



Strategic Asset Allocation for Life Insurers with Stochastic Liability[†]

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Abstract. The problem of strategic asset allocation and product mix choice of a life insurance company is considered where account is taken of the stochastic risk associated with both assets and liabilities. Using the methods of stochastic dynamic programming we derive equations for optimal weights of both risky and riskless assets under continuous time. The resulting equations can be solved exactly for some parameter values and utility functions.

When this is not possible a general perturbation expansion method is set up for which explicit solutions are derived for the first terms but the method can be generalized to any order in the expansion parameter.

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1. Introduction

The financial management of an insurance company involves the analysis of the assets and liabilities on a unified basis. The premiums paid by the policyholder should be prudently invested so that the company can honor the contractual obligation that comes with the policy with a high level of confidence. In its most basic form, asset-liability management (ALM) dictates the investment choice on the asset side and the choice of products marketed on the liability side. Moreover, ALM deals with the planning of financial resources with uncertainty about the economic, capital markets and actuarial conditions. The two controls available to the management of the insurance company are thus the

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allocation of the premiums received from the policyholders to the various invested assets as well as the decision to market different insurance products[†].

Insurers currently use cashflow projection of liabilities based on Monte Carlo simulation as a starting point for the strategic asset allocation exercise. Several papers e.g. Gerstner, Griebel and Holtz (2009) used deterministic numerical simulation to model Asset-Liability management (ALM). These numerical techniques are often cumbersome and lack the transparency offered by analytical method which we apply in this paper. Here, we show that if the utility of the company is of the constant relative risk aversion (CRRA) form then the solution can be found exactly for the case where the correlation ρ between the risky asset and the risky insurance liability is zero. For the non-zero correlation case, we find that we can solve for the solution provided that the interest rates for savings (r_a) and loans (r_l) are the same. To solve for the scenario where the interest rates on savings and loans differ only slightly, we devise a perturbation method and find a first and second order solution though the method can be generalized to any order.

This work is based on Merton's continuous time optimal portfolio selection and consumption rule, Merton (1990). Here an investor wishes to maximise the expected utility of consumption over his life time. For the case of the utility functions of the type constant relative risk aversion (CRRA), one can find the optimal allocation and consumption rule which will maximise the lifetime expected utility of the investor.

2. Governing Equations

2.1. Evolution of the Liability L

Let L be the liabilities of the company. L is made up of the risky insurance liability uL and riskless liability $(1 - u)L$. The risky insurance liability comes from the stochastic nature of the benefits outgo (claims payment, surrender, and maturity benefits). The riskless liability is defined to be liabilities that are deterministic in value such as debt issued by the company.

Suppose that we assume that the liability of a life insurance company evolves according to the stochastic differential equation

$$dL_t = (cuL_t - muL_t + (1 - u)L_tr_l)dt - qL_tudZ(t) \quad (1)$$

where L_t is the liability at time t , c (or m) represents the exponential growth (or decay) rate in L , qdZ is the stochastic component which represents the random nature of insurance, dZ is an infinitesimal Wiener increment. The stochastic term $-qL_tudZ(t)$ can increase or decrease the liability of the insurance company. The variable r_l represents the interest rate at which the riskless liability grows; equivalently this is the same as the interest rate charged for a loan. The control variable u allows the management to vary the proportion of the policyholder's liability between the deterministic (riskless) liability which grows at the rate r_l and the stochastic (risky) liability with some known probability distribution.

[†]Life insurance products have vastly different risk profiles. The decision to market a particular product should be based on the ability to find suitable assets to invest in.

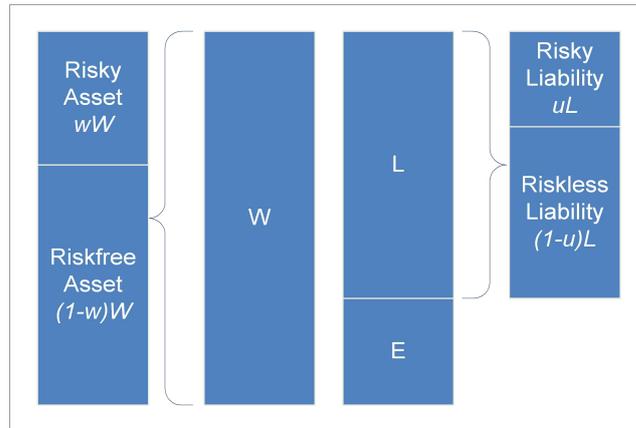


Figure 1: Illustration of the insurer's balance sheet and the associated control variables w and u .

In reality, this can be done through risk transfer techniques such as reinsurance or hedging with swaps. Intuitively, suppose that there is a death claim then the company's liability decreases (i.e. once the company has paid out the claim of this particular policy, the contract is terminated). Also, the liability can increase if the actuary decides to strengthen the reserve.

2.2. Evolution of the total Asset W

Let us assume that the company's assets are comprised of the risky asset, $S_1 (= wW)$ and the risk-free asset $S_0 (= (1 - w)W)$ with

$$dS_1 = \alpha S_1 dt + \sigma S_1 dX \tag{2}$$

$$dS_0 = r_a S_0 dt \tag{3}$$

then we can write

$$dW = [w(\alpha - r_a) + r_a] W dt + wW\sigma dX \tag{4}$$

where dX is an infinitesimal Wiener increment. The total asset of the company W is by definition

$$W \equiv L + E \tag{5}$$

where E stands for equity and represents the portion of assets owned by the shareholders once all the liabilities have been paid out. The illustration of the insurer's balance sheet is shown in Figure (1).

Then

$$dE = dW - dL \tag{6}$$

$$\begin{aligned}
 &= \{[w(\alpha - r_a) + r_a]W - cuL_t - ((1 - u)r_l - mu)L_t\} dt \\
 &+ qL_t u dZ + wW \sigma dX
 \end{aligned}
 \tag{7}$$

The problem for choosing the strategic asset allocation and insurance product mix strategy is formulated as follows;

$$\max E \left\{ \int_0^T U [W(t), L(t)] dt + B [W(T), L(T), T] \right\}$$

subject to the budget equation (4), and $W(t) > 0; W(0) = W_0 > 0$. Let us assume $U(W, L)$ to be a strictly concave utility function and $B[W(t), T]$ is a specified bequest valuation function. More details on the bequest valuation function can be found in Merton (1969). Here E is short for $E(0)$, the conditional expectation operator given W_0 is known.

The differential equations governing the optimal solution of this stochastic control problem for the portfolio selection (w is the weight of risky investment) and liability mix (u is the weight of the risky insurance liability) can be derived by the methods of stochastic dynamic programming see e.g. Merton (1990).

Define the value function $J(W(t), L(t), t)$ by

$$\begin{aligned}
 J &= \max E_t \left\{ \int_t^T U(W(t), L(t)) dt + B(W(T), L(T), T) \right\} \\
 &= \max E_t \left\{ \int_t^{t+\delta t} U(W(t), L(t)) dt \right. \\
 &\quad \left. + J(W(t + \delta t), L(t + \delta t), t + \delta t) \right\}
 \end{aligned}
 \tag{8}$$

where E_t is the conditional expectation operator given $W(t)$ is known. If one now expands the term in J on the right hand side of (8) by Taylor's series using equations (1) and (4), apply Ito's Lemma, and then take the limit as δt tends to zero one arrives at the Hamilton-Jacobi equation for the value function given by

$$\begin{aligned}
 0 &= \phi[w^*, u^*; W, L, t] \equiv \max_{\{u, w\}} \left\{ U(W, L) + \frac{\partial J}{\partial t} \right. \\
 &+ [w(\alpha - r_a) + r_a]W \frac{\partial J}{\partial W} + \frac{1}{2} \sigma^2 w^2 W^2 \frac{\partial^2 J}{\partial W^2} \\
 &+ \{cu - mu + (1 - u)r_l\} L \frac{\partial J}{\partial L} \\
 &\left. + qLWuw\sigma\rho \frac{\partial^2 J}{\partial L \partial W} + \frac{1}{2} q^2 L^2 u^2 \frac{\partial^2 J}{\partial L^2} \right\}
 \end{aligned}
 \tag{9}$$

The optimality condition may be rewritten as

$$\phi[w^*, u^*; W, L, t] = 0,
 \tag{10}$$

for fixed W and L at anytime t . The optimal strategy of (9) is given by w^* and u^* . We differentiate (10) with respect to w and u and set to zero to find the interior maximum.

This gives

$$(\alpha - r_a) W \frac{\partial J}{\partial W} + \sigma^2 w W^2 \frac{\partial^2 J}{\partial W^2} + q L W u \sigma \rho \frac{\partial^2 J}{\partial L \partial W} = 0 \tag{11}$$

and

$$\{c - m - r_l\} L \frac{\partial J}{\partial L} + q L W w \sigma \rho \frac{\partial^2 J}{\partial L \partial W} + u q^2 L^2 \frac{\partial^2 J}{\partial L^2} = 0 \tag{12}$$

Note that $w \frac{\partial \phi}{\partial w} + u \frac{\partial \phi}{\partial u}$ gives

$$\begin{aligned} & \frac{1}{2} \sigma^2 w^2 W^2 \frac{\partial^2 J}{\partial W^2} + \frac{1}{2} u^2 q^2 L^2 \frac{\partial^2 J}{\partial L^2} + q L W w u \sigma \rho \frac{\partial^2 J}{\partial L \partial W} \\ & = \frac{1}{2} \left\{ -\{c - m - r_l\} u L \frac{\partial J}{\partial L} - (\alpha - r_a) w W \frac{\partial J}{\partial W} \right\} \end{aligned} \tag{13}$$

Substituting this into $\phi = 0$ gives

$$\begin{aligned} 0 = \max_{\{u,w\}} & \left\{ U(W, L) + \frac{\partial J}{\partial t} + \left[\frac{1}{2} w (\alpha - r_a) + r_a \right] W \frac{\partial J}{\partial W} \right. \\ & \left. + \left\{ \frac{1}{2} c u - \frac{1}{2} m u + \left(1 - \frac{1}{2} u \right) r_l \right\} L \frac{\partial J}{\partial L} \right\} \end{aligned} \tag{14}$$

subject to the terminal condition $J[W(T), L(T), T] = B[W(T), L(T), T]$ and the solution being feasible.

3. Utility $U(W) = \frac{W^\gamma}{\gamma}$

For this particular case, the utility function is a function of W only. This suggests that the management is most interested in maximising the firm’s total asset. This is not as nonsensical as it may first seem as financial institutions are constantly seeking to reduce their expense ratio; increasing the size of the asset under management is one way of achieving this.

Suppose that $U(W) = \frac{W^\gamma}{\gamma}$ and assume that the value function has the form $J = W^\gamma F(L, t)$, where $F(L, t)$ is an arbitrary function independent of W , then the optimality conditions for w and u in equations (11) and (12) become

$$(\alpha - r_a) F + \sigma^2 w (\gamma - 1) F + q L u \sigma \rho \frac{\partial F}{\partial L} = 0 \tag{15}$$

and

$$(c - m - r_l + q w \sigma \rho \gamma) \frac{\partial F}{\partial L} + u q^2 L \frac{\partial^2 F}{\partial L^2} = 0. \tag{16}$$

The value function as given in (9) becomes

$$0 = \phi[w^*, u^*; W, L, t] \equiv \max_{\{u(s), w(s)\}} \left\{ \frac{1}{\gamma} + \frac{\partial F}{\partial t} \right.$$

$$\begin{aligned}
 & + \left(w(\alpha - r_a) + r_a + \frac{1}{2}\sigma^2 w^2(\gamma - 1) \right) \gamma F \\
 & + (cu - mu + (1 - u)r_l + quw\sigma\rho\gamma) L \frac{\partial F}{\partial L} \\
 & + \frac{1}{2}q^2 u^2 L^2 \frac{\partial^2 F}{\partial L^2} \}. \tag{17}
 \end{aligned}$$

3.1. Zero Correlation Case

In this particular scenario, we assume that the correlation between the risky asset and the risky liability is zero. Note that if $\rho = 0$ then from (15) and (16) we obtain

$$w^* = \frac{(\alpha - r_a)}{\sigma^2(1 - \gamma)} \tag{18}$$

and

$$u^* = \left(\frac{-(c - m - r_l) \frac{\partial F}{\partial L}}{q^2 L \frac{\partial^2 F}{\partial L^2}} \right) \tag{19}$$

Then from (17)

$$\begin{aligned}
 0 &= \frac{1}{\gamma} + \frac{\partial F}{\partial t} + \left(r_a + \frac{1(\alpha - r_a)^2}{2\sigma^2(1 - \gamma)} \right) \gamma F \\
 &+ \left(r_l - \frac{(c - m - r_l)^2 \frac{\partial F}{\partial L}}{2q^2 L \frac{\partial^2 F}{\partial L^2}} \right) L \frac{\partial F}{\partial L} \tag{20}
 \end{aligned}$$

If we choose

$$F(L, t) = A(t)L^\lambda + B(t), \tag{21}$$

then $\frac{\partial F}{\partial L} = A(t)\lambda L^{\lambda-1}$, $\frac{\partial^2 F}{\partial L^2} = A(t)\lambda(\lambda - 1)L^{\lambda-2}$, $\frac{\partial F}{\partial t} = \dot{A}(t)L^\lambda + \dot{B}(t)$. Substituting this into (20) gives

$$\begin{aligned}
 0 &= \frac{1}{\gamma} + \dot{A}(t)L^\lambda + \dot{B}(t) + \left(r_a + \frac{1(\alpha - r_a)^2}{2\sigma^2(1 - \gamma)} \right) \gamma (A(t)L^\lambda + B(t)) \\
 &+ \left(r_l - \frac{(c - m - r_l)^2}{2q^2(\lambda - 1)} \right) A\lambda L^\lambda. \tag{22}
 \end{aligned}$$

This implies that

$$0 = \frac{1}{\gamma} + \dot{B}(t) + \left(r_a + \frac{1(\alpha - r_a)^2}{2\sigma^2(1 - \gamma)} \right) \gamma B(t) \tag{23}$$

and

$$0 = \dot{A}(t) + \left(r_a + \frac{1(\alpha - r_a)^2}{2\sigma^2(1 - \gamma)} \right) \gamma A(t)$$

$$+ \left(r_l - \frac{(c - m - r_l)^2}{2q^2(\gamma - 1)} \right) \lambda A(t). \tag{24}$$

Therefore,

$$A(t) = A_0 \exp \left[- \left(r_a \gamma + \frac{1}{2} \frac{(\alpha - r_a)^2 \gamma}{\sigma^2(1 - \gamma)} + r_l \lambda - \frac{(c - m - r_l)^2 \lambda}{2q^2(\lambda - 1)} \right) t \right]. \tag{25}$$

We now solve for $A(t)$ and $B(t)$ with the terminal condition $F(L, T) = A(T)L^2 + B(T)$, with $A(t)$ given in (25) then

$$u^* = \frac{c - m - r_l}{q^2(1 - \lambda)}. \tag{26}$$

If this final condition is $F(L, T) = 0$ then from (21) we obtain $B(T) = 0$ and $A(T) = 0$ and then

$$B(t) = \frac{1}{\gamma^2} \left(r_a + \frac{1}{2} \frac{(\alpha - r_a)^2}{\sigma^2(1 - \gamma)} \right) \left(\exp \left[- \left\{ r_a + \frac{1}{2} \frac{(\alpha - r_a)^2}{\sigma^2(1 - \gamma)} \right\} \gamma (t - T) \right] - 1 \right)$$

$$A(t) \equiv 0.$$

Hence the solution is independent of u . This is aligned with our intuition that if the utility depends only on W and that the risky insurance liability is not related to the risky asset then we would expect to find that the strategic allocation of the risky asset, w , is the same as Merton’s solution and the optimal product mix u can take on any value.

4. Utility $U(W, L) = \frac{(W-L)^\gamma}{\gamma}$

The power utility function assumed in this paper belongs to the class of utility functions known as constant relative risk aversion (CRRA). Utility of this class are sensible because we observe that companies strive to link the size of the risk that they take to their capacity for risk absorption.

4.1. Non-zero correlation and $r_a = r_l$

By setting the utility of the form $\frac{(W-L)^\gamma}{\gamma}$, we are assuming that the management is interested in maximising the utility based on shareholder’s equity. It can be argued that a utility function that depends on the shareholder’s equity is more sensible than a utility which depends solely on the size of the company as shown in the earlier example.

Suppose that the utility function has the form $U(E) = \frac{E^\gamma}{\gamma} = \frac{(W-L)^\gamma}{\gamma}$ and we seek the solution to the value function of the form $J(W, L, t) = A(t) \frac{(W-L)^\gamma}{\gamma}$

Thus from (11) and (12) we have

$$\sigma^2 w W (\gamma - 1) - quL\sigma\rho(\gamma - 1) = -(\alpha - r_a)(W - L) \tag{27}$$

and

$$-qwW\sigma\rho(\gamma - 1) + q^2uL(\gamma - 1) = \{c - m - r_l\}(W - L) \tag{28}$$

which can be rewritten as

$$\begin{pmatrix} \sigma^2(\gamma - 1) & -q\sigma\rho(\gamma - 1) \\ -q\sigma\rho(\gamma - 1) & q^2(\gamma - 1) \end{pmatrix} \begin{pmatrix} wW \\ uL \end{pmatrix} = \begin{pmatrix} -(\alpha - r_a)(W - L) \\ \{c - m - r_l\}(W - L) \end{pmatrix}. \tag{29}$$

The equation (29) can then be solved to give

$$wW = Q_1(W - L) \tag{30}$$

$$uL = Q_2(W - L) \tag{31}$$

where

$$Q_1 = \frac{-(\alpha - r_a)q^2 + q\sigma\rho(c - m - r_l)}{\sigma^2q^2(\gamma - 1)(1 - \rho^2)} \tag{32}$$

and

$$Q_2 = \frac{-q\sigma\rho(\alpha - r_a) + \sigma^2(c - m - r_l)}{\sigma^2q^2(\gamma - 1)(1 - \rho^2)}. \tag{33}$$

Note that the weights w and u are time independent depending on the parameters $\alpha, r_a, r_l, q, \gamma, \sigma, \rho, c, m$. and also on the ratio $\frac{E}{W}$. Also these weights involve both r_a and r_l and they have only been assumed equal to solve the full Hamilton-Jacobi equation to this order.

When we go to the next order, order ε , we must keep r_a and r_l in the above equations.

With the value function of the form $J(W, L, t) = A(t) \frac{(W-L)^\gamma}{\gamma}$ and its respective derivatives, we can substitute (30) and (31) into (9), which gives

$$\begin{aligned} 0 = & \left\{ \frac{(W - L)^\gamma}{\gamma} + A' \frac{(W - L)^\gamma}{\gamma} \right. \\ & + (\alpha - r_a) Q_1 A (W - L)^\gamma + \frac{1}{2} \sigma^2 Q_1^2 (\gamma - 1) A (W - L)^\gamma \\ & - \{c - m - r_l\} Q_2 A (W - L)^\gamma + r_a W A (W - L)^{\gamma-1} - r_l L A (W - L)^{\gamma-1} \\ & \left. - q\sigma\rho Q_1 Q_2 (\gamma - 1) A (W - L)^\gamma + \frac{1}{2} q^2 Q_2^2 (\gamma - 1) A (W - L)^\gamma \right\} \tag{34} \end{aligned}$$

Dividing (34) by $(W - L)^\gamma$ gives

$$\begin{aligned} 0 = & \left\{ \frac{1}{\gamma} + \frac{A'}{\gamma} \right. \\ & + \left((\alpha - r_a) Q_1 + \frac{1}{2} \sigma^2 Q_1^2 (\gamma - 1) - \{c - m - r_l\} Q_2 \right. \\ & \left. - q\sigma\rho Q_1 Q_2 (\gamma - 1) + \frac{1}{2} q^2 Q_2^2 (\gamma - 1) \right) A \\ & \left. + \frac{(r_a W - r_l L) A}{(W - L)} \right\}. \tag{35} \end{aligned}$$

4.1.1. Case: $r_a \equiv r_l$

The equation above has a special solution when $r_a \equiv r_l$. In this particular case, we can cancel out $W - L$ and solve for $A(t)$. From a practical perspective, equating r_a to r_l implies that the risk-free interest rate on the asset side is assumed to be the same as the borrowing rate on the liability side. This situation is likely to be found in developed markets where policyholders exhibit a high level of financial literacy and hence will demand that the compensation for opportunity cost be on par with the observed riskfree rate. Hence, for this case, we can solve for $A(t)$.

To solve for $A(t)$, consider

$$0 = 1 + A' + \left((\alpha - r_a) Q_1 + \frac{1}{2} \sigma^2 Q_1^2 (\gamma - 1) - \{c - m - r_l\} Q_2 - q\sigma\rho Q_1 Q_2 (\gamma - 1) + \frac{1}{2} q^2 Q_2^2 (\gamma - 1) + r_a \right) \gamma A. \tag{36}$$

We can rewrite (34) by letting

$$\kappa = \left(\begin{array}{c} (\alpha - r_a) Q_1 + \frac{1}{2} \sigma^2 Q_1^2 (\gamma - 1) - \{c - m - r_l\} Q_2 \\ -q\sigma\rho Q_1 Q_2 (\gamma - 1) + \frac{1}{2} q^2 Q_2^2 (\gamma - 1) + r_a \end{array} \right) \gamma$$

then

$$0 = \frac{dA}{dt} + \kappa A + 1$$

and

$$A(t) = \frac{-1}{\kappa} + e^{-\kappa t} C_1.$$

4.1.2. Case: $r_a \neq r_l$

It is clear from equation (35) that the choice of $J(W, L, T) = A(T) \frac{(W-L)^\gamma}{\gamma}$ is a solution only in the case when $r_a = r_l$ because only then do W and L disappear in equation (35) leaving an ordinary differential equation for $A(t)$. However, if the difference between r_a and r_l is assumed small it is possible to solve equations (11), (12), and (13) as a power series in ε where ε is the difference between r_a and r_l . In less developed markets, the policyholders may not demand that the return on their policies to match the observed risk free asset unlike the previous case.

We assume here that $|\varepsilon| \ll r_a$ and describe the method for determining the order ε correction we generalise the method to get higher order terms in the appendix.

We write

$$J = J_0 + \varepsilon J_1$$

where J_0 is the value function determined in the last section. Note that when $\varepsilon = 0$, we have the previous case and the value function J . Now we write

$$wW = \bar{w},$$

$$uL = \bar{u}$$

and expand as

$$\begin{aligned} \bar{w} &= \bar{w}_0 + \varepsilon \bar{w}_1 + o(\varepsilon^2) \\ \bar{u} &= \bar{u}_0 + \varepsilon \bar{u}_1 + o(\varepsilon^2) \end{aligned}$$

where \bar{w}_0 and \bar{u}_0 are the solutions of equations (30) and (31) where $J = J_0$. With the above notation we can expand equations (11) and (12) to get for $O(1)$

$$A_0 \begin{pmatrix} \bar{w}_1 \\ \bar{u}_1 \end{pmatrix} = \begin{pmatrix} -(\alpha - r_a) J_{1W} - \sigma^2 \bar{w}_0 J_{1WW} - q\sigma\rho\bar{u}_0 J_{1LW} \\ -\bar{c} J_{1L} - q^2 \bar{u}_0 J_{1LL} - q\sigma\rho\bar{w}_0 J_{1LW} + J_{0L} \end{pmatrix}$$

where $\bar{c} = c - m - r_a$ and $A_0 = \begin{pmatrix} \sigma^2 \frac{\partial^2 J_0}{\partial W^2} & q\sigma\rho \frac{\partial^2 J_0}{\partial L \partial W} \\ q\sigma\rho \frac{\partial^2 J_0}{\partial L \partial W} & q^2 \frac{\partial^2 J_0}{\partial L^2} \end{pmatrix}$.

For the Hamilton-Jacobi equation (9) when $U(E) = \frac{E\gamma}{\gamma}$ it is best to use equation (14) i.e.

$$\begin{aligned} 0 &= U(E) + \frac{\partial J}{\partial t} + \frac{1}{2}(\alpha - r_a)\bar{w} \frac{\partial J}{\partial W} + \frac{1}{2}\bar{c}\bar{u} \frac{\partial J}{\partial L} \\ &+ r_a \left(W \frac{\partial J}{\partial W} + L \frac{\partial J}{\partial L} \right) + \varepsilon L \frac{\partial J}{\partial L} \end{aligned} \tag{37}$$

when $r_l = r_a + \varepsilon$.

Then for zero order we have

$$\begin{aligned} 0 &= U(E) + \frac{\partial J_0}{\partial t} + \frac{1}{2}(\alpha - r_a)\bar{w}_0 \frac{\partial J_0}{\partial W} + \frac{1}{2}\bar{c}\bar{u}_0 \frac{\partial J_0}{\partial L} \\ &+ r_a \left(W \frac{\partial J_0}{\partial W} + L \frac{\partial J_0}{\partial L} \right) \end{aligned} \tag{38}$$

which we can write as

$$U(E) + L_0(J_0) = 0.$$

For order ε , from (37) we also get

$$0 = L_0(J_1) + L \frac{\partial J_0}{\partial L} + \frac{1}{2}(\alpha - r_a)\bar{w}_1 \frac{\partial J_0}{\partial W} + \frac{1}{2}\bar{c}\bar{u}_1 \frac{\partial J_0}{\partial L} \tag{39}$$

We have solved the zero order problem for $U(E) = \frac{E\gamma}{\gamma} = \frac{(W-L)^\gamma}{\gamma}$ by choosing $J_0 = A(t) \frac{(W-L)^\gamma}{\gamma}$ and solving for $A(t)$. The extra terms in the $o(\varepsilon)$ equation depending on J_0 gives

$$L_0(J_1) + (W - L)^{\gamma-1} A(t) \left[-L + \frac{1}{2}(\alpha - r_a)\bar{w}_1 - \frac{1}{2}\bar{c}\bar{u}_1 \right] = 0. \tag{40}$$

We now look for J_1 in the form

$$J_1 = [B(t)L + B_1(t)W](W - L)^{\gamma-1} \tag{41}$$

so that

$$\begin{aligned}
 L_0(J_1) &= [\dot{B}(t)L + \dot{B}_1W] (W - L)^{\gamma-1} \\
 &+ (W - L)^{\gamma-1} \left\{ \left(\frac{1}{2}(\alpha - r_a)\bar{w}_0B_1(t) + \frac{1}{2}\bar{c}u_0B(t) + r_a(B_1W + BL) \right) \right\} \\
 &+ [BL + B_1W] \left\{ \left(\frac{1}{2}(\alpha - r_a)\bar{w}_0 - \frac{1}{2}\bar{c}u_0 \right) (\gamma - 1) \right\} E^{\gamma-2} \\
 &+ r_aE^{\gamma-1}(\gamma - 1)[BL + B_1W]
 \end{aligned}$$

but note that from the zero order equation (38) we have

$$0 = \left(\frac{1}{2}(\alpha - r_a)\bar{w}_0 - \frac{1}{2}\bar{c}u_0 \right) A(t)E^{\gamma-1} + r_aA(t)E^\gamma + \frac{\dot{A}(t)E^\gamma}{\gamma} + \frac{E^\gamma}{\gamma} \tag{42}$$

and since from equations (30) and (31), \bar{w}_0 and \bar{u}_0 are proportional to $E = W - L$, this equation reduces to

$$\begin{aligned}
 0 &= \left\{ \left[\frac{1}{2}(\alpha - r_a)Q_1 - \frac{1}{2}\bar{c}Q_2 \right] + r_a \right\} A(t) \\
 &+ \frac{\dot{A}(t)}{\gamma} + \frac{1}{\gamma}
 \end{aligned} \tag{43}$$

from which we can determine $A(t)$. This expression agrees with that of equation (36) when $r_a = r_l$ though they look superficially different.

Furthermore, using

$$\begin{aligned}
 \bar{w}_0 &= Q_1(W - L) \\
 \bar{u}_0 &= Q_2(W - L)
 \end{aligned}$$

in Equation (40) gives to order ε .

$$\begin{aligned}
 0 &= \left(\dot{B}(t)L + \dot{B}_1W \right) + \frac{1}{2}(\alpha - r_a)Q_1B_1(W - L) \\
 &+ \frac{1}{2}\bar{c}BQ_2(W - L) + r_a(B_1W + BL) \\
 &+ (\gamma - 1)(BL + B_1W) \left\{ \frac{1}{2}(\alpha - r_a)Q_1 - \frac{1}{2}\bar{c}Q_2 \right\} \\
 &+ r_a(\gamma - 1)(BL + B_1W) \\
 &+ A(t) \left[-L + \frac{1}{2}(\alpha - r_a)\bar{w}_1 - \frac{1}{2}\bar{c}u_1 \right]
 \end{aligned} \tag{44}$$

where $\bar{c} = c - m - r_l$. If we now equate to zero coefficients of L and W , we get coupled differential equations for $B(t)$ and $B_1(t)$ in terms of $A(t)$. To proceed further we need \bar{w}_1

and \bar{u}_1 which we have from the $o(\varepsilon)$ equations of (11) and (12). With the above choice of J_0 and J_1 these become

$$A(t) \begin{pmatrix} \sigma^2 & -q\sigma\rho \\ -q\sigma\rho & q^2 \end{pmatrix} \begin{pmatrix} \bar{w}_1 \\ \bar{u}_1 \end{pmatrix} = \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} \tag{45}$$

where

$$\begin{aligned} \Xi_1 = & -(\alpha - r_a) \left[BL + \frac{B_1\gamma W}{(\gamma - 1)} - \frac{B_1L}{(\gamma - 1)} \right] \\ & - \sigma^2 Q_1 [B_1 \{ \gamma W - 2L \} + (\gamma - 2) BL] \\ & - q\sigma\rho Q_2 [B \{ W - (\gamma - 1) L \} - (\gamma - 1) B_1 W + B_1 L] \end{aligned} \tag{46}$$

and

$$\begin{aligned} \Xi_2 = & -\bar{c} \left[\frac{BW}{(\gamma - 1)} - \frac{BL\gamma}{(\gamma - 1)} - B_1 W \right] \\ & - q^2 Q_2 [-2BW + \gamma BL + (\gamma - 2) B_1 W] \\ & - q\sigma\rho Q_1 [BW - (\gamma - 1) BL + B_1 L - (\gamma - 1) B_1 W] \\ & - A(t) \frac{W - L}{\gamma - 1}. \end{aligned} \tag{47}$$

From this equation we can see that $A(t)\bar{w}_1$ and $A(t)\bar{u}_1$ are expressions linear in W and L so when substituted in (44), we get two linear coupled differential equations in $B(t)$ and $B_1(t)$. Further as can be seen in (44) there is one term which is $-A(t)L$ which stands alone hence both $B(t)$ and $B_1(t)$ will depend on $A(t)$. Finally since the condition at $t = T$ has been satisfied by $A(t)$ we will have $B(T) = 0$ and $B_1(T) = 0$. The functional form of $A(t)$ is derived in the Appendix.

5. Conclusion & Final remarks

This paper investigates the optimal portfolio selection for insurers with stochastic liabilities. The model considers characteristics of the insurers balance sheet and the dependence on control variables w of the wealth investment (for which wW is the risky part see Figure 1) and u of the liability of which uL is the amount of risky liability. These controls are found for a variety of utility functions where portfolio selection is found via a maximum expected utility over a life time. For the simplest case (Section 3) where the utility $\left(U = \frac{W^\gamma}{\gamma} \right)$ depends only on wealth we find (Equation 18) that the optimal allocation to the risky asset is

$$w^* = \frac{(\alpha - r_a)}{\sigma^2(1 - \gamma)}$$

and u can take any value.

In section 4, we consider a utility function of the form

$$U(W, L) = \frac{(W - L)^\gamma}{\gamma}.$$

For this case if $r_a = r_l$ (i.e. asset interest rate is equal to liability interest rate) and for any correlation ρ , we have the optimal weights w and u given as w_0 and u_0 below which are exact in this case since $\varepsilon = r_l - r_a = 0$.

For the situation when $r_a \neq r_l$ but $|\varepsilon| \ll r_a$, we use an expansion such as that described below (see also the Appendix).

We have solved for the optimal weight w and u

$$\bar{w} = wW = \bar{w}_0 + \varepsilon\bar{w}_1 + \varepsilon^2\bar{w}_2 + O(\varepsilon^3) \tag{48}$$

$$\bar{u} = uL = \bar{u}_0 + \varepsilon\bar{u}_1 + \varepsilon^2\bar{u}_2 + O(\varepsilon^3) \tag{49}$$

This can be solved to give to $O(1)$ as

$$\bar{w}_0 = w_0W = Q_1(W - L) \tag{50}$$

$$\bar{u}_0 = u_0L = Q_2(W - L) \tag{51}$$

where

$$Q_1 = \frac{[-(\alpha - r_a)q^2 + q\sigma\rho\{c - m - r_l\}]}{\sigma^2q^2(\gamma - 1)(1 - \rho^2)} \tag{52}$$

and

$$Q_2 = \frac{[-q\sigma\rho(\alpha - r_a) + \sigma^2\{c - m - r_l\}]}{\sigma^2q^2(\gamma - 1)(1 - \rho^2)} \tag{53}$$

w_0 and u_0 have no explicit dependence of time. The $O(\varepsilon)$ approximation can be deduced from the equation

$$A(t) \begin{pmatrix} \sigma^2 & -q\sigma\rho \\ -q\sigma\rho & q^2 \end{pmatrix} \begin{pmatrix} \bar{w}_1 \\ \bar{u}_1 \end{pmatrix} = \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} \tag{54}$$

with

$$\begin{aligned} \Xi_1 = & -(\alpha - r_a) \left[BL + \frac{B_1\gamma W}{(\gamma - 1)} - \frac{B_1L}{(\gamma - 1)} \right] \\ & - \sigma^2Q_1 [B_1\{\gamma W - 2L\} + (\gamma - 2)BL] \\ & - q\sigma\rho Q_2 [B\{W - (\gamma - 1)L\} - (\gamma - 1)B_1W + B_1L] \end{aligned} \tag{55}$$

and

$$\begin{aligned} \Xi_2 = & -\bar{c} \left[\frac{BW}{(\gamma - 1)} - \frac{BL\gamma}{(\gamma - 1)} - B_1W \right] \\ & - q^2Q_2 [-2BW + \gamma BL + (\gamma - 2)B_1W] \\ & - q\sigma\rho Q_1 [BW - (\gamma - 1)BL + B_1L - (\gamma - 1)B_1W] \end{aligned}$$

$$- A(t) \frac{W - L}{\gamma - 1}, \tag{56}$$

where $A(t)$ follows from Equation (43) and B and B_1 can be deduced from a pair of first order coupled differential equation in time. From (54) we can write

$$A(t) \bar{w}_1 = \frac{q^2 \Xi_1}{\sigma^2 q^2 (1 - \rho^2)} + \frac{q \sigma \rho \Xi_2}{\sigma^2 q^2 (1 - \rho^2)} \equiv t_1 L + t_2 W \tag{57}$$

$$A(t) \bar{u}_1 = \frac{q \sigma \rho \Xi_1}{\sigma^2 q^2 (1 - \rho^2)} + \frac{\sigma^2 \Xi_2}{\sigma^2 q^2 (1 - \rho^2)} \equiv t_3 L + t_4 W \tag{58}$$

Equating coefficients of L and W in equation (44) gives

$$\begin{aligned} 0 = & \dot{B} - \frac{1}{2}(\alpha - r_a) Q_1 B_1 \\ & + B \left\{ r_a \gamma - \frac{1}{2} \bar{c} Q_2 \gamma + \frac{1}{2}(\gamma - 1)(\alpha - r_a) Q_1 \right\} \\ & + \frac{(\alpha - r_a)}{2} t_1 - \frac{1}{2} \bar{c} t_3 - A(t) \end{aligned} \tag{59}$$

and

$$\begin{aligned} 0 = & \dot{B}_1 - \frac{1}{2} \bar{c} Q_2 B \\ & + B_1 \left\{ r_a \gamma + \frac{1}{2}(\alpha - r_a) \gamma Q_1 - \frac{1}{2} \bar{c} Q_2 (\gamma - 1) \right\} \\ & + \frac{(\alpha - r_a)}{2} t_2 - \frac{1}{2} \bar{c} t_4 \end{aligned} \tag{60}$$

5.1. Example: Results for $r_a = r_l, \rho \neq 0$

For this case, the ratios $\frac{wW}{E}, \frac{uL}{E}, (E = W - L)$ are time independent for the example we consider $r_a = r_l$, we plot Q_1 and Q_2 for the parameters $\alpha = 0.1, r_a = 0.03, q = 0.25, \sigma = 0.3, c = 1.5, m = 1.3, r_l = 0.025, \gamma = 0.5$ but with ρ and \bar{c} varying, see Figure 2 and Figure 3.

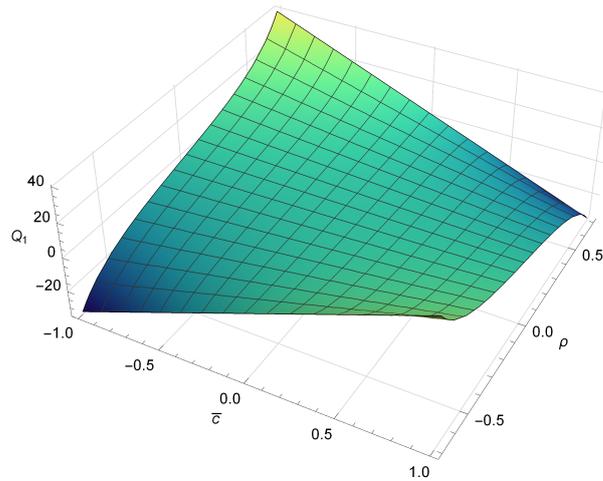


Figure 2: Plot of Q_1 when $r_a = r_l$.

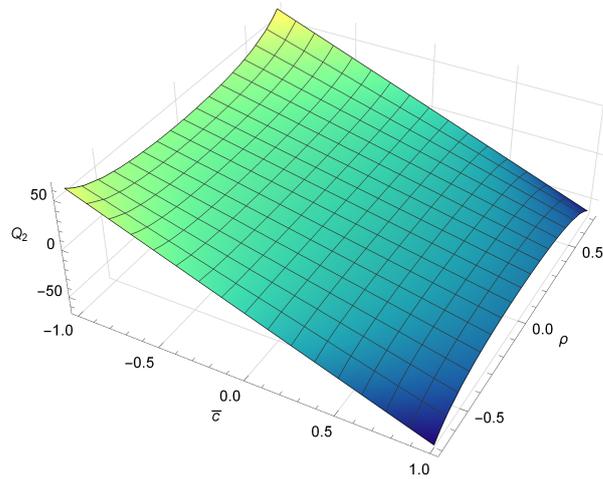


Figure 3: Plot of Q_2 when $r_a = r_l$.

As is evident from the plot of Q_1 , we must exclude regions where $Q_1 < 0$ since it is inadmissible given our short selling constraint of the risky asset. If we examine Q_1 when $\bar{c} = 0$, we see the convexity structure and if $\rho = 0$, then $Q_1 = 1.56$, which implies the amount of risky asset is 1.56 times that of the shareholders equity $W - L$, refer to Figure 4. For ease of explanation, we plot Q_2 (optimal allocation into risky liability) for the case where $\bar{c} = 0$, we see that the $Q_2 = 0$ when $\rho = 0$, refer to Figure 5.

This is sensible because the movements of the risky assets are not correlated to that of the risky liabilities hence the policy should suggest we minimise the unnecessary volatility this stochastic liability may cause. As the correlation increases, Q_2 increases monotonically to help match the movements of the assets.

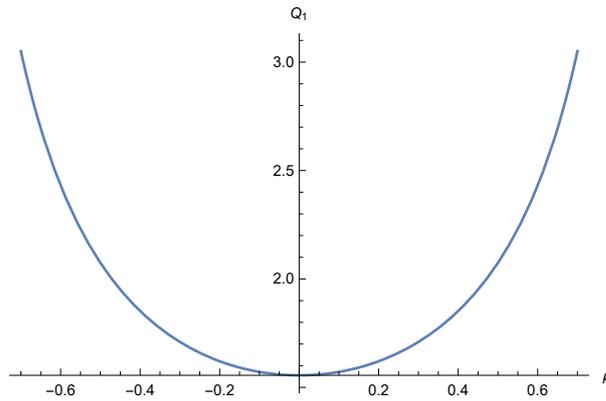


Figure 4: Plot of Q_1 when $r_a = r_l$ and $\bar{c} = 0$.

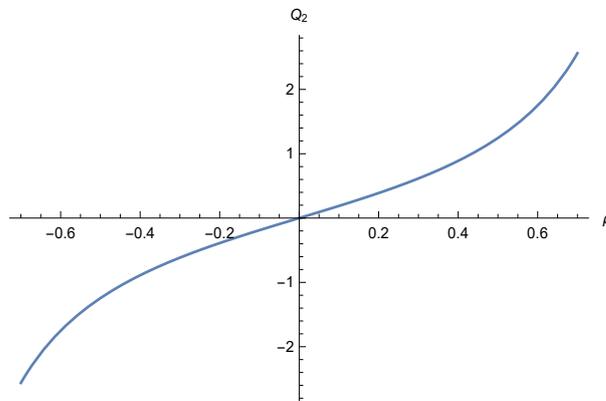


Figure 5: Plot of Q_2 when $r_a = r_l$ and $\bar{c} = 0$.

5.2. Example: Results for $r_a \neq r_l$ but $|\varepsilon| \ll r_a$

$O(1)$ solution as above but now $r_a \neq r_l$

$$\frac{w_0 W}{(W - L)} = Q_1$$

$$\frac{u_0 L}{(W - L)} = Q_2$$

To correct for order ε the behavior will depend on $B(t)$ and $B_1(t)$ and will potentially require us to correct for the optimal ratios as t evolves. The general result for \bar{w}_1 and \bar{u}_1 can be deduced from equations (54), (55) and (56). To this order we deduce

$$wW = Q_1 (W - L) + \varepsilon [P_1 (W - L) + P_2 L] + O(\varepsilon^2)$$

and

$$uL = Q_2 (W - L) + \varepsilon [P_3 (W - L) + P_4 W] + O(\varepsilon^2)$$

where P_1 and P_3 are constants depending on the various parameters of the problem. Note that higher order terms could be attained if necessary, see the appendix. Also a full numerical solution could be obtained by solving Equation (9) or Equation (14) as a backward diffusion equation together with the constraints Equations (11) and (12).

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Appendix

We assume here that $|\varepsilon| \ll r_a$ and show how the method for determining the order ε correction can be extended to obtain higher order terms.

$$\begin{aligned} J(W, L, t) &= J_0 + \varepsilon J_1 + \varepsilon^2 J_2 + O(\varepsilon^3) \\ wW &= \bar{w}_0 + \varepsilon \bar{w}_1 + \varepsilon^2 \bar{w}_2 + O(\varepsilon^3) \\ uL &= \bar{u}_0 + \varepsilon \bar{u}_1 + \varepsilon^2 \bar{u}_2 + O(\varepsilon^3) \end{aligned}$$

$$\begin{aligned} J(W, L, t) &= A(t) \frac{(W - L)^\gamma}{\gamma} + \varepsilon (BL + B_1W) (W - L)^{\gamma-1} \\ &+ \varepsilon^2 (C_1W^2 + C_2LW + C_3L^2) (W - L)^{\gamma-2} \end{aligned}$$

where $B(t)$, $B_1(t)$ are to be determined in terms of $A(t)$.

To this order, equations (11) and (12) (the conditions for the maximum) give

$$\begin{aligned} 0 &= (\alpha - r) [J_{0W} + \varepsilon J_{1W} + \varepsilon^2 J_{2W}] \\ &+ \sigma^2 [J_{0WW} + \varepsilon J_{1WW} + \varepsilon^2 J_{2WW}] [\bar{w}_0 + \varepsilon \bar{w}_1 + \varepsilon^2 \bar{w}_2] \\ &+ q\sigma\rho [J_{0LW} + \varepsilon J_{1LW} + \varepsilon^2 J_{2LW}] [\bar{u}_0 + \varepsilon \bar{u}_1 + \varepsilon^2 \bar{u}_2] \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} 0 &= \bar{c} [J_{0L} + \varepsilon J_{1L} + \varepsilon^2 J_{2L}] \\ &+ q^2 [J_{0LL} + \varepsilon J_{1LL} + \varepsilon^2 J_{2LL}] [\bar{u}_0 + \varepsilon \bar{u}_1 + \varepsilon^2 \bar{u}_2] \\ &+ q\sigma\rho [J_{0LW} + \varepsilon J_{1LW} + \varepsilon^2 J_{2LW}] [\bar{w}_0 + \varepsilon \bar{w}_1 + \varepsilon^2 \bar{w}_2]. \end{aligned} \tag{A.2}$$

So we have for $O(1)$

$$0 = (\alpha - r_a) J_{0W} + \sigma^2 \bar{w}_0 J_{0WW} + q\sigma\rho \bar{u}_0 J_{0LW}$$

and

$$0 = \bar{c} J_{0L} + q^2 \bar{u}_0 J_{0LL} + q\sigma\rho \bar{w}_0 J_{0LW}.$$

For $O(\varepsilon)$

$$\begin{aligned} 0 &= (\alpha - r_a) J_{1W} + \sigma^2 \bar{w}_0 J_{1WW} + q\sigma\rho \bar{u}_0 J_{1LW} \\ &+ \sigma^2 \bar{w}_1 J_{0WW} + q\sigma\rho \bar{u}_1 J_{0LW} \end{aligned}$$

and

$$\begin{aligned} 0 &= \bar{c} J_{1L} + q^2 \bar{u}_0 J_{1LL} + q\sigma\rho \bar{w}_0 J_{1LW} \\ &+ q^2 \bar{u}_1 J_{0LL} + q\sigma\rho \bar{w}_1 J_{0LW}. \end{aligned}$$

For $O(\varepsilon^2)$

$$\begin{aligned} 0 &= \bar{w}_2 \sigma^2 J_{0WW} + \bar{u}_2 q \sigma \rho J_{0LW} \\ &(\alpha - r_a) J_{2W} + \sigma^2 \bar{w}_0 J_{2WW} + q \sigma \rho \bar{u}_0 J_{2LW} \\ &+ \sigma^2 \bar{w}_1 J_{1WW} + q \sigma \rho \bar{u}_1 J_{1LW} \end{aligned}$$

and

$$\begin{aligned} 0 &= \bar{w}_2 q \sigma \rho J_{0LW} + \bar{u}_2 \sigma^2 J_{0LL} \\ &\bar{c} J_{2L} + q \sigma \rho \bar{w}_0 J_{2LW} + q^2 \bar{u}_0 J_{2LL} \\ &+ q \sigma \rho \bar{w}_1 J_{1LW} + q^2 \bar{u}_1 J_{1LL}. \end{aligned}$$

Our equation (9) can be written as (or alternatively one can use the version in (14)),

$$\begin{aligned} 0 &= U[E] + \frac{\partial J}{\partial t} + \bar{c} \bar{u} \frac{\partial J}{\partial L} + (\alpha - r_a) \bar{w} \frac{\partial J}{\partial W} \\ &+ \left(r_a W \frac{\partial J}{\partial W} + r_l L \frac{\partial J}{\partial L} \right) \\ &+ \frac{1}{2} \sigma^2 \bar{w}^2 \frac{\partial^2 J}{\partial W^2} + q \sigma \rho \bar{w} \bar{u} \frac{\partial^2 J}{\partial L \partial W} + \frac{1}{2} q^2 \bar{u}^2 \frac{\partial^2 J}{\partial L^2}, \end{aligned} \tag{A.3}$$

where $\bar{u} = uL$, $\bar{w} = wW$.

For $O(i)$

$$\begin{aligned} 0 &= \frac{1}{2} \sigma^2 \bar{w}_0^2 J_{iWW} + q \sigma \rho \bar{u}_0 \bar{w}_0 J_{iLW} + \frac{1}{2} q^2 \bar{u}_0^2 J_{iLL} \\ &+ \bar{c} \bar{u}_0 J_{iL} + (\alpha - r_a) \bar{w}_0 J_{iW} + J_{it} \\ &+ \text{other terms (involving } J_{i-1}, \text{ etc.)}. \end{aligned}$$

So write

$$\begin{aligned} \mathcal{L}(J_i) &= \left(\frac{1}{2} \sigma^2 \bar{w}_0^2 \frac{\partial^2}{\partial W^2} + q \sigma \rho \bar{u}_0 \bar{w}_0 \frac{\partial^2}{\partial L \partial W} + \frac{1}{2} q^2 \bar{u}_0^2 \frac{\partial^2}{\partial L^2} \right. \\ &\left. + \bar{c} \bar{u}_0 \frac{\partial}{\partial L} + (\alpha - r_a) \bar{w}_0 \frac{\partial}{\partial W} + \frac{\partial}{\partial t} \right) J_i \end{aligned}$$

and note if $r_l = r_a + \varepsilon$ then

$$r_a W \frac{\partial J}{\partial W} + r_l L \frac{\partial J}{\partial L} = r_a \left(W \frac{\partial J}{\partial W} + L \frac{\partial J}{\partial L} \right) + \varepsilon L \frac{\partial J}{\partial L}.$$

We have $U(E) = \frac{E^\gamma}{\gamma} = \frac{(W-L)^\gamma}{\gamma}$ with $J_0 = A(t) \frac{(W-L)^\gamma}{\gamma}$

$$W \frac{\partial J_0}{\partial W} + L \frac{\partial J_0}{\partial L} = A(t) (W - L)^\gamma$$

and

$$r_a W \frac{\partial J}{\partial W} + r_l L \frac{\partial J}{\partial L} = r_a \mathcal{L}_1(J) + \varepsilon L \frac{\partial J}{\partial L},$$

where $\mathcal{L}_1(J) \equiv (W \frac{\partial}{\partial W} + L \frac{\partial}{\partial L}) J$ so from our equation we have $O(1)$

$$U(E) + \mathcal{L}(J_0) + r_a \mathcal{L}_1(J_0) = 0$$

and $O(\varepsilon)$

$$\begin{aligned} 0 &= \mathcal{L}(J_1) + r_a \mathcal{L}_1(J_1) + L \frac{\partial J_0}{\partial L} \\ &\quad + \bar{c} \bar{u}_1 J_{0L} + (\alpha - r_a) \bar{w}_1 J_{0W} \\ &\quad + \frac{1}{2} \sigma^2 (2 \bar{w}_1 \bar{w}_0) J_{0WW} + q \sigma \rho (\bar{u}_0 \bar{w}_1 + \bar{u}_1 \bar{w}_0) J_{0LW} \\ &\quad + \frac{1}{2} q^2 (2 \bar{u}_1 \bar{u}_0) J_{0LL} \end{aligned}$$

Call this $\mathcal{L}_{ij}(J_0)$, therefore,

$$0 = \mathcal{L}_{10}(J_0)$$

$O(\varepsilon^2)$

$$\begin{aligned} 0 &= \mathcal{L}(J_2) + r_a \mathcal{L}_1(J_2) + L \frac{\partial J_1}{\partial L} \\ &\quad + r_a \mathcal{L}_{20}(J_0) + \mathcal{L}_{10}(J_1), \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}_{20}(J_0) &= \bar{c} \bar{u}_2 J_{0L} + (\alpha - r_a) \bar{w}_2 J_{0W} \\ &\quad + \sigma^2 \left(\bar{w}_2 \bar{w}_0 + \frac{\bar{w}_1^2}{2} \right) J_{0WW} + q \sigma \rho (\bar{u}_0 \bar{w}_2 + \bar{u}_2 \bar{w}_0 + \bar{u}_1 \bar{w}_1) J_{0LW} \\ &\quad + q^2 \left(\bar{u}_2 \bar{u}_0 + \frac{\bar{u}_1^2}{2} \right) J_{0LL} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{10}(J_0) &= \bar{c} \bar{u}_1 J_{0L} + (\alpha - r_a) \bar{w}_1 J_{0W} \\ &\quad + \sigma^2 \bar{w}_1 \bar{w}_0 J_{0WW} + q \sigma \rho (\bar{u}_0 \bar{w}_1 + \bar{u}_1 \bar{w}_0) J_{0LW} \\ &\quad + q^2 \bar{u}_1 \bar{u}_0 J_{0LL}. \end{aligned}$$

With the choice

$$J_0 = A(t) \frac{(W - L)^\gamma}{\gamma},$$

the $O(1)$ equation reduces to the equation we had previously, which has the form

$$\frac{\dot{A}(t)}{\gamma} + \frac{1}{\gamma} + DA = 0,$$

where D is constant and equals $\frac{\kappa}{\gamma}$. Thus

$$A(t) = -\frac{1}{\gamma D} + \frac{1}{\gamma D} e^{(-\kappa(t-T))}$$

with $A(T) = 0$.

From the $O(1)$ equations for \bar{w}_0 and \bar{u}_0 we find that \bar{w}_0 and \bar{u}_0 are both proportional to $(W - L)$.

When we come to the $O(\varepsilon)$ equations, we write

$$J_1 = (BL + B_1W)(W - L)^{\gamma-1}$$

and note that $\mathcal{L}_{10}(J_0)$ will have terms like $(W - L)^{\gamma-1}$ multiplying a linear combination of terms involving \bar{u}_1 and \bar{w}_1 .

The term $L \frac{\partial J_0}{\partial L}$ is again of the form $(W - L)^{\gamma-1}(L)$.

Then $\mathcal{L}_1(J_1)$ has this form also on account of

$$\left(W \frac{\partial}{\partial W} + L \frac{\partial}{\partial L} \right) (W - L)^{\gamma-1} = (\gamma - 1)(W - L)^{\gamma-1},$$

$\mathcal{L}(J_1)$ will also have this form. So if \bar{u}_1 and \bar{w}_1 turn out to be proportional to a linear combination of L and W we get the coupled ODE's for B and B_1 by equating to zero coefficients of L and W . This behaviour of \bar{u}_1 and \bar{w}_1 follows from the $O(\varepsilon)$ equations of (A.1) and (A.2).

For the $O(\varepsilon)$ equations, note that $\mathcal{L}_{10}(J_1)$ will be of the form $(W - L)^{\gamma-2}$. Similarly $L \frac{\partial J_1}{\partial L}$ has the same form.

$\mathcal{L}_{20}(J_0)$ has the form $\bar{u}_2() (W - L)^{\gamma-1} + \bar{w}_2() (W - L)^{\gamma-1}$ and the determined terms $\frac{\bar{w}_1^2}{2} J_{0WW}$ etc are of the quadratic $\times (W - L)^{\gamma-2}$.

If we assume

$$J_2 = (C_1W^2 + C_2LW + C_3L^2)(W - L)^{\gamma-2},$$

then $\mathcal{L}(J_2) = \text{quadratic} \times (W - L)^{\gamma-2} + (C'_1W^2 + C'_2LW + C'_3L^2)(W - L)^{\gamma-2}$

Similarly, $r\mathcal{L}_1(J_2)$ is quadratic $\times (W - L)^{\gamma-2}$. Finally the $O(\varepsilon^2)$ equation gives

$$(W - L)^{\gamma-2} \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix} \begin{pmatrix} \bar{w}_2 \\ \bar{u}_2 \end{pmatrix} = (\text{quadratic})(W - L)^{\gamma-3}.$$

This gives \bar{w}_2 and \bar{u}_2 as $\frac{\text{quadratic}}{(W-L)}$. Thus the term in $\mathcal{L}_{20}(J_0)$ has the form (quadratic) $(W - L)^{\gamma-2}$.

Finally equating coefficients of W^2 , LW and L^2 gives 3 ODE for C_1 , C_2 and C_3 .