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On C-co-epi-retractable modules

Abdoul Djibril Diallo[1], Papa Cheikhou Diop[2,*], Mamadou Barry[1]

[1] *Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, Dakar, Sénégal*
[2] *Département de Mathématiques , UFR Sciences et Technologies, Université de Thiès, Thiès, Sénégal*

Abstract. In this paper, we introduce the notion of c-co-epi-retractable modules. An R-module M is called c-co-epi-retractable if it contains a copy of its factor module by a complement submodule. The ring R is called c-co-pri if R_R is c-co-epi-retractable. Conditions are found under which, a c-co-epi-retractable module is extending, retractable, semi-simple, quasi-injective, injective and simple. Also, we investigate when c-co-epi-retractable modules have finite uniform dimension. Finally, right SI-rings, semi-simple artinian rings and quasi-Frobenius rings are characterized in termes of c-co-epi-retractable modules.

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1. Introduction

Throughout all rings are associative with identity and all modules are unitary right module. In [10], Ghorbani introduced the co-epi-retractable modules. An R-module M is called co-epi-retractable if it contains any of its factor modules. A ring R is called co-pri if R_R is a co-epi-retractable module. It is was shown in [10], that a ring R is co-pri iff its right ideals is the right annihilator of an element of R. Also co-pi-retractable modules have been investigated by Mostafanasab [15]. He studied the simplicity and the semi-simplicity of co-pi-retractable modules. Recall that a module M is called extending if every complement submodule is a direct summand. Motivated by the definition of a co-epi-retractable module and the definition of a extending module, we say that a module is c-co-epi-retractable if it contains a copy of its factor modules by a complement submodule. Every co-epi-retractable module and every extending module is c-co-epi-retractable. In particular uniform modules and semi-simple modules are c-co-epi-retractable. In this

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Email addresses: cheikpapa@yahoo.fr (P. C. Diop),
dialloabdoulaziz58@yahoo.fr (A. D. Diallo), mansabadion1@hotmail.com (M. Barry)

paper, we investigate when c-co-epi-retractable modules are extending, continuous, quasi-continuous, semi-simple, retractable, quasi-injective, injective and simple. Also, we prove under certain conditions that a c-co-epi-retractable module has finite uniform dimension. Finally, we characterize some well-known rings with the help of c-co-epi-retratacble modules.
Our paper is structured as follows:
In the second section, we give preliminary definitions and results which we will use throughout this paper.
In the third section, we define the c-co-epi-retractable modules. Our aim in this section is to work on the concept of c-co-epi-retractable modules. We show, among others, the following results.
(1) For an R-module with regular endomorphism ring, the properties, c-co-epi-retractable, extending, continuous and quasi-continuous are all equivalent.
(2) If M is a c-co-epi-retractable module with $Udim(M) = n \geq 2$, then M is retractable and for any $0 \neq C \subseteq_c M$, M/C is uniform.
(3) Let R be a right self-injective ring and M be a self-hereditary R-module. Then M is c-co-epi-retractable iff it is finitely generated semi-simple injective.
(4) Let M be a c-co-epi-retractable R-module such that S satisfies DCC for cyclic right ideals. If for any finitely generated right ideal $I \subseteq S$, $r(KerI) = I$ then M has finite uniform dimension.
(5) The following conditions are euivalent for a right SI-ring R:
(a) $R_R^{(\mathbb{N})}$ is extending.
(b) Every R-module is c-co-epi-retractable.
(c) Every R-module is extending.

For an R-module M, $S = End_R(M)$ denotes the endomorphism ring of M. For $\phi \in S$, $Ker\phi$ and $Im\phi$ stand for kernel and image of ϕ, respectively. The notations $N \leq M$, $N \leq_e M$ and $N \leq^{\oplus} M$ mean that N is a submodule of M, an essential submodule and a direct summand of M, respectively. Also $E(M)$ denotes the injective hull of M.

2. Preliminaries

In this section, we are going to give preliminary definitions and results which we will use throughout this paper.

Definition 1. 1. *An R-module M is called CS module if every complement submodule of M is a direct summand.*
2. *An R-module M is called continuous if it is a CS module and satisfies the following condition: $(C2)$ Every submodule of M that is isomorphic to a direct summand M is itself a direct summand of M.*
3. *An R-module M is called quasi-continuous if it is a CS module and satisfies the following condition: $(C3)$ If N and K are direct summands of M with $N \cap K = \{0\}$, then $N \oplus K$ is a direct summand of M.*

Definition 2. *Let M be an R-module, put $Z(M) = \{m \in M : ann_R(m) \leq_e R\}$. M is called nonsingular if $Z(M) = \{0\}$, and singular if $Z(M) = M$. The Goldie torsion submodule $Z_2(M)$ of M is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. M is Z_2-torsion if, $Z_2(M) = M$.*

3. C-co-epi-retractable modules and some applications

Definition 3. *An R-module M is called c-co-epi-retractable if, for every complement submodule N of M, there exists a monomorphism $f : M/N \longrightarrow M$. The ring R is called c-co-pri if R_R is c-co-epi-retractable.*

Remark 1. *Clearly, every co-epi-retractable module is c-co-epi-retractable while the converse is not true. For example $\mathbb{Q}$ as $\mathbb{Z}$-module is c-co-epi-retractable but it is not co-epi-retractable.*

Lemma 1. *The following statements are equivalent for an R-module M:*
(1) *M is a c-co-epi-retractable module.*
(2) *There exists $\varphi \in End_R(M)$ such that $Ker\varphi = N$ for any nonzero $N \subseteq_c M$.*

Proposition 1. *Let M be a c-co-epi-retractable R-module. Then a fully invariant complement submodule of a M is also c-co-epi-retractable.*

Proof.
Let M be a c-co-epi-retractable module and $N \subseteq_c M$ with N fully invariant. Let $K \subseteq_c N$. Then, $K \subseteq_c M$. So, there is an endomorphism $f : M \longrightarrow M$ such that $K = Kerf$. Then $f|_N : N \longrightarrow N$ and $K = Ker(f|_N)$. Therefore, N is a c-co-epi-retractable module.

Corollary 1. *Every fully invariant direct summand of a c-co-epi-retractable module is c-co-epi-retractable.*

Proposition 2. *Let M be an R-module with $S = End_R(M)$ regular.*
Then the following conditions are equivalent:
(1) *M is a c-co-epi-retractable module.*
(2) *M is an extending module.*

Proof.
(1) $\Rightarrow$ (2) Suppose M is a c-co-epi-retractable module and K a complement submodule of M. Then, there is $0 \neq g \in End_R(M)$ such that $Kerg = K$. By our assumption, $Kerg = K$ is a direct summand of M. Therefore, M is an extending module.
(2) $\Rightarrow$ (1) is obvious.

Corollary 2. *A ring R is regular c-co-pri if and only if it is right nonsingular right continuous.*

Proposition 3. *A ring R is c-co-pri if and only if every complement right ideal of R is the right annihilator of an element of R.*

Proof.
Let I be a right complement ideal of R. If R is c-co-pri, there is a monomorphism $f : R/I \longrightarrow R$. Set $x = f(1 + I)$, then $I = r(x)$, where $r(x)$ denotes the right annihilator of x. On the other hand, if $I = r(x)$ is a right complement ideal of R for an element $x \in R$, then $R/I \cong xR$.

Let M be an R-module. The left annihilator of $N \leq M$ in $S = End_R(M)$ is denoted by $L_S(N) = \{\phi \in S : \phi N = \{0\}\}$ and the right annihilator of a left ideal I of S is $r_M(I) = \{m \in M : Im = \{0\}\}$

Recall that an R-module is called Baer if, for all $N \leq M$, $L_S(N) = Se$, with $e^2 = e \in S$. Equivalenly, M is Baer if, for all ideal $I \leq_S S, r_M(I) = eM$ with $e^2 = e \in S$. An R-module M is called Rickart if any endomorphism of M has a direct summand kernel. A module M is called $\mathcal{K}$-nonsingular if, $\forall \varphi \in End(M)$,
$Ker\varphi \leq_e M$ implies $\varphi = 0$.

Lemma 2. *([16], Lemma 2.14)*
Any $\mathcal{K}$-nonsingular extending module is Baer.

In the two following results, we show that for a c-co-epi-retractable R-module or a module with c-co-pri endomorphism ring the properties Rickart and Baer are equivalent.

Proposition 4. *Let M be a c-co-epi-retractable R-module. Then M is Rickart if and only if M is Baer.*

Proof.
Suppose M is Rickart. Since, M is c-co-epi-retractable, it is also extending by Proposition 2. Now, suppose $Kerf \leq_e M$ for some $f \in End_R(M)$. The property of Rickart implies that $Kerf \leq^{\oplus} M$, and so $Kerf = M$. Consequently, $f = 0$. Therefore, according to Lemma 2, M is Baer. The converse implication is clear.

Corollary 3. *Let R be a c-co-pri ring. Then R is Baer if and only if R is right Rickart.*

Proposition 5. *Let M be an R-module for which S is c-co-pri. Then the following statements are equivalent:*
(1) *M is Baer.*
(2) *M is Rickart.*
(3) *S is right Rickart.*

Proof.
(1) $\Rightarrow$ (2) This is clear.
(2) $\Rightarrow$ (3) Follows from Proposition 2.2.1 in [13].
(3) $\Rightarrow$ (1) Let N be a submodule of M. Since S is right Rickart, it is also right nonsingular. Thus, $L_S(N)$ is a complement right ideal in S. Because S is c-co-pri, it follows from Proposition 2 that S is right extending. Therefore, $L_S(N) = S(1-e)$ for some $e = e^2 \in S$,

and hence M is Baer.

Recall that an R-module N is said to subgenerated by M if N is isomorphic to a submodule of an M-generated module, i.e N is a kernel of a morphism between M-generated modules.
We denote by $\sigma[M]$, the full subcategory of mod-R whose objects are all R-modules subgenerated by M.
Recall that an R-module M is self-hereditary if every submodule of M is projective in $\sigma[M]$.

Theorem 1. *Let R be a right self-injective ring and M be a self-hereditary R-module. Then the following conditions are equivalent:*
(1) *M is c-co-epi-retractable.*
(2) *M is extending.*
(3) *M is continuous.*
(4) *M is finitely generated semi-simple injective.*

Proof.
(1) $\Rightarrow$ (2) Suppose M is c-co-epi-retractable. Thus for any complement submodule C of M, there exists a submodule N of M such that $M/C \cong N$. Consequently, the property of self-hereditary implies that C is a direct summand of M. Hence M is extending.
(2) $\Rightarrow$ (3) Suppose M is extending. Thus, according to Theorem 10.5 in [7], M is nonsingular and has finite uniform dimension. Hence, there exists uniform independent submodules $U_1, ..., U_n$ of M such that $V = U_1 \oplus U_2 \oplus ... \oplus U_n$ is an essential submodule of M. Set $0 \neq u_i \in U_i, 1 \leq i \leq n$. Then, $U_i = u_iR$. It is easy to see that $M = V$. Therefore, M is finitely generated semi-simple injective. This means that M is continuous.
(3) $\Rightarrow$ (4) Follows from an argument similar to the one in (2) $\Rightarrow$ (3).
(4) $\Leftrightarrow$ (1) It is easy to see.

Corollary 4. *A right self-injective right hereditary ring is semi-simple artinian.*

Recall that a module M is said to be retractable if for any $0 \neq N \leq M$, there exists a nonzero homomorphism form M to N. A module M has finite uniform dimension n (written $Udim(M) = n$) if there is an essential submodule $V \leq_e M$ that is a direct sum of n uniform submodules.

Remark 2. *A c-co-epi-retractable module with finite uniform dimension need not be retractable. In fact, the $\mathbb{Z}$-module $\mathbb{Q}$ is c-co-epi-retractable with finite uniform dimension but it is not retractable. Clearly, a c-co-epi-retractable need not to have a finite uniform dimension. For example extending modules are c-co-epi-retractable which need not have finite uniform dimension.*

Proposition 6. *If M is a c-co-epi-retractable R-module with $Udim(M) = n \geq 2$, then the following assertions are verified:*
(1) *M is retractable.*
(2) *For every $0 \neq C \subseteq_c M$, M/C is uniform.*

Proof.
(1) Let $0 \neq N \leq M$. Since $Udim(N) < \infty$, N contains a uniform submodule, say U. After replacing U by an essential closure, we may assume that U is a complement submodule of M. By the c-co-epi-retractable condition on M, there exists a monomorphism $f : M/U \longrightarrow M$. Consider the inclusion map $i : U \longrightarrow M$. Thus, $f = ij$ is a monomorphism where $j : M/U \longrightarrow U$. Consequently, j is a monomorphism. Now, consider the inclusion map $i_1 : U \longrightarrow N$. So, $i_1 j\pi : M \longrightarrow N$ is a nonzero homomorphism where $\pi : M \longrightarrow M/U$ is the natural surjection. Therefore, M is retractable.
(2) Since $Udim(M) = n \geq 2$, there exist complements submodules $C_i \subseteq_c M(1 \leq i \leq n)$ such that each M/C_i is uniform and $C_1 \cap ... \cap C_n = 0$. Thus, there exists a monomorphism $f : M \longrightarrow \oplus_i^n M/C_i$. Let $0 \neq C \subseteq_c M$. Since M is c-co-epi-retractable, there exists a monomorphism $g : M/C \longrightarrow M$. Hence, $h = fg : M/C \longrightarrow \oplus_i^n M/C_i$ is a monomorphism. Consider the inclusion map $i : M/C_i \longrightarrow \oplus_i^n M/C_i$. Thus, $h = ii_1$ is a monomorphism where $i_1 : M/C \longrightarrow M/C_i$. Therefore, i_1 is a monomorphism. It follows that M/C is uniform.

Corollary 5. *An R-module M is simple if and only if M is Artinian c-co-epi-retractable and every endomorphisme of M is a monomorphism.*

Let N and M be R-modules and $S = End_R(M)$. We denote by $\mathcal{N}$ the set of R-submodules of N and by $\mathcal{H}$ the set of S-submodules of $Hom_R(N, M)_S$.
For $X \in \mathcal{H}$ we put:
$Ker(X) = \cap\{Kerg | g \in X\} \in \mathcal{N}$.

In the next result, we investigate when c-co-epi-retractable R-modules have finite uniform dimension.

Proposition 7. *Let M be a c-co-epi-retractable R-module such that S satisfies DCC for cyclic right ideals. If for any finitely generated right ideal $I \subseteq S$, $r(KerI) = I$, then has finite uniform dimension.*

Proof.
In view of Proposition 6.30 in [12], we need to show that the complements in M satisfy ACC. Now, let $C_1 \subseteq C_2 \subseteq ....$ be an ascending chain of complement submodules of M. By the c-co-epi-retractable condition on M, there is $f_i \in S$ such that each C_i is of the form $Kerf_i = Kerf_iS$. With applying $r(-)$ to this chain, we see that $f_1S \supseteq f_2S \supseteq .....$ By our assumption, there is some n such that $f_iS = f_nS$ for all $i \geq n$. Hence, $C_i = C_n$ for all $i \geq n$. Therefore, M has finite uniform dimension.

Recall that an R-module M is said to be have the summand sum property (SSP, for short) if, the sum of any two direct summands of M is again a direct summand of M.

Corollary 6. *Let M be a quasi-injective R-module such that $R \oplus M$ has the SSP. Assume that S satisfies DCC for cyclic right ideals. Then M is finitely generated semi-simple.*

Proof.
Since $R \oplus M$ has the SSP, we infer from Proposition 3.4 in [9] that every cyclic submodule of M is a direct summand of M. Since M is quasi-injective, $r(KerI) = I$ for any finitely generated right ideal $I \subseteq S$ by ([19], 28.1). But M is c-co-epi-retractable. Then, according to Proposition 7, M has finite uniform dimension. Therefore, M is finitely generated semi-simple.

Corollary 7. *If M is a quasi-injective R-module such that S satisfies DCC for cyclic right ideals, then M is a finite direct sum of uniform submodules.*

Proof.
Suppose M is quasi-injective such that S satisfies DCC for cyclic right ideals. Thus by ([19], 28.1), $r(KerI) = I$ for any finitely generated right ideal $I \subseteq S$. Therefore, according to Proposition 7, M is a finite direct sum of uniform submodules.

Corollary 8. *Let R be a right self-injective ring and M a nonsingular R-module such that S satisfies DCC for cyclic right ideals. Then the following conditions are equivalent.*
(1) *M is quasi-injective.*
(2) *M is semi-simple injective.*

Recall that an R-module M is called compressible if for every nonzero submodule N of M there is a monomorphism $f : M \longrightarrow N$.

Proposition 8. *An R-module is simple if and only if it is compressible c-co-epi-retractable and contains a maximal complement submodule.*

Proof.
The necessity is clear. Conversely, assume that M is compressible c-co-epi-retractable and contains a maximal complement submodule C. Then there is a submodule N of M such that $M/C \cong N$. Hence, N is simple. By the compressible condition on M, there is a monomorphism $f : M \longrightarrow N$. Thus, M is isomorphic to a submodule of M. As $f \neq 0$, $M = N$, and so M is simple.

Corollary 9. *An R-module is simple if and only if it is compressible finitely generayed c-co-epi-retractable.*

Recall that an R-module M is cohereditary if every factor module of M is injective. Now, let us introduce the following notion.

Definition 4. *An R-module module is called c-cohereditary if M/C is injective for each nonzero proper complement submodule C of M. The ring R is called c-cohereditary if R_R is c-cohereditary.*

Proposition 9. *A c-cohereditary c-co-epi-retractable R-module M is injective. Moreover, M is a direct sum of a nonsingular module and an injective module.*

Proof.
Suppose M is c-co-epi-retractable. It is well known that $Z_2(M)$ is a complement submodule of M. By the c-co-epi-retractable condition on M, there exists a nonzero endomorphism f of M such that $Kerf = Z_2(M)$. Hence, $M/Kerf \cong Imf$. By our assumption, Imf is injective, and so a direct summand of M. Thus, there exists a submodule K of M such that $M = Imf \oplus K$. By hypthesis again, $M/Imf \cong K$ is injective. Therefore, M is injective. The last part is clear since Imf is nonsingular.

Corollary 10. *A c-co-pri right c-cohereditary ring is right self-injective.*

Corollary 11. *A right extending right c-cohereditary ring is right self-injective.*

Corollary 12. *Any extending c-cohereditary R-module is injective*

We end this section with some applications of c-co-epi-retractable modules regarding the characterization of right SI, semi-simple artinian and quasi-Frobenius rings. Recall that a ring R is said to be right SI if every singular R-module is injective.

Lemma 3. *(([17], Lemma 3.1) and ([11], Theorem 3))*
If R is a right SI-ring, then R is right nonsingular right hereditary and every singular R-module is semi-simple.

Lemma 4. *([4], Corollary 3.2)*
Let M be an R-module having C_3-condition. If $M = M_1 \oplus M_2$ and $f : M_1 \longrightarrow M_2$ is a monomorphism, then $Imf \leq^{\oplus} M_2$.

Theorem 2. *The following conditions are equivalent for a ring R.*
(1) *R is a right SI-ring.*
(2) *Every Z_2-torsion c-co-epi-retractable R-module is injective.*
(3) *Every Goldie-torsion R-module has C_3-condition.*

Proof.
(1) $\Leftrightarrow$ (2) Follows from Theorem 3 in [11].
The implication (1) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1) Let M be a cyclic Z_2-torsion R-module. It is clear that $M \oplus E(M)$ is Goldie-torsion and has the C_3-condition by (4). Consider the inclusion map $i : M \longrightarrow E(M)$. Hence, $i(M) = M \leq^{\oplus} E(M)$ by Lemma 4. It follows that M is injective. This means that every cyclic singular R-module is injective. Therefore, according to ([7], 17.4), R is a right SI-ring.

The following lemmas are crucial in the establishement of the next theorem.

Lemma 5. *([7], Corollary 11.4)*
Let R be ring such that $R_R^{(A)}$ is extending, then the following statements hold true:
(1) *Every nonsingular R-module is extending.*
(2) *Every nonsingular R-module is projective.*

Lemma 6. *([7], 7.11)*
An R-module M is extending if and only if $M = Z_2(M) \oplus M'$, for some submodule M' of M, such that M' and $Z_2(M)$ are both extending and $Z_2(M)$ is M'-injective.

Now, we are able to prove the following result.

Theorem 3. *For a right SI-ring R, the following conditions are equivalent:*
(1) $R_R^{(\mathbb{N})}$ *is extending.*
(2) $R_R^{(\mathbb{N})}$ *is c-co-epi-retractable.*
(3) *Every R-module is extending.*
(4) *Every R-module is c-co-epi-retractable.*

Proof.
(1) ⇒ (2) It is easy to see.
(2) ⇒ (3) Suppose $R_R^{(\mathbb{N})}$ is c-co-epi-retracatble. Hence, for every complement right ideal I of $R^{(\mathbb{N})}$, there exists a right ideal J of $R^{(\mathbb{N})}$ such that $R^{(\mathbb{N})}/I \cong J$. Then it follows from Lemma 3 that $R^{(\mathbb{N})}$ is an extending R-module. Thus, by ([17], Propositions 3.4 and 3.9), R is right Noetherian. So, according to Corollary 11.12 in [7], $R_R^{(A)}$ is extending for any index set A. Now, let M be any R-module. Thus, by Lemma 5, $M = Z_2(M) \oplus N$ for some submodule N of M and clearly N is nonsingular. In view of Lemma 5 again, N is an extending module. Moroever, since R is right nonsingular, $Z_2(M) = Z(M)$ is singular. Hence, $Z_2(M)$ is injective. Therefore, according to Lemma 6, M is extending, as desired.
(3) ⇒ (4) is clear.
(4) ⇒ (1) By (4), $R_R^{(\mathbb{N})}$ is c-co-epi-retracatble. But R is right hereditary. Thus, by the proof of (2) ⇒ (3), $R_R^{(\mathbb{N})}$ is extending.

Corollary 13. *The the following conditions are equivalent for a ring R:*
(1) *R is semi-simple artinian.*
(2) *R is a regular right SI-ring and $R_R^{(\mathbb{N})}$ is c-co-epi-retractable.*
(3) *R is a regular right SI-ring and $R_R^{(\mathbb{N})}$ is extending.*
(4) *R is right SI-ring and $R_R^{(\mathbb{N})}$ is continuous.*
(5) *R is right SI-ring and $R_R^{(\mathbb{N})}$ is quasi-continuous.*

Proof.
(1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) are clear.
(5) ⇒ (1) Assume that R is a right SI-ring such that $R_R^{(\mathbb{N})}$ is quasi-continuous. In particular, $R_R^{(\mathbb{N})}$ is extending. Then, by Theorem 3 every R-module is extending. So by ([7], 13.5), R is an Artinian serial ring. Thus, by ([6], Proposition 6.1 (4)), $R_R^{(\mathbb{N})}$ is quasi-injective, and hence R_R is quasi-injective. Consequently, R is right self-injective by ([12], Remark 6.71(2B)). Since R is right Artinian, R_R has finite uniform dimension. Because R is right SI, it is right nonsingular by Lemma 3. Now, R_R is nonsingular extending and has finite uniform dimension. Thus, R_R is a finite direct sum of uniform submodules. But R is right self-injective. Then as in the proof of (2) ⇒ (3) in theorem 1, R_R is semi-simple. Therefore, R is semi-simple arinian.

Corollary 14. *If R is a right SI-ring such that $R_R^{(\mathbb{N})}$ is c-co-epi-retractable, then all R-modules with a regular endomorphism ring are quasi-injective.*

Proof.
Let M be an R-module with a regular endomorphism ring. Thus, by Theorem 3, M is extending. It follows that M is quasi-continuous. On the other hand, R is Artinian serial by ([7], 13.5). Hence by ([6], Proposition 6.1 (4)), M is quasi-injective.

Note that along the lines of the proof of the above Theorem we have shown that if R is a right SI-ring such that $R^{(\mathbb{N})}$ is right extending, then $R^{(A)}$ is right extending for any index set A.

Theorem 4. *The following conditions are equivalent for a ring R with $\overline{R} = R/Z_2(R_R)$.*
(1) *$\overline{R}$ is semi-simple.*
(2) *Every c-co-epi-retractable R-module is $\overline{R}$-injective.*
(3) *Every nonsingular R-module is quasi-injective.*
(4) *Every nonsingular R-module is quasi-continuous.*
(5) *Every nonsingular R-module has C_3-condition.*
(6) *Every submodule of a nonsingular R-module is a C_3-module.*
(7) *Every submodule of $\overline{R} \oplus \overline{R}$ is a C_3-module.*

Proof.
The implication (1) $\Leftrightarrow$ (2) follows from a similar proof to ([1], Theorem 4.5).
The implication (1) $\Rightarrow$ (3) is clear by ([3], Theorem 3.2).
Implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are easy to see.
(7) $\Rightarrow$ (1) By ([3], Theorem 3.2), we need to show that $\overline{R}$ is semi-simple. Let I be a right ideal of $\overline{R}$. Thus $I \oplus \overline{R}$, being a submodule of $\overline{R} \oplus \overline{R}$ is a C_3-module by (7). Now, let $i : I \longrightarrow \overline{R}$ be the inclusion map. By Lemma 4, I is a direct summand of $\overline{R}$. Therfore, $\overline{R}$ is semi-simple.

Corollary 15. *The following conditions are equivalent for a ring R with $R/Z_2(R) = \overline{R}$.*
(1) *R is quasi-Frobenius.*
(2) *Every c-co-epi-retractable R-module is $\overline{R}$-injective and $Z_2(R_R)$ is an Artinian injective R-module.*
(3) *Every c-co-epi-retractable R-module is $\overline{R}$-injective and $Z_2(R_R)$ is a Noetherian injective R-module.*

Proof.
(1) $\Rightarrow$ (2) Since R is quasi-Frobenius, it is right continuous. Hence, R is a continuous R-module. Thus, $R_R = Z_2(R_R) \oplus R'$ for a continuous R-module R'. It follows that $\overline{R}$ is right nonsingular right continuous. Consequently, $\overline{R}$ is regular. Since R is right Noetherian, $\overline{R}$, also, is right Noetherian. Thus, the property of regular implies that $\overline{R}$ is semi-simple. Thus, in view of Theorem 4, every c-co-epi-retractable R-module is $\overline{R}$-injective. The last part is clear since R_R is injective and Artinian.

(2) ⇒ (1) Assume that every c-co-epi-retractable R-module is $\overline{R}$-injective and $Z_2(R_R)$ is an Artinian injective ring. Since every c-co-epi-retractable R-module is $\overline{R}$-injective, we infer from theorem 4 that $\overline{R}$ is a semi-simple ring. Thus, $\overline{R}$ is semi-simple as an R-module. Since $\overline{R}$ is a nonsingular R-module, $\overline{R}$ is a projective R-module. So, $Z_2(R_R) \leq^{\oplus} R$, say $R = Z_2(R_R) \oplus R'$ where R' is semi-simple ring. By our assumption, R is right Artinian right self-injective. Consequently, R is quasi-Frobenius.
Similarly, (3) is equivalent to (1).

Proposition 10. *The following statements are equivalent for a ring R.*
(1) *R is an Artinian serial ring with $J^2(R) = 0$.*
(2) *Every submodule of a co-c-epi-retractable R-module is extending.*
(3) *Every submodule of an extending R-module is extending.*

Proof.
(1) ⇒ (2) follows from ([7], 13.5).
(2) ⇒ (1) Let M be any R-module. Then $M \oplus E(M)$, being a submodule of $E(M) \oplus E(M)$ is extending by (2). In view of Proposition 2.7 in [14], M is CS. Therefore, R is a Artinian serial ring with $J^2 = 0$ by ([7], 13.5).
Similarly, (1) and (3) are equivalent.

Proposition 11. *The following conditions are equivalent for a ring R.*
(1) *R is semi-simple artinian.*
(2) *Every c-co-epi-retractable R-module is semi-simple.*
(3) *Every c-co-epi-retractable R-module is injective.*
(4) *Every submodule of a c-co-epi-retractable R-module is quasi-continuous.*

Proof.
(1) ⇒ (2) This is clear.
(2) ⇒ (1) Let M be any R-module. By (2), $E(M)$ is semi-simple, and hence $M = E(M)$. Therefore, R is semi-simple artinian.
(1) ⇒ (4) is clear.
(4) ⇒ (1) Let M be any R-module. Then $M \oplus E(M)$, being a submodule of $E(M) \oplus E(M)$ is quasi-continuous by (2). Consequently, $M \oplus E(M)$ has C_3-condition. By Lemma 4, M injective and so R is semi-simple artinian.
(1) ⇔ (3) follows from Corollary 2 in [11].

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