On the nonchaotic nature of monotone dynamical systems

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Abstract. Two common types of dynamics, chaotic and monotone, are compared. It is shown that monotone maps in strongly ordered spaces do not have chaotic attractors.

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1. Introduction

This article reviews and contrasts two protean classes of dynamical systems frequently used applied fields, *chaotic* and *monotone*.

1.1. Terminology

"Space" means a topological space with a metric denoted by d. The distance from a point y to a set Q is

$$\text{dist}\,(y, Q) := \inf\{d(y, q)\colon\ q \in Q\}.$$

Maps are assumed continuous.

Aa *dynamical system* on a space S — the set of states— is a a family $F = \{F_t\}$ of maps between subsets of S, called the *dynamic*. The parameter t, representing time, varies over an appropriate set of real numbers. An initial state x evolves to $F_t x$ at $t \geq 0$, so that $F_s(F_t x) = F_{s+t}x$, and F_0 is the identity map on S. Every map $g\colon S \to S$ defines a dynamical system on S whose dynamic is the set of iterates $\{g^j\}$ where j runs over the integers ≥ 0, or all the integers when g is a homeomorphism of S. When F is clear from the context, $F_t(x)$ may be denoted by $x(t)$.

Let $T\colon Y \to Y$ denote a map. The *orbit* of $y \in Y$ under T is the set

$$\{y, Ty, T^2y, \dots\} \subset Y.$$

If $T^k y = y$ for some $k \geq 1$ then y and its orbit are *periodic*.

A nonempty set $A \subset Y$ *attracts* $y \in Y$ if

$$\lim_{k \to \infty} \mathsf{dist}\,(T^k y, A) = 0. \tag{1}$$

We call A an *attractor* provided:

- A is compact and nonempty,

- $TA \subset A$,

- A attracts every point in some neighborhood of A.

It follows that the set of points attracted to A is an open neighborhood N of A such that $A \subset TN \subset N$.

We call A a *global* attractor if it attracts every point of Y.

1.2. Chaotic dynamics

The hallmark of a chaotic dynamical system is this:

Sensitivity to Initial Conditions: There exists $\delta > 0$ with the following property: If x, y are distinct initial states, there exists a time $t > 0$ such that the corresponding future states satisfy:
$$d(x(t), y(t)) > \delta.$$

When such a dynamical system is used to model the evolution of a natural system, (e.g., the atmosphere, an economy, an ecology, a disease), accurate long-term predictions are not possible. This was discovered by the meteorologist Edward Lorenz in his seminal 1963 article, "*Deterministic Non-periodic Flow*" [26]. After drastically simplifying standard equations for fluid flow, Lorenz arrived at the system of differential equations:

$$
\begin{aligned}
\dot{x} &= 10(y - z),\\
\dot{y} &= 28x - y - xz,\\
\dot{z} &= xy - (8/3)z.
\end{aligned}
\tag{2}
$$

Despite the simple algebraic form of these equations, Lorenz found a disturbing feature in his extensive computations of solutions:

"...two states differing by imperceptible amounts may eventually evolve into two considerably different states. If, then, there is any error whatever in observing the present state— and in any real system such errors seem inevitable— an acceptable prediction of an instantaneous state in the distant future may well be impossible."

This unexpected problem for applied dynamics inspired a great many publications analyzing it from various points of view. The first mathematical proof of Lorenz's numerical discovery, computer-assisted but rigorous, is due to W. Tucker [41]. For interesting discussions of this work, see I. Stewart [36] and M. Viana [42].

R. Devaney [7, p.324] validated Lorenz's conclusions dynamically by constructing a Poincaré (or "first return") map $T \colon C \to C$ for Lorenz's differential equations (2), such that:

- $C \subset \mathbb{R}^3$ *is an affine open 2-cell, transverse to the solution curves of* (2).

- *T has a chaotic global attractor.*

His proof is based on estimates derived from (2) by simple algebra. R. WILLIAMS [43] showed that Lorenz's attractor is structurally stable, which means that all sufficiently small perturbations preserve the essential features of Lorenz's equation.

The term "chaos" is used in many ways in mathematics. In a widely accepted definition, R. DEVANEY [6] called a map $T \colon A \to A$ to be chaotic if it satisfies three axioms:

SENSITIVITY TO INITIAL CONDITIONS: *There exists $\delta > 0$ such that if $x, y \in A$ are distinct, then* $d(T^k x, T^k y) \geq \delta$ *for some $k > 0$.*

> DENSE PERIODIC POINTS: *Every nonempty open subset of Y contains a periodic point.*
>
> TOPOLOGICAL TRANSITIVITY: *If $U, V \subset A$ are nonempty open sets, $T^k U \cap T^k V \neq \varnothing$ for some $k \geq 0$.*
>
> Topological Transitivity holds if some orbit is dense, and the converse is proved in the Results section.

In order to avoid trivialities, I add a fourth axiom:[*]

> NONFINITENESS: *A is not a finite orbit*.

When all four axioms are satisfied, *A* and *T* are *chaotic*.

1.3. Remarks

- P. TOUHEY [38] found a remarkably simple condition equivalent to Devaney's definition:

  > SHARING OF PERIODIC ORBITS: *Every pair of nonempty open sets have a periodic point in common.*

- Sensitivity to Initial Conditions follows from Devaney's other two axioms (SILVERMAN [29]), so it is independent of the metric.

- In most applications Y is a complete, separable metric space. In this case the Baire Category Theorem can be used to show that Topological Transitivity is equivalent to existence of dense orbit (stated in DEVANEY [6]). A proof for Y compact is given in the Results section.

For some other approaches to chaos, see [4, 11, 22, 23, 37, 43].

Our main result, Theorem 2 below, shows that:

Monotone maps in strongly ordered spaces do not have chaotic attractors.

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[*]Suggested by DEVANEY [8].

1.4. Monotone dynamics

The state space of a monotone dynamical system is a space X endowed with a (partial) order denoted by $\preceq$. The set $\{(x, y) \in X \times X : x \preceq y\}$ is assumed to be closed. A map T between ordered spaces is *monotone* provided

$$x \preceq y \implies Tx \preceq Ty.$$

We use the standard notation:

$x \prec y \iff x \preceq y, \; x \neq y.$

If A and B are sets, $\quad A \prec B \iff a \prec b, \quad (a \in A, \, b \in B).$

$a \prec B \iff \{a\} \prec B.$

In the main result X is *strongly ordered*:

If $W \subset X$ is an open neighborhood of x, there are nonempty open sets $U, V \subset W$ such that $U \prec x \prec V$.

Examples.

- Euclidean space $\mathbb{R}^n$ is strongly ordered by the classical *vector order*:

$$x \preceq y \iff x_j \leq y_j, \; (j = 1, \ldots, n). \tag{3}$$

- Many Banach spaces of continuous real-valued functions on a space S are strongly ordered by the functional order:

$$f_1 \preceq f_2 \iff f_1 x \leq f_2 x, \; (x \in S).$$

In many dynamical models of natural systems the state space reflects the relative size of states— density, population, etc.

Scientific fields are often modeled by a dynamical system whose state space $S \subset \mathbb{R}^n$ has an order that reflects the relative "size" of states— density, population, etc. Each coordinate

A typical example of a monotone dynamical system is one that models an ecology of n species that is biologists call *commensual*: an increase in the growth rate of any species tends to increase the sizes of the others. The state space is the positive orthant $\mathbb{R}^n_+$ with the vector order; coordinate x_i is a measure of the size (e.g., the density) of population i, and dynamic is defined by a set of ordinary differential equations

$$\frac{dx_i}{dt} = x_i G_i(x_1, \ldots, x_n), \quad (i = 1, \ldots, n). \tag{4}$$

Monotonicity is established by assuming

$$i \neq j \implies \frac{\partial G_i}{\partial x_j} \geq 0. \tag{5}$$

If the species reproduce only once a year, the ecology can be modeled by a map $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$. The dynamic $\{T^k\}$, parametrized by integers $k \geq 0$, is monotone if the partial derivatives of T are nonnegative.

An opposite kind of ecology— for example, sheep and wolves— is modeled by a *competitive* system of differential equations (4), which instead of (5) satisfies:

$$i \neq j \implies \frac{\partial G_i}{\partial x_j} \leq 0. \tag{6}$$

Here an increase in the growth rate of one species tends to *decrease* the sizes of others. The theory of cooperative systems can be used to analyze competitive systems by "time reversal"— choosing $-t$ as the time parameter. This produces a cooperative system.

While cooperative systems cannot have chaotic attractors, by Theorem 2, many competitive systems do have them. They can be constructed using a theorem of S. SMALE [30]:

Let F be a dynamical system on the (n − 1)-simplex

$$\Delta := \{x \in \mathbb{R}^n_+ : \ \Sigma_1^n x_i = 1\},$$

determined by a continuously differentiable vector field. Then F extends to a dynamical system $\hat{F}$ in $\mathbb{R}^n_+$ determined by a suitable G in (4), and has Δ as a global attractor.

It follows that if $A \subset \Delta$ is a chaotic global attractor for F, then A is also a chaotic global attractor for $\hat{F}$.

Monotone dynamical systems often permit reliable predictions of long-term behavior. In many case it can be proved that typical trajectories tend toward fixed points or periodic orbits. See for example references [2, 3, 5, 9, 10, 12, 13, 15–17, 19, 20, 24, 25, 28, 31, 34, 44]. The recent survey by H. SMITH [35] has an extensive bibliography.

Monotonicity and chaos play quite different roles in dynamical models:

Monotonicity is easily verified for many standard classes of systems, and in many cases standard theory gives accurate predictions of long-term behavior.

Chaos is quite difficult to either prove or disprove, and it makes accurate long-term prediction impossible. But it is unavoidable: as Lorenz discovered, simple models of real systems exhibit chaos.

Results

Proposition 1. *If Y is a compact metric space and H: Y → Y is topologically transitive, then H has a dense orbit.*[†]

Proof. Y is covered by a family $U_1, U_2, \ldots$ of open sets whose diameters go to 0 as i goes to ∞. Using Topological Transitivity repeatedly, one finds a sequence K_i of compact sets satisfying satisfy:

$$U_1 \supset K_1 \supset K_2 \supset \cdots \tag{7}$$

[†]DEVANEY [6] points out that it suffices for Y to be any separable, complete metric space.

and

$$U_i \cap T^i(K_i) \text{ is not empty.} \tag{8}$$

By compactness of the K_i there exists $p \in \bigcap_i K_i$, whence (7) and (8) show that the orbit of p is dense. ∎

Theorem 1. *Assume $T: X \to X$ is a monotone map in a strongly ordered space, and A is an attractor for T. Then $T|A$ is not topologically transitive, hence A is not chaotic.*

Proof. Assume *per contra* that $T|A$ is topologically transitive. Then Proposition 1, with $H = T|A$, implies:

(i) *Some orbit is dense in A,*

which implies:

(ii) *No point is isolated in A.*

To reach a contradiction we rely on a *deus ex machina*, Corollary 6.4 of HIRSCH [12]:

If no point is isolated in A, then no finite union of orbits is dense in A.

Therefore (i) is false, proving that $T|A$ is topologically transitive.

Acknowledgements

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