



## Topologies Induced by Neighborhoods of a Graph Under Some Binary Operation

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**Abstract.** Let  $G = (V(G), E(G))$  be any undirected graph. Then  $G$  induces a topology  $\tau_G$  on  $V(G)$  with base consisting of sets of the form  $F_G[A] = V(G) \setminus N_G[A]$ , where  $N_G[A] = A \cup \{x : xa \in E(G) \text{ for some } a \in A\}$  and  $A$  ranges over all subsets of  $V(G)$ . In this paper, we describe the topologies induced by the corona, edge corona, disjunction, symmetric difference, Tensor product, and the strong product of two graphs by determining the subbasic open sets.

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### 1. Introduction

Let  $G = (V(G), E(G))$  be any undirected (simple) graph and let  $v \in V(G)$ . The open neighborhood of  $v$  is the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$  and its closed neighborhood is  $N_G[v] = \{v\} \cup N_G(v)$ . If  $A \subseteq V(G)$ , then the *open neighborhood* of  $A$  is the set  $N_G(A) = \cup_{v \in A} N_G(v)$ . The *closed neighborhood* of  $A$  is  $N_G[A] = A \cup N_G(A)$ . Clearly, if  $A = \{v\}$ , then  $N_G(A) = N_G(v)$  and  $N_G[A] = N_G[v]$ . The degree of  $v$ , denoted by  $deg_G(v)$ , is equal to  $|N_G(v)|$ . The distance between vertices  $u$  and  $v$  of  $G$ , denoted by  $d_G(u, v)$ , is the length of a shortest path connecting  $u$  and  $v$  (or length of a shortest  $u$ - $v$  path).

A way to relate graph theory to topology is to find a way of constructing a topological space from a given graph or devise a method of generating a graph from a given (finite) topological space. In 1983, Gervacio and Diesto in [1] introduced a way of constructing

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a topological space from a given undirected graph. Specifically, they used the closed neighborhood subsets of the vertex set of a graph to obtain a base for some topology on its vertex set. This type of construction of a topological space was also studied by Guerrero and Gervacio in [2] where they characterized those graphs which induce the indiscrete topology and the discrete topology. Canoy and Lemence in [4] investigated further this construction and obtained a subbase for the generated topology. Using this particular result, they described the subbases of the topologies induced by a path, fan, complement of a graph, and graphs resulting from the join, Cartesian product, and composition of two graphs.

Recently, Nianga and Canoy in [7] used the hop neighborhoods of a graph to define a topology on its vertex set. In [8] they describe the subbasic open sets in graphs under some unary and binary operations.

In this paper, we revisit the construction of a topological space given in [1] and describe the topologies induced by the corona, edge corona, disjunction, symmetric difference, Tensor product, and the strong product of two graphs. It can be observed from the subbasic open sets that, generally, it's not easy to determine the basic open sets and exact topologies these graphs induced.

For some basic concepts in graph theory and topology, we refer readers to [3] and [6].

## 2. Results

If  $A \subseteq V(G)$ , we denote by  $F_G(A)$  and  $F_G[A]$  the complements of  $N_G(A)$  and  $N_G[A]$ , respectively, that is,  $F_G(A) = V(G) \setminus N_G(A)$  and  $F_G[A] = V(G) \setminus N_G[A]$ . If  $A = \{v\}$ , then we write  $F_G(A) = F_G(v)$  and  $F_G[A] = F_G[v]$ . Clearly,  $F_G(v) = F_G[v] \cup \{v\}$ .

The first two results are found in [1] and [5], respectively, and play vital roles in the next results.

**Theorem 1.** *Let  $G$  be a graph. Then  $\mathcal{B}_G = \{F_G[A] : A \subseteq V(G)\}$  is a base for some topology on  $V(G)$ .*

Throughout this paper, we denote by  $\tau_G$  the topology on  $V(G)$  generated by the family  $\mathcal{B}_G$  in Theorem 1. This topology is also called the *topology induced by  $G$* .

**Theorem 2.** *Let  $G$  be a graph. Then  $\mathcal{S}_G = \{F_G[v] : v \in V(G)\}$  is a subbase for  $\tau_G$ .*

**Definition 1.** [3] *The corona  $G \circ H$  of graphs  $G$  and  $H$  is the graph obtained by taking one copy of  $G$  and  $|V(G)|$  copies  $H$  and then forming the join  $\langle v \rangle + H^v = v + H^v$  for each  $v \in V(G)$ , where  $H^v$  is a copy of  $H$  corresponding to the vertex  $v$ .*

We now describe the subbasic open sets in the space  $(V(G \circ H), \tau_{G \circ H})$ .

**Theorem 3.** *Let  $K = G \circ H = (V(K), E(K))$ , and let  $a \in V(K)$ .*

(i) *If  $a \in V(G)$ , then*

$$F_K[a] = F_G[a] \cup \left[ \bigcup_{u \in V(G) \setminus \{a\}} V(H^u) \right].$$

(ii) If  $a \in V(H^w)$  for some  $w \in V(G)$ , then

$$F_K[a] = [V(G) \setminus \{w\}] \cup F_{H^w}[a] \cup \left[ \bigcup_{z \in V(G) \setminus \{w\}} V(H^z) \right]$$

*Proof.* (i) Suppose  $a \in V(G)$ . Then

$$N_K[a] = N_G[a] \cup V(H^a)$$

by Definition 1. Hence,

$$\begin{aligned} F_K[a] &= V(K) \setminus (N_G[a] \cup V(H^a)) \\ &= F_G[a] \cup \left[ \bigcup_{u \in V(G) \setminus \{a\}} V(H^u) \right]. \end{aligned}$$

(ii) Suppose  $a \in V(H^w)$  for some  $w \in V(G)$ . Then  $N_K[a] = \{w\} \cup N_{H^w}[a]$  by Definition 1. Thus,

$$\begin{aligned} F_K[a] &= V(K) \setminus (\{w\} \cup N_{H^w}[a]) \\ &= [V(G) \setminus \{w\}] \cup F_{H^w}[a] \cup \left[ \bigcup_{z \in V(G) \setminus \{w\}} V(H^z) \right] \end{aligned}$$

This proves the assertion. □

**Definition 2.** [3] The edge corona  $G \diamond H$  of graphs  $G$  and  $H$  is the graph obtained by taking one copy of  $G$  and  $|E(G)|$  copies  $H$  and joining each of the end vertices  $u$  and  $v$  of every edge  $uv$  to every vertex of the copy  $H^{uv}$  of  $H$  (that is, forming the join  $\langle \{u, v\} \rangle + H^{uv}$  for each  $uv \in E(G)$ ).

**Theorem 4.** Let  $K = G \diamond H = (V(K), E(K))$  and let  $a \in V(G)$ .

(i) If  $a \in V(G)$ , then

$$F_K[a] = F_G[a] \cup \left[ \bigcup_{u, v \neq a} V(H^{uv}) \right].$$

(ii) If  $a \in V(H^{wz})$  for some  $wz \in E(G)$ , then

$$F_K[a] = V(G) \setminus \{w, z\} \cup F_{H^{wz}}[a] \cup \left[ \bigcup_{pq \in E(G) \setminus \{wz\}} V(H^{pq}) \right]$$

*Proof.* (i) Suppose  $a \in V(G)$ . Then, by Definition 2,

$$N_K[a] = N_G[a] \cup \left[ \bigcup_{z \in N_G(a)} V(H^{az}) \right].$$

Consequently,

$$F_K[a] = F_G[a] \cup \left[ \bigcup_{u,v \neq a} V(H^{uv}) \right].$$

(ii) Suppose that  $a \in V(H^{wz})$  for some  $wz \in E(G)$ . Then, by Definition 2,

$$N_K[a] = \{w, z\} \cup N_{H^{wz}}[a].$$

Therefore,

$$F_K[a] = V(G) \setminus \{w, z\} \cup F_{H^{wz}}[a] \cup \left[ \bigcup_{pq \in E(G) \setminus \{wz\}} V(H^{pq}) \right],$$

showing the desired equality. □

**Definition 3.** [3] The Tensor product  $G \boxtimes H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(u, v)$  is adjacent with  $(u', v')$  whenever  $uu' \in E(G)$  and  $vv' \in E(H)$ .

**Theorem 5.** Let  $K = G \boxtimes H = (V(K), E(K))$ , where  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$  are non trivial graphs. Then, for each  $(v, a) \in V(K)$ ,

$$F_K[(v, a)] = [F_G(v) \times (V(H) \setminus \{a\})] \cup [(V(G) \setminus \{v\}) \times F_H(a)].$$

*Proof.* Suppose  $(v, a) \in V(G \boxtimes H) = V(K)$ . By Definition 3,

$$\begin{aligned} N_K[(v, a)] &= \{(x, y) : x \in N_G(v) \text{ and } y \in N_H(a)\} \cup \{(v, a)\} \\ &= [(N_G(v) \times V(H)) \cap (V(G) \times N_H(a))] \cup \{(v, a)\} \\ &= (N_G(v) \times N_H(a)) \cup \{(v, a)\}. \end{aligned}$$

Hence,

$$\begin{aligned} F_K[(v, a)] &= [(F_G(v) \times V(H)) \cup ((V(G) \times F_H(a))] \setminus \{(v, a)\} \\ &= [F_G(v) \times (V(H) \setminus \{a\})] \cup [(V(G) \setminus \{v\}) \times F_H(a)], \end{aligned}$$

showing the desired result. □

**Corollary 1.** Let  $G$  be any graph and let  $(v, a) \in V(G \boxtimes K_n)$ . Then

$$F_{G \boxtimes K_n}[(v, a)] = [F_G(v) \times (V(K_n) \setminus \{a\})] \cup [(V(G) \setminus \{v\}) \times \{a\}].$$

*Proof.* Since  $a \in V(K_n)$ ,  $F_{K_n}[a] = \emptyset$ . Hence,  $F_{K_n}(a) = \{a\}$ . Therefore, by Theorem 5,

$$F_{G \boxtimes K_n}[(v, a)] = [F_G(v) \times (V(K_n) \setminus \{a\})] \cup [(V(G) \setminus \{v\}) \times \{a\}].$$

This proves the assertion. □

**Definition 4.** [3] *The disjunction  $G \vee H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(u, v)$  is adjacent with  $(u', v')$  whenever  $uu' \in E(G)$  or  $vv' \in E(H)$ .*

**Theorem 6.** *Let  $K = G \vee H = (V(K), E(K))$ , where  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$ . Then, for each  $(v, a) \in V(K)$ ,*

$$F_K[(v, a)] = [\{v\} \times F_H[a]] \cup [F_G[v] \times F_H(a)].$$

*Proof.* Let  $A = [\{v\} \times F_H[a]] \cup [F_G[v] \times F_H(a)]$ . Suppose  $(v, a) \in V(G \vee H) = V(K)$  and  $(x, q) \in F_K[(v, a)]$ . Then  $(v, a) \neq (x, q)$  and  $d_K((v, a), (x, q)) \neq 1$ . Consider the following cases:

Case 1. Assume that  $x = v$ . Then  $q \neq a$  and  $d_H(q, a) = d_K((x, q), (x, a)) \neq 1$ . Thus,  $q \in F_H[a]$  and so,  $(x, q) \in \{v\} \times F_H[a]$ .

Case 2. Assume that  $x \neq v$ . Suppose  $q = a$ . Then  $d_G(x, v) = d_K((v, a), (x, a)) \neq 1$ . It follows that  $x \in F_G[v]$  and  $(x, q) \in F_G[v] \times \{a\}$ . Suppose  $q \neq a$ . Observe that  $q \notin N_H[a]$  and  $x \notin N_G[v]$ . Hence,  $q \in F_H[a]$  and  $x \in F_G[v]$  which implies that  $(x, q) \in F_G[v] \times F_H[a]$ . Thus,

$$\begin{aligned} F_K[(v, a)] &\subseteq [\{v\} \times F_H[a]] \cup [F_G[v] \times \{a\}] \cup [F_G[v] \times F_H[a]] \\ &= [\{v\} \times F_H[a]] \cup [F_G[v] \times F_H(a)] = A. \end{aligned}$$

Next, let  $(u, z) \in A$ . If  $(u, z) \in \{v\} \times F_H[a]$ . Then  $u = v, a \neq z$  and  $az \notin E(H)$ . Hence,  $(u, z) \neq (v, a)$  and by Definition 4,  $(u, z)(v, a) \notin E(K)$ . Thus,  $(u, z) \in F_K[(v, a)]$ . Also if  $(u, z) \in F_G[v] \times F_H(a), u \neq v, uv \notin E(G)$ , and  $az \notin E(H)$ . This implies that  $(u, z) \neq (v, a)$  and by Definition 4,  $(u, z)(v, a) \notin E(K)$ . Thus  $(u, z) \in F_K[(v, a)]$ . Consequently,  $A \subseteq F_K[(v, a)]$ . Therefore,  $F_K[(v, a)] = A$ . □

**Corollary 2.** *Let  $G$  be any graph and let  $(v, a) \in V(G \vee K_n)$ . Then  $F_{G \vee K_n}[(v, a)] = F_G[v] \times \{a\}$ .*

*Proof.* Again, since  $a \in V(K_n)$ ,  $F_{K_n}(a) = \{a\}$  and  $\{v\} \times F_{K_n}[a] = \emptyset$ . Hence, by Theorem 6,  $F_{G \vee K_n}[(v, a)] = F_G[v] \times \{a\}$ . □

**Definition 5.** [3] *The symmetric difference  $G \oplus H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(u, v)$  is adjacent with  $(u', v')$  whenever  $[uu' \in E(G)]$  or  $[vv' \in E(H)]$  but not both.*

**Theorem 7.** *Let  $K = G \oplus H = (V(K), E(K))$ , where  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$ . Then, for each  $(v, a) \in V(K)$ ,*

$$F_K[(v, a)] = [F_G(v) \times F_H[a]] \cup [N_G(v) \times N_H(a)] \cup [F_G[v] \times \{a\}].$$

*Proof.* Let  $W = [F_G(v) \times F_H[a]] \cup [N_G(v) \times N_H(a)] \cup [F_G[v] \times \{a\}]$ . Suppose  $(v, a) \in V(G \oplus H) = V(K)$  and let  $(x, q) \in F_K[(v, a)]$ . Then  $(v, a) \neq (x, q)$  and  $d_K((v, a), (x, q)) \neq 1$ . Now, consider the following cases:

Case 1. Assume  $x = v$ . Then  $q \neq a$  and  $d_H(q, a) = d_K((x, q), (x, a)) \neq 1$ . Hence,  $q \in F_H[a]$  and so,  $(x, q) \in \{v\} \times F_H[a]$ .

Case 2. Assume that  $x \neq v$ . Suppose  $q = a$ . Then  $d_G(x, v) = d_K((v, a), (x, a)) \neq 1$ . This implies that  $x \in F_G[v]$  and  $(x, q) \in F_G[v] \times \{a\}$ . Suppose  $q \neq a$ . If  $x \in N_G(v)$ , then  $q \in N_H(a)$ . If  $x \in F_G[v]$ , then  $q \notin N_H(a)$  and so,  $q \in F_H[a]$ . Hence,

$$\begin{aligned} F_K[(v, a)] &\subseteq [\{v\} \times F_H[a]] \cup [F_G[v] \times \{a\}] \cup [N_G(v) \times N_H(a)] \cup [F_G[v] \times F_H[a]] \\ &= [F_G(v) \times F_H[a]] \cup [N_G(v) \times N_H(a)] \cup [F_G[v] \times \{a\}] = W. \end{aligned}$$

Conversely, let  $(b, c) \in W$ . If  $(b, c) \in F_G(v) \times F_H[a]$ , then  $c \neq a$ ,  $bv \notin E(G)$ , and  $ac \notin E(H)$ . Hence,  $(b, c) \neq (v, a)$  and, by Definition 5,  $(b, c)(v, a) \notin E(K)$ . Thus,  $(b, c) \in F_K[(v, a)]$ . Also, if  $(b, c) \in N_G(v) \times N_H(a)$ , then  $(b, c) \neq (v, a)$  and by Definition 5,  $(b, c)(v, a) \notin E(K)$ . Hence,  $(b, c) \in F_K[(v, a)]$ . Finally, if  $(b, c) \in F_G[v] \times \{a\}$ , then  $b \neq v$ ,  $bv \notin E(G)$ , and  $ac \notin E(H)$ . It follows that  $(b, c) \neq (v, a)$  and by Definition 5,  $(b, c)(v, a) \notin E(K)$  which shows that  $(b, c) \in F_K[(v, a)]$ . Thus,  $W \subseteq F_K[(v, a)]$ . Therefore,  $F_K[(v, a)] = W$ .  $\square$

**Corollary 3.** *Let  $G$  be any graph and let  $(v, a) \in V(G \oplus K_n)$ . Then  $F_{G \oplus K_n}[(v, a)] = (N_G(v) \times [V(K_n) \setminus \{a\}]) \cup (F_G[v] \times \{a\})$ .*

*Proof.* Since  $F_{K_n}[a] = \emptyset$  and  $N_{K_n}(a) = V(K_n) \setminus \{a\}$ , it follows from Theorem 7 that  $F_{G \oplus K_n}[(v, a)] = (N_G(v) \times [V(K_n) \setminus \{a\}]) \cup (F_G[v] \times \{a\})$ .  $\square$

**Definition 6.** [3] *The strong product  $G \otimes H$  of graphs  $G$  and  $H$  is the graph with vertex set  $V(G) \times V(H)$  and  $(u, v)$  is adjacent with  $(u', v')$  whenever  $[uu' \in E(G) \text{ and } v = v']$  or  $[vv' \in E(H) \text{ and } u = u']$  or  $[uu' \in E(G) \text{ and } vv' \in E(H)]$ .*

**Theorem 8.** *Let  $K = G \otimes H = (V(K), E(K))$  where  $G = (V(G), E(G))$  and  $H = (V(H), E(H))$ . Then, for each  $(v, a) \in V(K)$ ,*

$$F_K[(v, a)] = [F_G[v] \times V(H)] \cup [N_G[v] \times F_H[a]].$$

*Proof.* Let  $Z = [F_G[v] \times V(H)] \cup [N_G[v] \times F_H[a]]$ . Suppose  $(v, a) \in V(G \otimes H) = V(K)$ . Then  $(x, q) \in F_K[(v, a)]$  if and only if  $(v, a) \neq (x, q)$  and  $d_K((v, a), (x, q)) \neq 1$ . Consider the following cases:

Case 1. Assume that  $x = v$ . Then  $q \neq a$  and  $d_H(q, a) = d_K((x, q), (x, a)) \neq 1$ . Thus  $q \in F_H[a]$  and so,  $(x, q) \in \{v\} \times F_H[a]$ .

Case 2. Assume that  $x \neq v$ . Suppose  $q = a$ . Then  $d_G(x, v) = d_K((v, a), (x, a)) \neq 1$ . It follows that  $x \in F_G[v]$  and  $(x, q) \in F_G[v] \times \{a\}$ . Suppose  $q \neq a$ . If  $x \in N_G(v)$ , then  $q \notin N_G(a)$  and so,  $q \in F_G[a]$ . Suppose  $x \in F_G[v]$ . Since  $q \neq a$ ,  $q \in V(H) \setminus \{a\}$ . Therefore,

$$F_K[(v, a)] \subseteq [\{v\} \times F_H[a]] \cup [F_G[v] \times \{a\}] \cup [N_G(v) \times F_G[a]] \cup [F_G[v] \times V(H) \setminus \{a\}]$$

$$= [N_G[v] \times F_G[a]] \cup [F_G[v] \times V(H)] = Z.$$

Next, let  $(w, z) \in Z$ . If  $(w, z) \in F_G[v] \times V(H)$  then  $w \neq v$  and  $wv \notin E(G)$ . Hence  $(w, z) \neq (v, a)$  and by Definition 6,  $(w, z) \notin N_K((v, a))$ . Thus,  $(w, z) \in F_G[(v, a)]$ . If  $(w, z) \in N_G[v] \times F_H[a]$ , then  $z \neq a$ ,  $az \notin E(H)$  and either  $w = v$  or  $wv \in E(G)$ . This means that  $(w, z) \neq (v, a)$  and by Definition 6,  $(w, z) \notin N_K((v, a))$ . It follows that  $(w, z) \in F_K[(v, a)]$ . This shows that  $Z \subseteq F_K[(v, a)]$ . Therefore,  $F_K[(v, a)] = Z$ .  $\square$

**Corollary 4.** *Let  $G$  be any graph and let  $(v, a) \in V(G \otimes K_n)$ . Then  $F_{G \otimes K_n}[(v, a)] = F_G[v] \times V(K_n)$ .*

*Proof.* Since  $F_{K_n}[a] = \emptyset$ , Theorem 8 would imply that  $F_{G \otimes K_n}[(v, a)] = F_G[v] \times V(K_n)$ .  $\square$

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