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Boundary sentinel with given sensitivity in population dynamics problem and parameters identification

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Abstract. The notion of sentinels with given sensitivity was introduced by J.L.Lions [11] in order to identify parameters in the problem of pollution ruled by a parabolic equation. He proves that the existence of such sentinels is reduced to the solution of exact controllability problem with constraints on the state. In population dynamics model, we reconsider this notion of sentinels in a more general framework. We prove the existence of the boundary sentinels by solving a boundary null-controllability problem with constraint on the control. Our results use Carleman inequality which is adapted to the constraint.

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1. Introduction

The notion of sentinel was introduced by J.L.Lions to study systems with incomplete data [11]. The notion permits us to distinguish and to analyse two types of incomplete data: the so-called pollution terms at which we look for information, independently of the other type of incomplete data which is the missing terms and that we do not want to identify.

Typically, the Lions's sentinel is a functional defined on an open set O where we consider three functions: the "observation" y_{obs} corresponding to measurements, a given "mean" function h_0, and a control function w to be determined.

Let us remind that Lions's sentinel theory [11] relies on the following three features: the state equation y which is governed by a partial differential equation, the observation system and some particular evaluation function: the sentinel itself. More precisely, we consider a linear model (1) describing the dynamics of population with age dependence, spatial structure with incomplete data.

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Let Ω be an open and bounded domain of $\mathbb{R}^N$, $N \in \{1, 2, 3\}$, with boundary Γ of C^∞. For the time $T > 0$ and the life expectancy of an individual $A > 0$, we set $U = (0, T) \times (0, A)$, $Q = U \times \Omega$, $Q_A = (0, A) \times \Omega$, $Q_T = (0, T) \times \Omega$, $\Sigma = U \times \Gamma$, $\Sigma_1 = U \times \Gamma_1$, where Γ_1 is a non-empty open subset of Γ. Then consider the following two stroke problem:

$$\begin{cases} \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \Delta y + \mu y = 0 & \text{in} \quad Q \\ y(0, a, x) = y^0 + \tau \hat{y}^0 & \text{in} \quad Q_A \\ y(t, 0, x) = \int_0^A \beta(t, a, x) y(t, a, x) da & \text{in} \quad Q_T \\ y = \begin{cases} \xi + \sum_{i=1}^{M} \lambda_i \hat{\xi}_i & \text{on} \quad \Sigma_1 \\ 0 & \text{on} \quad \Sigma \setminus \Sigma_1 \end{cases} & \end{cases} \tag{1}$$

where :

- $y(t, a, x)$ is the distribution of a-year old individuals at time t at the point $x \in \Omega$.
- $\beta(t, a, x) \geq 0$ and $\mu(t, a, x) \geq 0$ are respectively the natural fertility and the natural death rate of age a at time t and position $x \in \Omega$.
- Thus, the formula $\int_0^A \beta(t, a, x) y(t, a, x) da$ denotes the distribution of newborn individuals at time t and location x.
- The boundary condition is unknown on a part Σ_1 of the boundary and represents a pollution with a structure of the form $\xi + \sum_{i=1}^{M} \lambda_i \hat{\xi}_i$. In this structure, the functions ξ and $\hat{\xi}_i$, $i = 1, \ldots M$ are known whereas the real λ_i, $i = 1, \ldots M$ are unknown.
- The initial distribution of individuals is unknown and its structure is of the form $y^0 + \tau \hat{y}^0$ where the function y^0 is known and the term $\tau \hat{y}^0$ is unknown.

System (1) is a system with incomplete data because the information on the boundary condition as well as on the initial condition are partially or completely unknown. Here, the pollution is isolated on the boundary $\Gamma \setminus \Gamma_1$. The missing term is located in the initial conditions. In what follows, we assume as in [8] that:

$$(H1) \quad : \quad \begin{cases} \beta \in L^\infty(Q), \quad \beta(t, a, x) \geq 0 \text{ a.e. in } Q; \\ \sup_{(t,x) \in]0,T[\times \Omega} \int_{]0,A[} (\|\beta^2(t, a, x)\| + \|\nabla \beta\|^2(t, a, x) da); \\ \exists\, \delta \in (0,\, A)\, s.t.\ \beta(a, .,) = 0 \text{ for a} \in (\delta,\, A); \end{cases}$$

$$(H2) \quad : \quad \mu \in C([0,\, T] \times [0,\, A] \times \bar{\Omega}),\ \mu(t, a, x) \geq 0 \text{ a.e} \quad \text{in} \quad Q$$

$$(H3) \quad : \quad \begin{cases} \forall t,\ 0 < t < A, \quad \forall\, x \in \Omega,\ \lim\limits_{a \longrightarrow A} \int_0^A \mu(\iota, a - t + \iota, x) d\iota = +\infty\,; \\ \forall t,\ A < t < T, \quad \forall\, x \in \Omega,\ \lim\limits_{a \longrightarrow A} \int_0^a \mu(t - a + \alpha, \alpha, x) d\alpha = +\infty; \\ \nabla \mu \in [L^\infty(Q)]^n. \end{cases}$$

We also assume that:

- y^0 and $\hat{y}^0$ belong to $L^2(Q_A)$, ξ and $\hat{\xi}_i$ belong to $L^2(\Sigma)$,
- the reals τ, λ_i $1 \le i \le M$ are sufficiently small and $\|\hat{y}^0\|_{L^2(Q_A)} \le 1$, and we set $\lambda = (\lambda_1, \ldots, \lambda_M)$.

Under the above assumptions on the data, one can prove as in [17] that problem (1) has a unique solution in $L^2(Q)$. For the sake of simplicity, we denote

$$y(t, a, x; \lambda, \tau) \tag{2}$$

the unique solution of (1). Therefore, the map

$$(\lambda, \tau) \mapsto y(\lambda, \tau) \text{ is in } C^1(\mathbb{R} \times \mathbb{R}; L^2(Q)). \tag{3}$$

For more literature on the model describing the dynamics of population with age dependence and spatial structure as well as for some existence results on such problem, we refer for instance to [1, 3, 8, 17] and the reference therein. Recently S. Sawadogo [16] use the sentinel method to control the migration of a single species population subjected to a migratory phenomenon.

For the model (1), we are interested in identifying the parameters λ_i without any attempt at computing $\tau \hat{y}^0$.

To identify these parameters, we use the theory of sentinel in a general framework. More precisely, Let O be a nonempty open subset of $\Gamma \backslash \Gamma_1$ and let $y = y(t, a, x; \lambda, \tau) = y(\lambda, \tau)$ be the solution of (1). Then for any non-empty open subset γ of $\Gamma \backslash \Gamma_1$ such that $O \cap \gamma \neq \emptyset$, we look for a function $S(\lambda, \tau)$ solution to the following problem : given $h_0 \in L^2(U \times O)$, find $w \in L^2(U \times \gamma)$ such that

i) the function S defined by

$$S(\lambda, \tau) = \int_U \int_O h_0 \frac{\partial y}{\partial \nu}(\lambda, \tau) dt da d\Gamma + \int_U \int_\gamma w \frac{\partial y}{\partial \nu}(\lambda, \tau) dt da d\Gamma, \tag{4}$$

satisfies :

- S is stationary to the first order with respect to missing term $\tau\hat{y}^0$

$$\frac{\partial S}{\partial \tau}(0,0) = 0 \qquad \forall\, \hat{y}^0 \tag{5}$$

- S is sensitive to the first order with respect to pollution terms $\lambda_i\hat{\xi}_i$:

$$\frac{\partial S}{\partial \lambda_i}(0,0) = c_i \qquad 1 \leq i \leq M, \tag{6}$$

where c_i, $1 \leq i \leq M$, are given constants not all identically zero.

ii) The control w is of minimal norm in $L^2(U\times\gamma)$ among " the admissible controls", i.e.

$$\|w\|^2_{L^2(U\times\gamma)} = \min_{\tilde{w}\in E}\|\tilde{w}\|^2_{L^2(U\times\gamma)}, \tag{7}$$

where

$$E = \left\{\tilde{w} \in L^2(U\times\gamma), \text{such that } \ (\tilde{w}, S(\tilde{w})) \ \text{ satisfies } \ (4)-(7)\right\}. \tag{8}$$

Remark 1. *J.L.Lions refers to the function S as a sentinel with given sensitivity c_i. In (6), the c_i are chosen according to the importance which is conferred to the component ξ_i of the pollution.*

Remark 2. *Notice that for the J.L.Lions's sentinels defined by (4)-(7), the observatory $O \subset (\Gamma\setminus\Gamma_1)$ is also the support of the control function w.*

For more information on the theory of sentinel, we refer to [9–11, 14, 15, 20] and the reference therein. We set $y_0 = y(0,\,0) \in L^2(Q)$, the solution of (1) when $\lambda = 0$ and $\tau = 0$ and we denote respectively by y_τ and y_{λ_i}, the derivatives of y at (0, 0) with respect to τ and λ_i, i.e. :

$$y_\tau = \lim_{\tau\to 0}\frac{y(0,\tau) - y(0,0)}{\tau}$$

and

$$y_{\lambda_i} = \lim_{\lambda_i\to 0}\frac{y(\lambda_i,0) - y(0,0)}{\lambda}.$$

Then y_τ and y_{λ_i} are respectively solutions of

$$\begin{cases} \frac{\partial y_\tau}{\partial t} + \frac{\partial y_\tau}{\partial a} - \triangle y_\tau + \mu y_\tau &= 0 & \text{in} \quad Q, \\ y_\tau(0,a,x) &= \hat{y}^0 & \text{in} \quad Q_A, \\ y_\tau(t,0,x) &= \displaystyle\int_0^A \beta(t,a,x)y_\tau(t,a,x)da & \text{in} \quad Q_T, \\ y_\tau &= 0 & \text{on} \quad \Sigma, \end{cases} \tag{9}$$

and

$$\begin{cases} \frac{\partial y_{\lambda_i}}{\partial t} + \frac{\partial y_{\lambda_i}}{\partial a} - \triangle y_{\lambda_i} + \mu y_{\lambda_i} & = & 0 & \text{in} & Q, \\ y_{\lambda_i}(0,a,x) & = & 0 & \text{in} & Q_A, \\ y_{\lambda_i}(t,0,x) & = & \int_0^A \beta y_{\lambda_i}(t,a,x)da & \text{in} & Q_T, \\ y_{\lambda_i} & = & \hat{\xi}_i \chi_{\Sigma_1} & \text{on} & \Sigma, \end{cases} \tag{10}$$

where χ_X denote now and in the sequel, the characteristic function of the set X. Under the assumptions $(H_1)-(H_4)$, the systems (9) and (10) have respectively a unique solution $y_\tau \in L^2(Q)$ and $y_{\lambda_i} \in L^2(Q)$ (see [8, 17]). From now on, we make the following assumptions:

- The functions
$$\hat{\xi}_{i.\chi_{\Sigma_1}}, 1 \leq i \leq M \quad \text{are linearly independent} \tag{11}$$

- Any function ρ such that

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} - \Delta\rho + \mu\rho & = & 0 & \text{in} & Q, \\ \rho & = & 0 & \text{in} & U \times \gamma, \\ \frac{\partial \rho}{\partial \nu} & = & 0 & \text{on} & U \times \gamma, \end{cases} \tag{12}$$

is identically zero.

and we set
$$Y = \text{Span}\{\frac{\partial y_{\lambda_1}}{\partial \nu}\chi_\gamma, ..., \frac{\partial y_{\lambda_M}}{\partial \nu}\chi_\gamma\}. \tag{13}$$

The vector subspace of $L^2(U \times \gamma)$ generated by M functions $\{\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma\}_{i=1}^M$.

$$Y_\theta = \frac{1}{\theta} Y$$

The vector subspace of $L^2(U \times \gamma)$ generated by M functions $\{\frac{1}{\theta}\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma\}_{i=1}^M$,
where θ is the positive function precisely defined later on by (31).

Remark 3. *We will prove in **Lemma 1** that the function* $\{\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma\}_{i=1}^M$ *and* $\{\frac{1}{\theta}\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma\}_{i=1}^M$ *are linearly independent.*

We now consider the following boundary null-controllability problem :

given $h_0 \in L^2(U \times O)$, $w_0 \in Y_\theta$, find $v \in L^2(U \times \gamma)$ such that

$$v \in Y^\perp, \tag{14}$$

and if $q = q(t, a, x; v)$ *is solution of*

$$\begin{cases} -\frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - \triangle q + \mu q &= \beta q(t,0,x) \quad \text{in} \quad Q, \\ q &= h_0\chi_O + (w_0 - v)\chi_\gamma \quad \text{on} \quad \Sigma, \\ q(T,a,x) &= 0 \quad \text{in} \quad Q_A, \\ q(t,A,x) &= 0 \quad \text{in} \quad Q_T, \end{cases} \tag{15}$$

q satisfy

$$q(0,a,x;v) = 0 \qquad \text{in} \quad Q_A. \tag{16}$$

Remark 4. *Let us notice that if v exists, the set*

$$\mathcal{E} = \{\overline{v} \in Y^\perp \textit{such} \quad \textit{that} \quad (\overline{v}, \overline{q} = q(t,a,x;\overline{v})) \quad \textit{satisfies} \quad (15)-(16)\} \tag{17}$$

is a non-empty closed, and convex set in $L^2(U\times\gamma)$. Therefore there exists $v \in \mathcal{E}$ of minimal norm.

The problem $(14)-(16)$ is a null boundary controllability problem with constraint on the control. When $Y^\perp = L^2(U \times \gamma)$, this problem becomes a null controllability problem without constraint on the control. This kind of problem has been studied by many authors with various methods [2, 5]. In this paper we solve the boundary null controllability problem with constraint on the control $(14)-(16)$, this allows us to prove the existence of the sentinel with given sensitivity $(4)-(7)$. More precisely, we have the following results:

Theorem 1. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary Γ of class C^∞. Let Γ_1 be a non-empty open subset of Γ. Let also O and γ be two non empty subsets of $\Gamma\setminus\Gamma_1$, such that $O\cap\gamma \neq 0$. Assume that the assumptions of the data of the system* (1) *are satisfied. Assume also that* (11) *and (12) holds. Then the existence of sentinel* $(4)-(7)$ *holds if and only if, the boundary null-controllability problem with constraints on the control* $(14)-(16)$ *has a solution.*

To prove the boundary null-controllability problem with constraints on the control $(14)-(16)$, we use an inequality of Carleman adapted to the constraint that we establish by means of a global Carleman inequality. More precisely we prove the following results.

Theorem 2. *Assume that the hypotheses of Theorem 1 are satisfied. Then there exists a positive real weight function θ (a precise definition of θ will be given later on* (31)*) such that, for any function $h_0 \in L^2(U\times O)$ with $\theta h_0 \in L^2(U\times O)$ there exists a unique control $\hat{v} \in L^2(U\times\gamma)$ such that $(\hat{v}, \hat{q})$ with $\hat{q} = q(\hat{v})$ is solution of null boundary controllability problem with constraint on the control* $(14)-(16)$ *and provides a control $\hat{w} = w_0\chi_\gamma - \hat{v}$ of the sentinel problem satisfying* (7) *. Moreover, the control $\hat{w}$ is given by*

$$\hat{w} = P(w_0) + (I-P)(\frac{\partial\hat{\rho}}{\partial\nu}\chi_\gamma), \tag{18}$$

where P is the orthogonal projection operator from $L^2(U \times \gamma)$ into Y, $w_0 \in Y_\theta$ depends on h_0 and c_i, $i \in \{1, \ldots, M\}$, and will be precisely determined in (27) *and $\hat{\rho}$ satisfies*

$$\begin{cases} \frac{\partial \hat{\rho}}{\partial t} + \frac{\partial \hat{\rho}}{\partial a} - \triangle \hat{\rho} + \mu \hat{\rho} & = & 0 & \text{in} & Q, \\ \hat{\rho} & = & 0 & \text{on} & \Sigma \\ \hat{\rho}(t,0,x) & = & \displaystyle\int_0^A \beta(t,a,x)\hat{\rho}(t,a,x)da & \text{in} & Q_T, \\ \hat{\rho}(0,.,.) & = & 0 & \text{in} & Q_A, \end{cases} \tag{19}$$

The rest of the paper is organized as follows: section 2 is devoted to the equivalence between the sentinel problem and the null boundary controllability problem with constraint on the control. In this section we give the proof of Theorem 1. In section 3, we establish Carleman inequalities necessary to solve the boundary null-controllability problem with constraint on the control $(14) - (16)$. In subsection 3.2 we give the proof of Theorem 2. In section 4, we formulate the sentinel and we identify the parameters.

2. Equivalence between the sentinel problem and the null boundary controllability problem with constraint on the control

In this subsection we prove **Theorem 1**. But before going further, we need the following result:

Lemma 1. *Assume that (11) and (12) holds. Then the functions $\frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma$, $1 \le i \le M$ are linearly independent. Moreover the functions $\frac{1}{\theta}\ \frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma$, $1 \le i \le M$, are also linearly independent.*

Proof. Let $\alpha_i \in \mathbb{R}$, $1 \le i \le M$ be such that $\displaystyle\sum_i^M \alpha_i \frac{\partial y_{\lambda_i}}{\partial \nu}\chi_\gamma = 0$. Set $k = \displaystyle\sum_i^M \alpha_i y_{\lambda_i}$, using (10) k is solution of

$$\begin{cases} \frac{\partial k}{\partial t} + \frac{\partial k}{\partial a} - \Delta k + \mu k & = & 0 & \text{in} & Q, \\ k(0,a,x) & = & 0 & \text{in} & Q_A, \\ k(t,0,x) & = & \displaystyle\int_0^A \beta k(t,a,x)da & \text{in} & Q_T, \\ k & = & \displaystyle\sum_{i=1}^M \alpha_i \hat{\xi}_i.\chi_{\Sigma_1} & \text{on} & \Sigma, \\ \frac{\partial k}{\partial \nu} & = & 0 & \text{on} & U \times \gamma. \end{cases} \tag{20}$$

Assumption (12) allows us to say that $k = 0$ in Q. Therefore, we deduce that $\displaystyle\sum_{i=1}^M \alpha_i \widehat{\xi_i}.\chi_{\Sigma_1} = 0$ on Σ. Then it follows from (11) that $\alpha_i = 0$ for $1 \le i \le M$. The second assertion of the lemma follows immediately.

Now, let us prove Theorem 1. To this end, we interpret (5) and (6). Actually, in view of (4), the stationary condition (5) and respectively the sensitivity conditions (6) hold if and only if

$$\int_U\int_O h_0\frac{\partial y_\tau}{\partial\nu}dtdad\Gamma+\int_U\int_\gamma w\frac{\partial y_\tau}{\partial\nu}dtdad\Gamma=0.\,\forall\hat{y}^0,\|\hat{y}^0\|_{L^2(Q_A)}\leq 1 \tag{21}$$

and

$$\int_U\int_O h_0\frac{\partial y_{\lambda_i}}{\partial\nu}dtdad\Gamma+\int_U\int_\gamma w\frac{\partial y_{\lambda_i}}{\partial\nu}dtdad\Gamma=c_i, 1\leq i\leq M. \tag{22}$$

Therefore, in order to transform equation (21), we consider the following adjoint equation

$$\begin{cases} -\frac{\partial q}{\partial t}-\frac{\partial q}{\partial a}-\triangle q+\mu q &= \beta q(t,0,x) \quad \text{in} \quad Q,\\ q &= h_0\chi_O+w\chi_\gamma \quad \text{on} \quad \Sigma,\\ q(T,a,x) &= 0 \quad \text{in} \quad Q_A,\\ q(t,A,x) &= 0 \quad \text{in} \quad Q_T, \end{cases} \tag{23}$$

Since $h_0\chi_O+w\chi_\gamma\in L^2(\Sigma)$, the assumptions $(H1)-(H2)$ ensure that that (23) has a unique solution $q\in L^2(Q)$. Now multiplying both sides of the differential equation in (23) by y_τ solution of (9) and integrating by parts in Q, we get

$$\int_U\int_O h_0\frac{\partial y_\tau}{\partial\nu}dtdad\Gamma+\int_U\int_\gamma w\frac{\partial y_\tau}{\partial\nu}dtdad\Gamma=\int_0^A\int_\gamma q(0,a,x)\hat{y}^0dadx\;\forall\;\hat{y}^0\in L^2(Q_A) \tag{24}$$

Thus, the condition (5) or (21) holds if and only if

$$q(0,a,x;v)=0 \qquad \text{in } Q_A. \tag{25}$$

Then, multiplying both sides of the differential equation in (23) by y_{λ_i} solution of (10) and integrating by parts in Q, we have

$$\int_U\int_O h_0\frac{\partial y_{\lambda_i}}{\partial\nu}dtdad\Gamma+\int_U\int_\gamma w\frac{\partial y_{\lambda_i}}{\partial\nu}dtdad\Gamma=\int_{\Sigma_1}\frac{\partial q}{\partial\nu}\hat{\xi}_i.\chi_{\Gamma_1}dtda,\quad 1\leq i\leq M.$$

Thus, the condition the condition (6) or (22) is equivalent to

$$\int_{\Sigma_1}\frac{\partial q}{\partial\nu}\hat{\xi}_i.\chi_{\Gamma_1}dtda=c_i,\quad 1\leq i\leq M. \tag{26}$$

Now, consider the matrix

$$\left(\int_0^T\int_0^A\int_\gamma\frac{1}{\theta}\frac{\partial y_{\lambda_j}}{\partial\nu}\frac{\partial y_{\lambda_i}}{\partial\nu}dtdad\Gamma\right)_{1\leq i,j\leq M}.$$

Since this matrix is symmetric positive definite therefore, there exists a unique $w_0 \in Y_\theta$ such that

$$c_i - \int_U \int_O h_0 \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma = \int_U \int_\gamma w_0 \frac{\partial y_{\lambda_i}}{\partial \nu} dtdad\Gamma. \quad 1 \leq i \leq M \tag{27}$$

Consequently, combining (27) with (22), we observe that condition (6) (or the constraints (26)) holds if and only if

$$w - w_0 = -v \in Y^\perp,$$

where Y is given by (13). Replacing w by $w_0 - v$ in the second expression of (23), we obtain (15). We just have proved that the sentinel problem $(4) - (7)$ hold if and only if null controllability problem with constraint on the control $(14) - (16)$ has a solution.

Remark 5. *If $\mathcal{E}$ is the set of admissible control $v \in L^2(U \times \gamma)$ such that* $(14) - (16)$ *is satisfied, then $\mathcal{E}$ is a closed convex subset of $L^2(U \times \gamma)$. Since $w_0 - \mathcal{E}$ is also a closed convex subset of $L^2(U \times \gamma)$, we can obtain w to be of minimum norm in $L^2(U \times \gamma)$ by minimizing the norm of $w_0 - v$ when $v \in \mathcal{E}$. Then the pair $(v, q(v))$ satisfying* $(14) - (16)$ *necessarily provides a control w satisfying* (7)

3. Study of the boundary null-controllability problem with constraint on the control

In this section, we prove existence of the solution of the boundary null controllability problem $(14) - (16)$ and of course uniqueness if we want the control to be of minimal norm among admissible controls. The main tool we use is an observability inequality adapted to the constraint (14) which itself is a consequence of a global Carleman inequality.

3.1. An adapted Carleman inequality

The observability inequality we are looking for is a consequence of the global Carleman's inequality. We consider an auxiliary function an auxillary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions :

$$\begin{array}{rcl} \psi(x) & > & 0, \forall x \in \Omega, \\ \nabla\psi & > & \alpha, \forall x \in \overline{\Omega}, \\ \psi(x) & = & 0, \forall x \in \Gamma \setminus \gamma, \\ \dfrac{\partial \psi}{\partial \nu} & < & 0, \forall x \in \Gamma \setminus \gamma. \end{array} \tag{28}$$

Such a function exists according to A. Fursikov and O. Yu. Imanuvilov [7]. For any positive parameter value λ we define the following weight functions :

$$\varphi(t, a, x) = \frac{e^{\lambda(m|\psi|_\infty + \psi(x))}}{at\,(A-a)\,(T-t)}, \tag{29}$$

$$\eta(t,a,x) = \frac{e^{2\lambda m|\psi|_\infty} - e^{\lambda(m|\psi|_\infty + \psi(x))}}{at\,(A-a)\,(T-t)}, \tag{30}$$

with $m \geq 1$. Since φ does not vanish in Q, for all $s > 0$ and $\lambda > 0$, we set

$$\frac{1}{\theta^2} = \min\left[e^{-2s\eta}\left(\varphi^{-1}, \varphi, \varphi^3, \varphi, |\frac{\partial\psi}{\partial\nu}|\right)\right] \tag{31}$$

and we adopt the following notations :

$$\left\{\begin{array}{rcl} L &=& \dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} - \Delta + \mu I \\ L^* &=& -\dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial a} - \Delta + \mu I \\ \mathcal{V} &=& \left\{\rho \in C^\infty\left(\overline{Q}\right), \rho = 0 \text{ on } \Sigma\right\} \end{array}\right. \tag{32}$$

Using the notations given by (32) and the definition of θ given by (31), we have the following boundary Carleman inequality:

Proposition 1. *[**Global Carleman inequality**] Let ψ, φ and η be defined respectively by $(28) - (30)$. Then, there exists numbers $\lambda_0 = \lambda_0(\gamma,\mu) > 1, s_0 = s_0(\gamma,\mu,T) > 1, C_0 = C_0(\gamma,\mu) > 0$ and $C_1 = C_1(\gamma,\mu) > 0$ such that for any $\lambda \geq \lambda_0$, for any $s \geq s_0$, for any $\rho \in \nu$, the following estimate holds :*

$$\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(|\rho_t + \rho_a|^2 + |\Delta\rho|^2\right) dadtdx + \int_Q e^{-2s\eta}\left(s\lambda^2\varphi|\nabla\rho|^2 + s^3\lambda^4\varphi^3|\rho|^2\right) dtdadx$$
$$+C_0 \int_0^T \int_0^A \int_{\Gamma\setminus\gamma} se^{-2s\eta}\varphi(-\frac{\partial\psi}{\partial\nu})|\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma$$
$$\leq C_1\left[\int_Q e^{-2\eta}|L\rho|^2 dtdadx + \int_0^T \int_0^A \int_\gamma se^{-2\eta}\varphi|\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma\right]. \tag{33}$$

Proof. See [18]

As ψ belong to $C^2(\overline{\Omega})$ and $\varphi e^{-2s\eta}$ is bounded, then $\frac{1}{\theta}$ is also bounded in Q. Hence, from Proposition 1, we have this other inequality :

Proposition 2. *Let θ be defined by (31). Then, there exists numbers $\lambda_0 = \lambda_0(\Omega, \gamma, \mu) > 1$, $s_0 = s_0(\Omega, \gamma, \mu, T) > 1$, $C_0 = C_0(\Omega, \gamma, \mu) > 0$, and $C_1 = C_1(\Omega, \gamma, \mu) > 0$ such that, for any $\lambda \geq \lambda_0$, for any $s \geq s_0$, and for any $\rho \in \mathcal{V}$,*

$$\int_U \int_\Omega \frac{1}{\theta^2}\left(|\frac{\partial\rho}{\partial\nu}|^2 + |\Delta\rho|^2 + |\nabla\rho|^2 + |\rho|^2\right) dtdad\Gamma + C_0 \int_U \int_\Gamma \frac{1}{\theta^2}|\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma$$
$$\leq C_1\left[\int_Q |L\rho|^2 dtdadx + \int_U \int_\gamma |\frac{\partial\rho}{\partial\nu}|^2 dtdad\Gamma\right]. \tag{34}$$

Lemma 2. *Under the assumptions of Lemma 1. Let Y be the real vector subspace of $L^2(U\times\gamma)$ of finite dimension defined in* (13). *Then any function ρ such that*

$$\begin{cases} \frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial a}-\Delta\rho+\mu\rho &= 0 \quad \text{in} \quad Q,\\ \rho(0,.,.) &= 0 \quad \text{in} \quad Q_A,\\ \rho &= 0 \quad \text{on} \quad \Sigma\setminus\Sigma_1,\\ \frac{\partial\rho}{\partial\nu}|_\gamma\in Y, & \end{cases} \tag{35}$$

is identically zero.

Proof. For any ρ verifying (35) there exists $\alpha_i\in\mathbb{R}$, $1\leq i\leq M$, such that $\frac{\partial\rho}{\partial\nu}=\sum_{i=1}^{M}\alpha_i\frac{\partial y_i}{\partial\nu}$. We set $z=\rho-\sum_{i=1}^{M}\alpha_i y_i$. Using (10), we have

$$\begin{cases} \frac{\partial z}{\partial t}+\frac{\partial z}{\partial a}-\Delta z+\mu z &= 0 \quad \text{in} \quad Q,\\ z(0,.,.) &= 0 \quad \text{in} \quad Q_A,\\ z &= 0 \quad \text{on} \quad \Sigma\setminus\Sigma_1,\\ \frac{\partial z}{\partial\nu} &= 0 \quad \text{on} \quad U\times\gamma. \end{cases} \tag{36}$$

As $\gamma\subset\Gamma\setminus\Gamma_1$, we have $z=0$ and $\frac{\partial z}{\partial\nu}=0$ in $U\times\gamma$. Then it follows from (12) that $z=0$ in Q. Consequently, we deduce on the one hand that $\rho=\sum_{i=1}^{M}\alpha_i y_i$ and on the other hand that $\sum_{i=1}^{M}\alpha_i\widehat{\xi_i}=0\,\text{on}\,\Sigma_1$. Hence, it follows from assumption (11) that $\alpha_i=0$ for $1\leq i\leq M$.Thus, $\rho=0$ in Q.

Proposition 3 (**Adapted Carleman inequality**). *Under the Assumption of Lemma1 . Let Y be the real vector subspace of $L^2(U\times\gamma)$ of finite dimension defined in* (13) *and P be the orthogonal projection operator from $L^2(U\times\gamma)$ into Y. Let also θ be the function defined by* (31). *Then, there exists numbers $\lambda_0=\lambda_0(\Omega,\gamma,\mu)>1$, $s_0=s_0(\Omega,\gamma,\mu,T)>1$, $C_0=C_0(\Omega,\gamma,\mu)>0$ and $C_1=C_1(\Omega,\gamma,\mu)>0$ such that, for any $\lambda\geq\lambda_0$, for any $s\geq s_0$, and for any $\rho\in\mathcal{V}$,*

$$\int_U\int_\Omega\frac{1}{\theta^2}|\frac{\partial\rho}{\partial\nu}|^2dtdad\Gamma\leq C_1\left[\int_Q|L\rho|^2dtdadx+\int_U\int_\gamma|P\frac{\partial\rho}{\partial\nu}-\frac{\partial\rho}{\partial\nu}|^2dtdad\Gamma\right]. \tag{37}$$

Proof. As in [9], we use a well known compactness-uniqueness argument and the inequality (34). Indeed, suppose that (37) does not hold. Then for any $j\in\mathbb{N}$, there exists $\rho_j\in\mathcal{V}$ such that

$$\int_U\int_\Omega|L\rho_j|^2dtdadx\leq\frac{1}{j}, \tag{38}$$

$$\int_U \int_\gamma |P\frac{\partial \rho_j}{\partial \nu} - \frac{\partial \rho_j}{\partial \nu}\chi_\gamma|^2 dtdad\Gamma \leq \frac{1}{j}, \tag{39}$$

$$\int_U \int_\Gamma \frac{1}{\theta^2}|\frac{\partial \rho_j}{\partial \nu}|^2 dtdad\Gamma = 1. \tag{40}$$

In what follows, we prove in three steps that (38) − (40) yields contradiction.

Step 1. We have

$$\begin{aligned}\int_U \int_\gamma \frac{1}{\theta^2}|P\frac{\partial \rho_j}{\partial \nu}|^2 dtdad\Gamma \leq\ & 2\int_U \int_\gamma \frac{1}{\theta^2}|P\frac{\partial \rho_j}{\partial \nu}|^2 dtdad\Gamma \\ & + 2\int_U \int_\gamma \frac{1}{\theta^2}|P\frac{\partial \rho_j}{\partial \nu} - \frac{\partial \rho_j}{\partial \nu}\chi_\gamma|^2 dtdad\Gamma.\end{aligned} \tag{41}$$

Since $\frac{1}{\theta^2}$ is bounded, using (38) and (39), it follows that there exists a positive constant C such that

$$\forall j \in \mathbb{N}, \quad \int_U \int_\gamma |P\frac{\partial \rho_j}{\partial \nu}|^2 dtdad\Gamma \leq C. \tag{42}$$

As $\frac{\partial \rho}{\partial \nu}\chi_\gamma = P\frac{\partial \rho}{\partial \nu}\chi_\gamma + (\frac{\partial \rho}{\partial \nu}\chi_\gamma - P\frac{\partial \rho}{\partial \nu}\chi_\gamma)$, using (40) and (42), we obtain

$$\|\frac{\partial \rho_j}{\partial \nu}\|^2_{L^2(U\times\gamma)} \leq C. \tag{43}$$

Step 2. Let $L^2\left(\frac{1}{\theta}, U\times\gamma\right) = \{\rho \in L^2(U\times\Omega); \int_U \int_\Gamma \frac{1}{\theta^2}\left|\frac{\partial \rho_j}{\partial \nu}\right|^2 dtdad\Gamma < \infty\}$.

Then in view of (40) and (43), we deduce from (34) that , $\left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a}\right)$, $(\frac{\partial \rho_j}{\partial \nu})$, $(\nabla \rho_j)$, (ρ_j) and $(\Delta \rho_j)$ are bounded in $L^2\left(\frac{1}{\theta}, U\times\gamma\right)$. Let us the take a subsequence still denoted by (ρ_j) such that

$$\rho_j \rightharpoonup \rho \quad \textit{weakly in } L^2(\frac{1}{\theta}, U\times\gamma), \tag{44}$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad \textit{weakly in } L^2(\frac{1}{\theta}, U\times\gamma). \tag{45}$$

Then follows from (28) − (30) and the definition of $\frac{1}{\theta}$ given by (31) that (ρ_j) and $(\Delta\rho_j)$ are bounded in $L^2(]\beta, T-\beta[\times]\alpha, A-\alpha[\times\Omega)$ for any $\beta > 0$ and any $\alpha > 0$. In particular, for all $\beta > 0$ and any $\alpha > 0$, we have

$$\rho_j \rightharpoonup \rho \quad \textit{weakly in } L^2(]\beta, T-\beta[\times]\alpha, A-\alpha[\times\Omega)$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad \textit{weakly in } L^2(]\beta, T-\beta[\times]\alpha, A-\alpha[\times\Sigma).$$

which implies that

$$\rho_j \rightharpoonup \rho \quad weakly\ in\ D'(Q)$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad weakly\ in\ D'(\Sigma).$$

Therefore, we get from (38) and (43) that

$$L\rho_j \longrightarrow L\rho = 0 \quad \text{strongly in } L^2(U \times \Omega), \tag{46}$$

$$\frac{\partial \rho_j}{\partial \nu} \rightharpoonup \frac{\partial \rho}{\partial \nu} \quad \text{strongly in } L^2(U \times \gamma). \tag{47}$$

And, since P is a compact operator, we deduce from (47) that

$$P\frac{\partial \rho_j}{\partial \nu} \longrightarrow P\frac{\partial \rho}{\partial \nu} \quad \text{strongly in } L^2(U \times \gamma). \tag{48}$$

In view of (39), we also have

$$\frac{\partial \rho_j}{\partial \nu} - P\frac{\partial \rho_j}{\partial \nu} \longrightarrow 0 \quad \text{strongly in } L^2(U \times \gamma). \tag{49}$$

Thus combining (48) and (49), we get

$$P\frac{\partial \rho_j}{\partial \nu} \longrightarrow \frac{\partial \rho_j}{\partial \nu} \quad \text{strongly in } L^2(U \times \gamma). \tag{50}$$

Thanks to the uniqueness of the limit in $L^2(U \times \gamma)$, the convergence relations (48)-(49) and (50) imply that $P\frac{\partial \rho}{\partial \nu} = \frac{\partial \rho}{\partial \nu}\chi_\gamma$. This means that $\frac{\partial \rho}{\partial \nu}\chi_\gamma \in Y$. We thus have proved that ρ verifies (35). Hence thanks to Lemma 2, ρ is identically zero.

Therefore, (50) becomes

$$\frac{\partial \rho_j}{\partial \nu} \longrightarrow 0 \quad \text{strongly in } L^2(U \times \gamma). \tag{51}$$

Step 3. Since $\rho_j \in \nu$, it follows from the observability inequality (34) that

$$\int_U \int_\Omega \frac{1}{\theta^2} |\frac{\partial \rho_j}{\partial \nu}|^2 dtdad\Gamma \leq C_1 \left[\int_Q |L\rho_j|^2 \, dtdadx + \int_U \int_\gamma |\frac{\partial \rho_j}{\partial \nu}|^2 dtdad\Gamma \right].$$

Therefore passing this latter inequality to the limit while using (46) and (51), we obtain

$$\lim_{j \longrightarrow \infty} \int_U \int_\Omega |\frac{\partial \rho_j}{\partial \nu}|^2 dtdadx = 0.$$

The contradiction occurs with (40).

3.2. Proof of Theorem 2

In this subsection, we are concerned with the proof of Theorem 2. That is, the optimality system for the control $\hat{v}$ such that the pair $(\hat{v};\hat{q})$ verifies $(14)-(16)$. Since a classical way to derive this optimality system is the method of penalization due to J.L.Lions [11], here we use this method.

Step 1. Let w_0 be defined by (27). If $v \in Y^{\perp}$ and q is solution of (15) then $q(0,.,.) \in L^2(Q_A)$ and we can define the functional

$$J_\epsilon(v) = \frac{1}{2}\|w_0 - v\|^2_{L^2(U\times\gamma)} + \frac{1}{2\epsilon}\|q(0,.,.)\|^2_{L^2(Q_A)}. \tag{52}$$

We consider the optimal control problem: Find $v_\epsilon \in Y^{\perp}$ such that

$$J_\epsilon(v_\epsilon) \quad = \quad \min_{v\in Y^{\perp}} J_\epsilon(v) \tag{53}$$

Since $Y^{\perp}$ is a closed and convex subset of $L^2(U\times\gamma)$, it is classical to prove that there exists a unique solution to (53). If we write q_ϵ the solution of (15) corresponding to v_ϵ using an adjoint state ρ_ϵ , we have that the triplet
$(q_\epsilon\ ,\ \rho_\epsilon\ v_\epsilon\ )$ is solution of the first order optimality system:

$$\left\{\begin{array}{rclcl} L^*q_\epsilon & = & \beta q_\epsilon(t,0,x) & \text{in} & Q, \\ q_\epsilon(T,a,x) & = & 0 & \text{in} & Q_A, \\ q_\epsilon(t,A,x) & = & 0 & \text{in} & Q_T, \\ q_\epsilon & = & h_0\chi_O + (w_0 - v_\epsilon)\chi_\gamma & \text{on} & \Sigma, \end{array}\right. \tag{54}$$

$$\left\{\begin{array}{rclcl} L\rho_\epsilon & = & 0 & \text{in} & Q, \\ \rho_\epsilon(0,a,x) & = & \frac{1}{\epsilon}q_\epsilon(0,a,x) & \text{in} & Q_A, \\ \rho_\epsilon(t,0,x) & = & \displaystyle\int_0^A \beta(t,a,x)\rho_\epsilon(t,a,x)da & \text{in} & Q_T, \\ \rho_\epsilon & = & 0 & \text{on} & \Sigma, \end{array}\right. \tag{55}$$

$$v_\epsilon = (w_0\chi_\gamma - \frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma) - P(w_0\chi_\gamma - \frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma) \in Y^{\perp}. \tag{56}$$

Step 2.

Multiplying the state equation (54) by ρ_ϵ and integrating by parts over Q, we get

$$\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} = \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma + \int_U\int_\gamma (w_0 - v_\epsilon)\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma.$$

Which in view of (56) and the fact that $v_\epsilon \in Y^\perp$ give

$$\begin{aligned}\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} &= \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma \\ &+ \int_U\int_\gamma (w_0 - v_\epsilon)(w_0 - v_\epsilon - P(w_0\chi_\gamma - \frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma))dtd\Gamma. \\ &= \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma \\ &- \|w_0 - v_\epsilon\|_{L^2(U\times\gamma)} + \|Pw_0\chi_\gamma\|_{L^2(U\times\gamma)} + \int_U\int_\gamma w_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma.\end{aligned}$$

As on $U\times\gamma$

$$w_0 - v_\epsilon = Pw_0\chi_\gamma + (I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma.$$

We have that

$$\|w_0 - v_\epsilon\|_{L^2(U\times\gamma)} = \|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} + \|Pw_0\chi_\gamma\|^2_{L^2(U\times\gamma)}$$

so that

$$\begin{aligned}\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} &= \int_U\int_O h_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma \\ &+ \int_U\int_\gamma w_0\frac{\partial\rho_\epsilon}{\partial\nu}dtd\Gamma.\end{aligned}$$

This implies that

$$\begin{aligned}\frac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} &\le (\int_U\int_O(\theta h_0)^2dtd\Gamma)^{\frac{1}{2}}(\int_U\int_\gamma\frac{1}{\theta^2}\frac{\partial\rho_\epsilon}{\partial\nu}^2dtd\Gamma)^{\frac{1}{2}} \\ &+ (\int_U\int_O(\theta w_0)^2dtd\Gamma)^{\frac{1}{2}}(\int_U\int_\gamma\frac{1}{\theta^2}\frac{\partial\rho_\epsilon}{\partial\nu}^2dtd\Gamma)^{\frac{1}{2}}.\end{aligned} \tag{57}$$

If we apply the adapted Carleman inequality (37) to ρ_ϵ we obtain

$$\int_U\int_\Gamma\frac{1}{\theta^2}|\frac{\partial\rho_\epsilon}{\partial\nu}|^2dtd\Gamma \le C\int_U\int_\gamma|(I-P)\frac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma|^2dtda\Gamma, \tag{58}$$

where $C>0$ is independent of ϵ. From (57), the choice of $w_0\in Y_\theta$ and the hypothesis on h_0, we deduce that

$$\begin{aligned}&\tfrac{1}{\epsilon}\|q_\epsilon(0,.,.)\|^2_{L^2(Q_A)} + \tfrac{1}{2}\|(I-P)\tfrac{\partial\rho_\epsilon}{\partial\nu}\chi_\gamma\|^2_{L^2(U\times\gamma)} \\ &\qquad\le C(\int_U\int_\omega\theta^2|w_0|^2dtdad\Gamma + \int_U\int_O\theta^2|h_0|^2dtdad\Gamma)^{\frac{1}{2}}\end{aligned} \tag{59}$$

and then

$$\|v_\epsilon\|^2_{L^2(U\times\omega)} \le C(\int_U\int_\gamma\theta^2|w_0|^2dtdad\Gamma + \int_U\int_O\theta^2|h_0|^2dtdad\Gamma)^{\frac{1}{2}} \tag{60}$$

$$\|q_\epsilon \chi_\omega\|^2_{L^2(U\times\gamma)} \leq C(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma)^{\frac{1}{2}} \tag{61}$$

In view of (58) and (59), we get

$$\|\frac{1}{\theta}\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(\Sigma)} \leq C(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma)^{\frac{1}{2}} \tag{62}$$

and using (59) and the fact that $\frac{1}{\theta}$ is bounded, we have

$$\|\frac{1}{\theta}P\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(U\times\gamma)} \leq C(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma)^{\frac{1}{2}}$$

Therefore, Y being a finite dimensional vector subspace of $L^2(U\times\gamma)$,we deduce that

$$\|P\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(U\times\gamma)} \leq C(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma)^{\frac{1}{2}} \tag{63}$$

from which we deduce by using (59) that

$$\|\frac{\partial\rho_\epsilon}{\partial\nu}\|_{L^2(U\times\gamma)} \leq C(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma)^{\frac{1}{2}} \tag{64}$$

Using Proposition 2 , we have that

$$\begin{aligned}&\int_U\int_\Omega \frac{1}{\theta^2}(|\frac{\partial\rho_\epsilon}{\partial\nu}|^2 + |\Delta\rho_\epsilon|^2 + |\nabla\rho_\epsilon|^2 + |\rho_\epsilon|^2)dtdad\Gamma \\ &\leq C(\int_U\int_\gamma \theta^2|w_0|^2 dtdad\Gamma + \int_U\int_O \theta^2|h_0|^2 dtdad\Gamma)^{\frac{1}{2}}\end{aligned} \tag{65}$$

Step 3.

We prove the convergence of $(v_\epsilon,\, q_\epsilon)_\epsilon$ and ρ_ϵ towards $\hat{v}$, $\hat{q}$ and ρ as $\epsilon \longrightarrow 0$. According to (60), (61) and (62) we can extract subsequences of $(v_\epsilon,\, q_\epsilon)_\epsilon$ (still called $(v_\epsilon,\, q_\epsilon)_\epsilon$) such that

$$v_\epsilon \rightharpoonup \widetilde{v} \quad \text{weakly in } L^2(U\times\gamma), \tag{66}$$

$$q_\epsilon \rightharpoonup \widetilde{q} \quad \text{weakly in } L^2(U; H^1_0(\Omega)), \tag{67}$$

$$\frac{1}{\theta}\rho_\epsilon \rightharpoonup \widetilde{\rho} \quad \text{weakly in } L^2(\frac{1}{\theta}, Q). \tag{68}$$

As v_ϵ belong to $Y^\perp$ which is closed vector subspace of $L^2(U\times\gamma)$, we have

$$\widetilde{v} \in Y^\perp. \tag{69}$$

The traces $(\widetilde{q}(0,.,.), \widetilde{q}(.,0,.))$, $(\widetilde{q}(T,.,.), \widetilde{q}(.,A,.))$ and $\frac{\partial \widetilde{q}}{\partial \nu}$ exists and belong respectively to $(L^2(Q_A))^2 \times (L^2(Q_T))^2$ and $L^2(\Sigma)$(see [8]).

So, using (66) and (67) while passing (54) to the limit as $\epsilon \longrightarrow 0$, we can prove that

$\widetilde{q}$ is solution of

$$\begin{cases} L^*\tilde{q} = \beta\tilde{q}(t,0,x) & \text{in} \quad Q, \\ \tilde{q}(T,a,x) = 0 & \text{in} \quad Q_A, \\ \tilde{q}(t,A,x) = 0 & \text{in} \quad Q_T, \\ \tilde{q} = h_0\chi_O + (w_0 - \tilde{v})\chi_\gamma & \text{on} \quad \Sigma, \end{cases} \tag{70}$$

and it follows from (59) that

$$q_\epsilon(0,.,.) \rightharpoonup \widetilde{q}(0,.,.) = 0 \text{ weakly in } L^2(Q). \tag{71}$$

In view of (69), (70) and (71), $(\widetilde{v}, \widetilde{q})$ verifies the null controllability $(14)-(16)$ and there exists a solution to the boundary null controllability problem. Moreover, it is clear from (68) that $\widetilde{\rho}$ satisfies

$$\begin{cases} L\widetilde{\rho} = 0 & \text{in} \quad Q \\ \tilde{\rho}(t,0,x) = \displaystyle\int_0^A \beta(t,a,x)\tilde{\rho}(t,a,x)da & \text{in} \quad Q_T, \\ \widetilde{\rho} = 0 & \text{on} \quad \Sigma \end{cases}$$

From (64)

$$\frac{\partial \rho_\epsilon}{\partial \nu} \rightharpoonup \frac{\partial \widetilde{\rho}}{\partial \nu} \qquad \text{weakly in } L^2(U\times\gamma) \tag{72}$$

We know on the one hand that $(\widetilde{v}, \widetilde{q})$ is solution to null controllability $(14)-(16)$, and on the other other hand that, there exists a unique $\hat{v} \in \varepsilon$ such that $(w_0 - \hat{v})$ is of minimal norm in $L^2(U \times \gamma)$. If we denote by $\hat{q}$ the corresponding solution to (15), we have $\hat{q}(0,.,.) = 0$ and, as $\widetilde{v} \in \mathcal{E}$,

$$\frac{1}{2}\|w_0 - v_\epsilon\|^2_{L^2(U\times\gamma)} \leq J_\epsilon(v_\epsilon) \leq J_\epsilon(\hat{v}) = \frac{1}{2}\|w_0 - \hat{v}\|^2_{L^2(U\times\gamma)}$$

and

$$\frac{1}{2}\|w_0 - \hat{v}\|^2_{L^2(U\times\gamma)} \leq \frac{1}{2}\|w_0 - v_\epsilon\|^2_{L^2(U\times\gamma)}$$

Using (66)

$$\liminf_{\epsilon\longrightarrow 0} \frac{1}{2}\|w_0 - v_\epsilon\|^2_{L^2(U\times\gamma)} \geq \frac{1}{2}\|w_0 - \hat{v}\|^2_{L^2(U\times\gamma)}$$

Hence,

$$\widetilde{v} = \widehat{v}$$

and

$$v_\epsilon \rightharpoonup \widetilde{v} \quad strongly\ in\ L^2(U \times \gamma).$$

Writing $\widetilde{\rho} = \widehat{\rho}$, we obtain

$$\widehat{v} = (I - P)\left(w_0\chi_\gamma - \frac{\partial \widehat{\rho}}{\partial \nu}\chi_\gamma\right).$$

4. Formulation of the sentinel with given sensitivity and identification of parameters λ_i

According to Theorem 2, if we replace in (4) w by

$$\hat{w} = P(w_0) + (I - P)\left(\frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma\right),$$

the function S defined by

$$S(\lambda, \tau) = \int_U \int_O h_0 \frac{\partial y}{\partial \nu}(\lambda, \tau) dt da d\Gamma + \int_U \int_\gamma (P(w_0) + (I - P)(\frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma))\frac{\partial y}{\partial \nu}(\lambda, \tau) dt da d\Gamma,$$

is such that $(\hat{w}, S(\hat{w}))$ verified the sentinel problem $(4) - (7)$. To estimate the parameters λ_i , one proceeds as follows: assume that the solution of (1) when $\lambda = 0$ and $\tau = 0$ is known. Then, one has the following information

$$S(\lambda, \tau) - S(0, 0) \approx \sum_{i=1}^{M} \lambda_i \frac{\partial S}{\partial \lambda_i}(0, 0).$$

Therefore, fixing $i \in \{1, \dots, M\}$ and choosing

$$\frac{\partial S}{\partial \lambda_j}(0,\ 0) = 0\ for\ j \neq i\ and\ \frac{\partial S}{\partial \lambda_i}(0,\ 0) = c_i,$$

one obtains the following estimate of the parameter λ_i:

$$\lambda_i \approx \frac{1}{c_i}\left(S(\lambda, \tau) - S(0, 0)\right),$$

we deduce that

$$\begin{aligned}\lambda_i \approx \frac{1}{c_i}&\left\{\int_U \int_O h_0 \left(m_0 - \frac{\partial y_0}{\partial \nu} dt da d\Gamma\right)\right\} \\ &+ \frac{1}{c_i}\left\{\int_U \int_\gamma \left(P(w_0) + (I - P)\left(\frac{\partial \hat{\rho}}{\partial \nu}\chi_\gamma\right)\right)\left(m_0 - \frac{\partial y_0}{\partial \nu}\right) dt da d\Gamma\right\},\end{aligned}$$

where m_0 is a measure of the flux of the population taken on the observatory $O \cup \gamma$ and y_0 is solution of (1) when $\lambda = 0$ and $\tau = 0$.

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