \gamma(U)$, $y \in V$ and $x \notin \gamma(V)$.
(ii) f_γ-T_1^* if for each pair of distinct points x, y in X, there exist two f_γ-open sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Definition 5.3. A fine space (X, τ, τ_f) with an operation γ on τ_f is said to be
(i) f_γ-T_2 if for any two distinct points x, y in X, there exist two fine-open sets U and V such that $x \in U$, $y \in V$ and $\gamma(U) \cap \gamma(V) = \phi$.
(ii) f_γ-T_2^* if for each pair of distinct points x, y in X, there exist f_γ-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Definition 5.4. A fine space (X, τ, τ_f) with an operation γ on τ_f is said to be f_γ-$T_{\frac{1}{2}}$ if every $f_\gamma g$.closed set in X is f_γ-closed set.

Theorem 5.5. *For any fine space (X, τ, τ_f) with an operation γ on τ_f. Then (X, τ, τ_f) is f_γ-$T_{\frac{1}{2}}$ if and only if for each point $x \in X$, the set $\{x\}$ is f_γ-closed or f_γ-open.*

Proof. Let X be a f_γ-$T_{\frac{1}{2}}$ space and let $\{x\}$ is not f_γ-closed set in (X, τ, τ_f). By Corollary 4.8, $X\backslash\{x\}$ is $f_\gamma g$.closed. Since (X, τ, τ_f) is f_γ-$T_{\frac{1}{2}}$, then $X\backslash\{x\}$ is f_γ-closed set which means that $\{x\}$ is f_γ-open set in X.

Conversely, let F be any $f_\gamma g$.closed set in the fine space (X, τ, τ_f). We have to show that F is f_γ-closed (that is $fcl_\gamma(F) = F$ (by Lemma 3.13 (4b))). It is sufficient to show that $fcl_\gamma(F) \subseteq F$. Let $x \in fcl_\gamma(F)$. By hypothesis $\{x\}$ is f_γ-closed or f_γ-open for each $x \in X$. So we have two cases:

Case (1): If $\{x\}$ is f_γ-closed set. Suppose $x \notin F$, then $x \in fcl_\gamma(F) \backslash F$ contains a non-empty f_γ-closed set $\{x\}$. A contradiction since F is $f_\gamma g$.closed set and according to the Theorem 4.3. Hence $x \in F$. This follows that $fcl_\gamma(F) \subseteq F$ and hence $fcl_\gamma(F) = F$. This means from by Lemma 3.13 (4b) that F is f_γ-closed set in (X, τ, τ_f). Thus (X, τ, τ_f) is f_γ-$T_{\frac{1}{2}}$ space.

Case (2): If $\{x\}$ is f_γ-open set. Then by Theorem 3.12, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So $fcl_\gamma(F) \subseteq F$. Thus by Lemma 3.13 (4b), F is f_γ-closed. Therefore, (X, τ, τ_f) is f_γ-$T_{\frac{1}{2}}$ space.

Theorem 5.6. *For any fine space (X, τ, τ_f) with an operation γ on τ_f, we have*

(i) Let γ be a fine-open operation on τ_f. Then a space X is a f_γ-T_0 space if and only if $fcl_\gamma(\{x\}) \neq fcl_\gamma(\{y\})$, for every pair x, y of X with $x \neq y$.

(ii) A space X is f_γ-T_0^* if and only if $\tau_{f\gamma}$-$cl(\{x\}) \neq \tau_{f\gamma}$-$cl(\{y\})$, for every pair of distinct points x, y of X.

Proof. (1) Let x, y be any two distinct points of a f_γ-T_0 space (X, τ, τ_f). Then by definition, we assume that there exists a f_γ-open set U such that $x \in U$ and $y \notin \gamma(U)$. Since γ is a fine-open operation on τ_f, then there exists a f_γ-open set W such that $x \in W$ and $W \subseteq \gamma(U)$. Hence $y \in X \backslash \gamma(U) \subseteq X \backslash W$. Since $X \backslash W$ is a f_γ-closed set in (X, τ, τ_f). Then we obtain that $fcl_\gamma(\{y\}) \subseteq X \backslash W$ and therefore $fcl_\gamma(\{x\}) \neq fcl_\gamma(\{y\})$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, we have $fcl_\gamma(\{x\}) \neq fcl_\gamma(\{y\})$. Now, we assume that there exists $z \in X$ such that $z \in fcl_\gamma(\{x\})$, but $z \notin fcl_\gamma(\{y\})$. If $x \in fcl_\gamma(\{y\})$, then $\{x\} \subseteq fcl_\gamma(\{y\})$, which implies that $fcl_\gamma(\{x\}) \subseteq fcl_\gamma(\{y\})$ (by Lemma 3.13 (5)). This implies that $z \in fcl_\gamma(\{y\})$. This contradiction shows that $x \notin fcl_\gamma(\{y\})$. This means that by Definition 3.10, there exists a fine-open set U such that $x \in U$ and $\gamma(U) \cap \{y\} = \phi$. Thus, we have that $x \in U$ and $y \notin \gamma(U)$. It gives that the fine space (X, τ, τ_f) is f_γ-T_0.

(2) Let X be a f_γ-T_0^* space and x, y be any two distinct points of X. Then there exists a f_γ-open set G containing x or y (say x, but not y). So $X \backslash G$ is a f_γ-closed set, which does not contain x, but contains y. Since $\tau_{f\gamma}$-$cl(\{y\})$ is the smallest f_γ-closed set containing y, $\tau_{f\gamma}$-$cl(\{y\}) \subseteq X \backslash G$, and so $x \notin \tau_{f\gamma}$-$cl(\{y\})$. Therefore, $\tau_{f\gamma}$-$cl(\{x\}) \neq \tau_{f\gamma}$-$cl(\{y\})$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, $\tau_{f\gamma}$-$cl(\{x\}) \neq \tau_{f\gamma}$-$cl(\{y\})$. Now, let $z \in X$ such that $z \in \tau_{f\gamma}$-$cl(\{x\})$, but $z \notin \tau_{f\gamma}$-$cl(\{y\})$. Now, we claim that $x \in \tau_{f\gamma}$-$cl(\{y\})$. For, if $x \in \tau_{f\gamma}$-$cl(\{y\})$, then $\{x\} \subseteq \tau_{f\gamma}$-$cl(\{y\})$, which implies that $\tau_{f\gamma}$-$cl(\{x\}) \subseteq \tau_{f\gamma}$-$cl(\{y\})$. This is a contradiction to the fact that $z \notin \tau_{f\gamma}$-$cl(\{y\})$. Hence x belongs to the f_γ-open set $X \backslash \tau_{f\gamma}$-$cl(\{y\})$ to which y does not belong. It gives that X is f_γ-T_0^* space.

Corollary 5.7. *Suppose that γ is a fine-open operation on τ_f. A fine space (X, τ, τ_f) is f_γ-T_0 if and only if (X, τ, τ_f) is f_γ-T_0^*.*

Proof. This follows from Theorem 5.6 and the fact that $fcl_\gamma(A) = \tau_{f\gamma}\text{-}cl(A)$ for any $A \subseteq X$ holds under the assumption that γ is a fine-open operation on τ_f (see Theorem 3.15).

Theorem 5.8. *For a fine space (X, τ, τ_f) with an operation γ on τ_f. Then the following statements are true:*

(i) (X, τ, τ_f) *is* f_γ-T_1.

(ii) For every point $x \in X$, the set $\{x\}$ is f_γ-closed.

(iii) (X, τ, τ_f) *is* f_γ-T_1^*.

Proof. (1) $\Rightarrow$ (2) Let x be a point of an f_γ-T_1 space (X, τ, τ_f). Then for any point $y \in X$ such that $x \neq y$, there exists a fine-open set V_y such that $y \in V_y$ but $x \notin \gamma(V_y)$. Thus, $y \in \gamma(V_y) \subseteq X \backslash \{x\}$. This implies that $X \backslash \{x\} = \cup\{\gamma(V_y) : y \in X \backslash \{x\}\}$. It is shown that $X \backslash \{x\}$ is f_γ-open set in (X, τ, τ_f). Hence $\{x\}$ is f_γ-closed set in (X, τ, τ_f).

(2) $\Rightarrow$ (3) Suppose every singleton set in X is f_γ-closed. Let $x, y \in X$ such that $x \neq y$. This implies that $x \in X \backslash \{y\}$. By hypothesis, we get $X \backslash \{y\}$ is a f_γ-open set contains x but not y. Similarly $X \backslash \{x\}$ is a f_γ-open set contains y but not x. Therefore, X is f_γ-T_1^* space.

(3) $\Rightarrow$ (1) It is shown that if $x \in U$, where $U \in \tau_{f\gamma}$, then there exist a fine-open set V such that $x \in V \subseteq \gamma(V) \subseteq U$. Applying the part (3), we obtain (X, τ, τ_f) is f_γ-T_1.

Theorem 5.9. *For any fine space (X, τ, τ_f) and any operation γ on τ_f, the following properties hold.*

(i) Every f_γ-T_2 *space is* f_γ-T_1.

(ii) Every f_γ-T_1 *space is* f_γ-$T_{\frac{1}{2}}$.

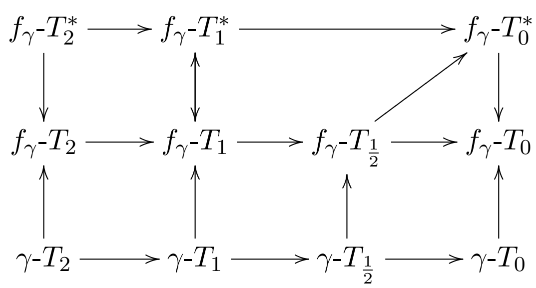
(iii) Every f_γ-$T_{\frac{1}{2}}$ *space is* f_γ-T_0^*.

(iv) Every f_γ-T_n^* *space is* f_γ-T_{n-1}^**, where* $n \in \{2, 1\}$.

(v) Every f_γ-T_n^* *space is* f_γ-T_n*, where* $n \in \{2, 0\}$.

Proof. Follows directly from their definitions.

Remark 5.10. *By Theorem 5.9 and Theorem 5.8, we obtain the following diagram of implications. Moreover, the following Examples 5.11, 5.12, 5.13 and 5.14 below show that the reverse implications are not true in general.*



Example 5.11. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. *Then* $\tau_f = \tau \cup \{\{a, c\}, \{b, c\}\}$.

(i) Define an operation γ *on* τ_f *as follows: For every* $A \in \tau_f$

$$\gamma(A) = \begin{cases} \{a,\ b\} & \text{if } A = \{b\} \\ \{b,\ c\} & \text{if } A = \{c\} \text{ or } \{b, c\} \\ X & \text{otherwise} \end{cases}$$

Obviously, the space (X, τ, τ_f) *is* f_γ-T_0, *but it is not* f_γ-T_0^*. *Hence the fine space* (X, τ, τ_f) *is not* f_γ-$T_{\frac{1}{2}}$.

(ii) Let $\gamma \colon \tau_f \to P(X)$ *be an operation on* τ_f *defined as follows:*
For every set $A \in \tau_f$

$$\gamma(B) = \begin{cases} B & \text{if } B = \{a\} \text{ or } \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\} \\ X & \text{otherwise} \end{cases}$$

Thus, $\tau_{f\gamma} = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. *Clearly, the space* (X, τ, τ_f) *is* f_γ-$T_{\frac{1}{2}}$, *but it is not* f_γ-T_1.

Example 5.12. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, X, \{b\}\}$. *Then* $\tau_f = \{\phi,\ X,\ \{b\},\ \{a, b\},\ \{a, c\}\}$. *Let* $\gamma \colon \tau_f \to P(X)$ *be an operation on* τ_f *defined as follows:*
For every set $A \in \tau_f$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{b\} \text{ or } \{a, b\} \\ f_{cl}(A) & \text{otherwise} \end{cases}$$

Thus, $\tau_{f\gamma} = \{\phi, X, \{b\}, \{a, b\}\}$. *Then the fine space* (X, τ, τ_f) *is* f_γ-T_0^*, *but it is not* f_γ-$T_{\frac{1}{2}}$. *Since* $\{b, c\}$ *is* $f_\gamma g$*.closed set in* (X, τ, τ_f), *but* $\{b, c\}$ *is not* f_γ*-closed set in* (X, τ, τ_f). *Therefore,* (X, τ, τ_f) *is not a* f_γ-T_1^* *space.*

Example 5.13. *Suppose* $X = \{a, b, c\}$ *and* $\tau =$ *all subsets of* X. *Define an operation* γ *on* τ_f *as follows: For every* $A \in \tau_f$

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\} \\ X & \text{otherwise} \end{cases}$$

Therefore, (X, τ, τ_f) *is* f_γ-T_1^* *space, and by Theorem* 5.8, *it is* f_γ-T_1, *but* (X, τ, τ_f) *is not* f_γ-T_2 *and hence it is not* f_γ-T_2^*.

Example 5.14. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\phi, X, \{a, b\}\}$. *Then* $\tau_f = \tau \cup \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. *Define an operation* $\gamma \colon \tau_f \to P(X)$ *by* $\gamma(A) = A$ *for all* $A \in \tau_f$. *Here,* $\tau_{f\gamma} = \tau_f$ *and* $\tau_\gamma = \tau$. *Then the fine space* (X, τ, τ_f) *is* f_γ-T_i, *but it is not* γ-T_i *for* $i = 0, \frac{1}{2}, 1, 2$.

6. $f_{\gamma\beta}$-Continuous Functions

Throughout Section 6 and Section 7, let (X, τ, τ_f) and (Y, σ, σ_f) be two fine spaces and let $\gamma\colon \tau_f \to P(X)$ and $\beta\colon \sigma_f \to P(Y)$ be operations on τ_f and σ_f respectively. In this section, we introduce a new class of functions called $f_{\gamma\beta}$-continuous. Some characterizations and properties of this function are investigated.

Definition 6.1. A function $h\colon (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ is said to be $f_{\gamma\beta}$-continuous if for each $x \in X$ and each fine-open set V containing $h(x)$, there exists a fine-open set U containing x such that $h(\gamma(U)) \subseteq \beta(V)$.

Theorem 6.2. *Let $h\colon (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ be a $f_{\gamma\beta}$-continuous function, then,*

(i) $h(fcl_\gamma(A)) \subseteq fcl_\beta(h(A))$, for every $A \subseteq (X, \tau, \tau_f)$.

(ii) $h^{-1}(F)$ is f_γ-closed set in (X, τ, τ_f), for every f_β-closed set F of (Y, σ, σ_f).

Proof. (1) Let $y \in h(fcl_\gamma(A))$ and V be any fine-open set containing y. Then by hypothesis, there exists $x \in X$ and fine-open set U containing x such that $h(x) = y$ and $h(\gamma(U)) \subseteq \beta(V)$. Since $x \in fcl_\gamma(A)$, we have $\gamma(U) \cap A \neq \phi$. Hence $\phi \neq h(\gamma(U) \cap A) \subseteq h(\gamma(U)) \cap h(A) \subseteq \beta(V) \cap h(A)$. This implies that $y \in fcl_\beta(h(A))$. Therefore, $h(fcl_\gamma(A)) \subseteq fcl_\beta(h(A))$.

(2) Let F be any f_β-closed set of (Y, σ, σ_f). By using (1), we have $h(fcl_\gamma(h^{-1}(F))) \subseteq fcl_\beta(F) = F$. Therefore, $fcl_\gamma(h^{-1}(F)) = h^{-1}(F)$. Hence $h^{-1}(F)$ is f_γ-closed set in (X, τ, τ_f).

Theorem 6.3. *In Theorem 6.2, the properties of $f_{\gamma\beta}$-continuity of f, (1) and (2) are equivalent to each other if either the fine space (Y, σ, σ_f) is f_β-regular or the operation β is fine-open.*

Proof. It follows from the proof of Theorem 6.2 that we know the following implications: "$f_{\gamma\beta}$-continuity of h" $\Rightarrow$ (1) $\Rightarrow$ (2). Thus, when the fine space (Y, σ, σ_f) is f_β-regular, we prove the implication: (2) $\Rightarrow$ $f_{\gamma\beta}$-continuity of h. Let $x \in X$ and let $V \in \sigma_f$ such that $h(x) \in V$. Since (Y, σ, σ_f) is a f_β-regular space, then by Theorem 3.6, $V \in \sigma_{g\beta}$. By using (2) of Theorem 6.2, $h^{-1}(V) \in \tau_{f\gamma}$ such that $x \in h^{-1}(V)$. So there exists a fine-open set U such that $x \in U$ and $\gamma(U) \subseteq h^{-1}(V)$. This implies that $h(\gamma(U)) \subseteq V \subseteq \beta(V)$. Therefore, h is $f_{\gamma\beta}$-continuous.

Now, when β is a fine-open operation, we show the implication: (2) $\Rightarrow$ $f_{\gamma\beta}$-continuity of h. Let $x \in X$ and let $V \in \sigma_f$ such that $h(x) \in V$. Since β is a fine-open operation, then there exists $W \in \sigma_{g\beta}$ such that $h(x) \in W$ and $W \subseteq \beta(V)$. By using (2) of Theorem 6.2, $h^{-1}(W) \in \tau_{f\gamma}$ such that $x \in h^{-1}(W)$. So there exists a fine-open set U such that $x \in U$ and $\gamma(U) \subseteq h^{-1}(W) \subseteq h^{-1}(\beta(V))$. This implies that $h(\gamma(U)) \subseteq \beta(V)$. Hence h is $f_{\gamma\beta}$-continuous.

Definition 6.4. A function $h\colon (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ is said to be

(i) $f_{\gamma\beta}$-closed if the image of each f_γ-closed set of X is f_β-closed in Y.

(ii) f_β-closed if the image of each fine-closed set of X is f_β-closed in Y.

Theorem 6.5. *Suppose that a function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is both $f_{\gamma\beta}$-continuous and f_β-closed, then:*

(i) For every $f_\gamma g$.closed set A of (X,τ,τ_f), the image $h(A)$ is $f_\beta g$.closed in (Y,σ,σ_f).

(ii) For every $f_\beta g$.closed set B of (Y,σ,σ_f), the inverse set $h^{-1}(B)$ is $f_\gamma g$.closed in (X,τ,τ_f).

Proof. (1) Let G be any f_β-open set in (Y,σ,σ_f) such that $h(A) \subseteq G$. Since h is $f_{\gamma\beta}$-continuous function, then by using Theorem 6.2 (2), $h^{-1}(G)$ is f_γ-open set in (X,τ,τ_f). Since A is $f_\gamma g$.closed and $A \subseteq h^{-1}(G)$, we have $fcl_\gamma(A) \subseteq h^{-1}(G)$, and hence $h(fcl_\gamma(A)) \subseteq G$. Thus, by Lemma 3.13 (1), $fcl_\gamma(A)$ is fine-closed set and since h is f_β-closed, then $h(fcl_\gamma(A))$ is f_β-closed set in Y. Therefore, $fcl_\beta(h(A)) \subseteq fcl_\beta(h(fcl_\gamma(A))) = h(fcl_\gamma(A)) \subseteq G$. This implies that $h(A)$ is $f_\beta g$.closed in (Y,σ,σ_f).

(2) Let H be any f_γ-open set of a fine space (X,τ,τ_f) such that $h^{-1}(B) \subseteq H$. Let $C = fcl_\gamma(h^{-1}(B)) \cap (X\backslash H)$, then by Lemma 3.13 (1), C is fine-closed set in (X,τ,τ_f). Since h is f_β-closed function. Then $h(C)$ is f_β-closed in (Y,σ,σ_f). Since h is $f_{\gamma\beta}$-continuous function, then by using Theorem 6.2 (1), we have $h(C) = h(fcl_\gamma(h^{-1}(B))) \cap h(X\backslash H) \subseteq fcl_\beta(B) \cap h(X\backslash H) \subseteq fcl_\beta(B) \cap (Y\backslash B) = fcl_\beta(B)\backslash B$. This implies from Theorem 4.3 that $h(C) = \phi$, and hence $C = \phi$. So $fcl_\gamma(h^{-1}(B)) \subseteq H$. Therefore, $h^{-1}(B)$ is $f_\gamma g$.closed in (X,τ,τ_f).

Theorem 6.6. *Let $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ be an injective, $f_{\gamma\beta}$-continuous and f_β-closed function. If (Y,σ,σ_f) is f_β-$T_{\frac{1}{2}}$, then (X,τ,τ_f) is f_γ-$T_{\frac{1}{2}}$.*

Proof. Let G be any $f_\gamma g$.closed set of (X,τ,τ_f). Since h is $f_{\gamma\beta}$-continuous and f_β-closed function. Then by Theorem 6.5 (1), $h(G)$ is $f_\beta g$.closed in (Y,σ,σ_f). Since (Y,σ,σ_f) is f_β-$T_{\frac{1}{2}}$, then $h(G)$ is f_β-closed in Y. Again, since h is $f_{\gamma\beta}$-continuous, then by Theorem 6.2 (2), $h^{-1}(h(G))$ is f_γ-closed in X. Hence G is f_γ-closed in X since h is injective. Therefore, (X,τ,τ_f) is a f_γ-$T_{\frac{1}{2}}$ space.

Theorem 6.7. *Let a function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ be surjective, $f_{\gamma\beta}$-continuous and f_β-closed. If (X,τ,τ_f) is f_γ-$T_{\frac{1}{2}}$, then (Y,σ,σ_f) is f_β-$T_{\frac{1}{2}}$.*

Proof. Let H be a $f_\beta g$.closed set of (Y,σ,σ_f). Since h is $f_{\gamma\beta}$-continuous and f_β-closed function. Then by Theorem 6.5 (2), $h^{-1}(H)$ is $f_\gamma g$.closed in (X,τ,τ_f). Since (X,τ,τ_f) is f_γ-$T_{\frac{1}{2}}$, then we have, $h^{-1}(H)$ is f_γ-closed set in X. Again, since h is f_β-closed function, then $h(h^{-1}(H))$ is f_β-closed in Y. Therefore, H is f_β-closed in Y since h is surjective. Hence (Y,σ,σ_f) is f_β-$T_{\frac{1}{2}}$ space.

Theorem 6.8. *If a function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is injective $f_{\gamma\beta}$-continuous and the fine space (Y,σ,σ_f) is f_β-T_2, then the fine space (X,τ,τ_f) is f_γ-T_2.*

Proof. Let x_1 and x_2 be any distinct points of a fine space (X,τ,τ_f). Since h is an injective function and (Y,σ,σ_f) is f_β-T_2. Then there exist two fine-open sets U_1 and U_2 in Y such that $f(x_1) \in U_1$, $h(x_2) \in U_2$ and $\beta(U_1) \cap \beta(U_2) = \phi$. Since h is $f_{\gamma\beta}$-continuous, there exist fine-open sets V_1 and V_2 in X such that $x_1 \in V_1$, $x_2 \in V_2$, $h(\gamma(V_1)) \subseteq \beta(U_1)$ and $h(\gamma(V_2)) \subseteq \beta(U_2)$. Therefore $\beta(U_1) \cap \beta(U_2) = \phi$. Hence (X,τ,τ_f) is f_γ-T_2.

Theorem 6.9. *If a function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is injective $f_{\gamma\beta}$-continuous and the fine space (Y,σ,σ_f) is f_β-T_i, then the fine space (X,τ,τ_f) is f_γ-T_i for $i \in \{0,1\}$.*

Proof. The proof is similar to Theorem 6.8.

Definition 6.10. A function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is said to be $f_{\gamma\beta}$-homeomorphism if h is bijective, $f_{\gamma\beta}$-continuous and h^{-1} is $f_{\beta\gamma}$-continuous.

Theorem 6.11. *Assume that a function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is $f_{\gamma\beta}$-homeomorphism. If (X,τ,τ_f) is f_γ-$T_{\frac{1}{2}}$, then (Y,σ,σ_f) is f_β-$T_{\frac{1}{2}}$.*

Proof. Let $\{y\}$ be any singleton set of (Y,σ,σ_f). Then there exists an element x of X such that $y = h(x)$. So by hypothesis and Theorem 5.5, we have $\{x\}$ is f_γ-closed or f_γ-open set in X. By using Theorem 6.2, $\{y\}$ is f_β-closed or f_β-open set. Hence the fine space by Theorem 5.5, (Y,σ,σ_f) is f_β-$T_{\frac{1}{2}}$.

7. Functions with f_β-Closed Graphs

For a function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$, the subset $\{(x,h(x)) : x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of h and is denoted by $G(h)$ [9]. In this section, we further investigate general operator approaches of closed graphs of functions. Let $\lambda\colon (\tau \times \sigma)_f \to P(X \times Y)$ be an operation on $(\tau \times \sigma)_f$.

Definition 7.1. The graph $G(h)$ of $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is called f_β-closed if for each $(x,y) \in (X \times Y)\backslash G(h)$, there exist fine-open sets $U \subseteq X$ and $V \subseteq Y$ containing x and y, respectively, such that $(U \times \beta(V)) \cap G(h) = \phi$.

The proof of the following lemma follows directly from the above definition.

Lemma 7.2. *A function $h\colon (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ has f_β-closed graph if and only if for each $(x,y) \in (X \times Y)\backslash G(h)$, there exist $U \in \tau_f$ containing x and $V \in \sigma_f$ containing y such that $h(U) \cap \beta(V) = \phi$.*

Definition 7.3. An operation $\lambda\colon (\tau \times \sigma)_f \to P(X \times Y)$ is said to be fine-associated with γ and β if $\lambda(U \times V) = \gamma(U) \times \beta(V)$ holds for each $U \in \tau_f$ and $V \in \sigma_f$.

Definition 7.4. The operation $\lambda\colon (\tau \times \sigma)_f \to P(X \times Y)$ is said to be fine-regular with respect to γ and β if for each $(x, y) \in X \times Y$ and each fine-open set W containing (x, y), there exist fine-open sets U in X and V in Y such that $x \in U$, $y \in V$ and $\gamma(U) \times \beta(V) \subseteq \lambda(W)$.

Theorem 7.5. *Let $\lambda\colon (\tau \times \tau)_f \to P(X \times X)$ be a fine-associated operation with γ and γ. If $h\colon (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ is a $f_{\gamma\beta}$-continuous function and (Y, σ, σ_f) is a f_β-T_2 space, then the set $A = \{(x, y) \in X \times X : h(x) = h(y)\}$ is a f_λ-closed set of $(X \times X, \tau \times \tau)$.*

Proof. We want to prove that $fcl_\lambda(A) \subseteq A$. Let $(x, y) \in (X \times X)\backslash A$. Since (Y, σ, σ_f) is f_β-T_2. Then there exist two fine-open sets U and V in (Y, σ, σ_f) such that $h(x) \in U$, $h(y) \in V$ and $\beta(U) \cap \beta(V) = \phi$. Moreover, for U and V there exist fine-open sets R and S in (X, τ, τ_f) such that $x \in R$, $y \in S$ and $h(\gamma(R)) \subseteq \beta(U)$ and $h(\gamma(S)) \subseteq \beta(V)$ since h is $f_{\gamma\beta}$-continuous. Therefore we have $(x, y) \in \gamma(R) \times \gamma(S) = \lambda(R \times S) \cap A = \phi$ because $R \times S \in (\tau \times \tau)_f$. This shows that $(x, y) \notin fcl_\lambda(A)$.

Corollary 7.6. *Suppose $\lambda\colon (\tau \times \tau)_f \to P(X \times X)$ is fine-associated operation with γ and γ, and it is fine-regular with γ and γ. A fine space (X, τ, τ_f) is f_γ-T_2 if and only if the diagonal set $\triangle = \{(x, x) : x \in X\}$ is f_λ-closed of $(X \times X, \tau \times \tau)$.*

Theorem 7.7. *Let $\lambda\colon (\tau \times \sigma)_f \to P(X \times Y)$ be a fine-associated operation with γ and β. If $h\colon (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ is $f_{\gamma\beta}$-continuous and (Y, σ, σ_f) is f_β-T_2, then the graph of h, $G(h) = \{(x, h(x)) \in X \times Y\}$ is a f_λ-closed set of $(X \times Y, \tau \times \sigma)$.*

Proof. The proof is similar to Theorem 7.5.

Definition 7.8. Let (X, τ, τ_f) be a fine space and γ be an operation on τ_f. A subset S of X is said to be f_γ-compact if for every fine-open cover $\{U_i, i \in \mathbb{N}\}$ of S, there exists a finite subfamily $\{U_1, U_2, ..., U_n\}$ such that $S \subseteq \gamma(U_1) \cup \gamma(U_2) \cup ... \cup \gamma(U_n)$.

Theorem 7.9. *Suppose that γ is fine-regular and $\lambda\colon (\tau \times \sigma)_f \to P(X \times Y)$ is fine-regular with respect to γ and β. Let $h\colon (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ be a function whose graph $G(h)$ is f_λ-closed in $(X \times Y, \tau \times \sigma)$. If a subset S is f_β-compact in (Y, σ, σ_f), then $h^{-1}(S)$ is f_γ-closed in (X, τ, τ_f).*

Proof. Suppose that $h^{-1}(S)$ is not f_γ-closed then there exist a point x such that $x \in fcl_\gamma(h^{-1}(S))$ and $x \notin h^{-1}(S)$. Since $(x, s) \notin G(h)$ and each $s \in S$ and $fcl_\lambda(G(h)) \subseteq G(h)$, there exists a fine-open set W of $(X \times Y, \tau \times \sigma)$ such that $(x, s) \in W$ and $\beta(W) \cap G(h) = \phi$. By fine-regularity of λ, for each $s \in S$ we can take two fine-open sets $U(s)$ and $V(s)$ in (Y, σ, σ_f) such that $x \in U(s)$, $s \in V(s)$ and $\gamma(U(s)) \times \beta(V(s)) \subseteq \lambda(W)$. Then we have $h(\gamma(U(s))) \cap \beta(V(s)) = \phi$. Since $\{V(s) : s \in S\}$ is fine-open cover of S, then by f_γ-compactness there exists a finite number $s_1, s_2, ..., s_n \in S$ such that $S \subseteq \beta(V(s_1)) \cup \beta(V(s_2)) \cup ... \cup \beta(V(s_n))$. By the fine-regularity of γ, there exist a fine-open set U such that $x \in U$, $\gamma(U) \subseteq \gamma(U(s_1)) \cap \gamma(U(s_2)) \cap ... \cap \gamma(U(s_n))$. Therefore, we have $\gamma(U) \cap h^{-1}(S) \subseteq U(s_i) \cap h^{-1}(\beta(V(s_i))) = \phi$. This shows that $x \notin fcl_\gamma(h^{-1}(S))$. This is a contradiction. Therefore, $h^{-1}(S)$ is f_γ-closed.

Theorem 7.10. *Suppose that the following condition hold:*

(i) $\gamma\colon \tau_f \to P(X)$ *is fine-open*

(ii) $\beta\colon \sigma_f \to P(Y)$ *is fine-regular, and*

(iii) $\lambda\colon (\tau \times \sigma)_f \to P(X \times Y)$ *is associated with* γ *and* β*, and* λ *is fine-regular with respect to* γ *and* β*.*

Let $h\colon (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ *be a function whose graph* $G(h)$ *is* f_λ*-closed in* $(X \times Y, \tau \times \sigma)$*. If every cover of* A *by* f_γ*-open sets of* (X, τ, τ_f) *has finite sub cover, then* $h(A)$ *is* f_β*-closed in* (Y, σ, σ_f)*.*

Proof. Similar to Theorem 7.9.

8. Conclusion

In the present paper, the concepts of an operation γ on τ_f are introduced. Also, the concept of f_γ-open sets are defined, and some of their properties are studied via this operation. Moreover, the concept of $f_\gamma g$.closed sets are studied. Furthermore, some types of f_γ-separation axioms and $f_{\gamma\beta}$-continuous functions are investigated. In addition, some basic properties of functions with f_β-closed graphs are obtained.

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