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A characterization of derivations in prime rings with involution

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Abstract. The purpose of this paper is to investigate $*$-differential identities satisfied by pair of derivations on prime rings with involution. In particular, we prove that if a 2-torsion free noncommutative ring R admit nonzero derivations d_1, d_2 such that $[d_1(x), d_2(x^*)] = 0$ for all $x \in R$, then $d_1 = \lambda d_2$, where $\lambda \in C$. Finally, we provide an example to show that the condition imposed in the hypothesis of our results are necessary.

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Key Words and Phrases: Prime ring, derivation, involution, $*$-differential identity

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1. Introduction

In all that follows, R will represent an associative ring with center $Z(R)$. We denote by Q and C the maximal ring of quotient and the extended centroid of a prime ring, respectively. For the explanation of Q and C we refer the reader to [4]. We denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator of x and y. A ring is said to 2-torsion free if $2x = 0$ (where $x \in R$) implies $x = 0$. A ring R is said to be prime if $aRb = (0)$ (where $a, b \in R$) implies either $a = 0$ or $b = 0$, and is called semiprime ring if $aRa = (0)$ (where $a \in R$) implies $a = 0$. An additive mapping $* : R \to R$ is called an involution if $*$ is an anti-automorphism of order 2; that is, $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. A ring equipped with an involution is known as ring with involution or $*$-ring. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case,

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$S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [14], where further references can be found.

A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. Over the last 30 years, several authors have investigated the relationship between commutativity of the ring R and certain special types of maps on R. The first result in this direction is due to Divinsky [12], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [18] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, many authors have refined and extended these results in various directions (see for example [3, 5–7, 9] where further references can be looked).

In [13], Herstein proved that if R is a prime ring of characteristic not two admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative. Further, Daif [10] showed that a 2-torsion free semiprime ring R admits a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in I$, where I is a nonzero ideal of R, then R contains a nonzero central ideal. In [15], Lanski prove that if L is a noncommutative Lie ideal of a 2-torsion free prime ring R and d, h are nonzero derivations of R such that $[d(x), h(x)] \in C$ for all $x \in L$, then $h = \lambda d$, where $\lambda \in C$. Very recently, the first author together with Dar [11] proved the following result: Let R be a prime ring with involution $*$ of the second kind such that $char(R) \neq 2$. If R admits a nonzero derivation d such that $[d(x), d(x^*)] = 0$ for all $x \in R$, then R is commutative. In the last three decades many authors have generalized the above mention result in several ways (viz.; [1, 2, 8, 11, 15, 17, 19] where further references can be found).

Motivated by the above results, here we continue this line of investigation by considering more general situations. Besides proving some other results, the main result is the following theorem.

Main Theorem. Let R be a 2-torsion free noncommutative prime ring with involution $*$ of the second kind and d_1, d_2 be two nonzero derivations on R such that

$$[d_1(x), d_2(x^*)] = 0 \quad \text{for all} \quad x \in R.$$

Then $d_1 = \lambda d_2$, where $\lambda \in C$.

2. Main results

In order to prove our results, we need the following lemma.

Lemma 1. *Let R be a 2-torsion free noncommutative prime ring with involution $*$ of the second kind and d_1, d_2 be two nonzero derivations on R. If one of the following conditions holds:*

(i) $[d_1(x), d_2(y)] = x \circ y$ *for all* $x, y \in R$,

(ii) $[d_1(x), d_2(y)] = -x \circ y$ *for all* $x, y \in R$,

then $d_1 = \lambda d_2$, *where* $\lambda \in C$.

Proof. (i) We consider the case

$$[d_1(x), d_2(y)] = x \circ y \quad \text{for all} \quad x, y \in R. \tag{1}$$

Substituting yh for y, where $h \in H(R) \cap Z(R)$, we get

$$[d_1(x), y]d_2(h) = 0 \quad \text{for all} \quad x, y \in R. \tag{2}$$

Using the primeness of R we have either $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$ or $[d_1(x), y] = 0$ for all $x, y \in R$. If $[d_1(x), y] = 0$ for all $x, y \in R$, then by Posner's result [18] R is commutative, a contradiction. Therefore we are left with the case $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replacing y by yx in (1), we obtain

$$d_2(y)[d_1(x), x] + [d_1(x), d_2(y)]x + y[d_1(x), d_2(x)] + [d_1(x), y]d_2(x) = xyx + yx^2 \tag{3}$$

for all $x, y \in R$. Multiplying (1) by x from right side and subtracting it from (3), we arrive at

$$d_2(y)[d_1(x), x] + y[d_1(x), d_2(x)] + [d_1(x), y]d_2(x) = 0 \quad \text{for all } x, y \in R.$$

Now taking h for y where $h \in H(R) \cap Z(R)$, we get $h[d_1(x), d_2(x)] = 0$ for all $x \in R$. Now using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we finally arrive at $[d_1(x), d_2(x)] = 0$ for all $x \in R$. Thus in view of [15, Theorem 4] we get $d_1 = \lambda d_2$, where $\lambda \in C$.

(ii) Using a similar approach with necessary variations, we can prove that the same conclusion holds for the case $[d_1(x), d_2(y)] = -x \circ y$ for all $x, y \in R$.

Proof of Main Theorem. By the given assumption, we have

$$[d_1(x), d_2(x^*)] = 0 \tag{4}$$

for all $x \in R$. A linearization of (4) yields that

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] = 0 \tag{5}$$

for all $x, y \in R$. Replacing y by hy in (5), where $y \in R$ and $h \in H(R) \cap Z(R)$, we get

$$h([d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]) + d_2(h)[d_1(x), y^*] + d_1(h)[y, d_2(x^*)] = 0.$$

Using (5), we get

$$d_2(h)[d_1(x), y^*] + d_1(h)[y, d_2(x^*)] = 0 \tag{6}$$

for all $x, y \in R$ and $h \in H(R) \cap Z(R)$. Substituting ky for y in (6), where $k \in S(R) \cap Z(R)$, we have

$$-d_2(h)[d_1(x), y^*k] + d_1(h)[ky, d_2(x^*)] = 0.$$

This further implies that

$$-d_2(h)k[d_1(x), y^*] + d_1(h)k[y, d_2(x^*)] = 0. \tag{7}$$

Multiplying (6) by k and comparing with (7), we obtain

$$2d_1(h)k[y, d_2(x^*)] = 0.$$

Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, the above expression gives

$$d_1(h)[y, d_2(x^*)] = 0 \tag{8}$$

for all $x, y \in R$ and $h \in H(R) \cap Z(R)$. Invoking the primeness of R, we get $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$ or $[y, d_2(x^*)] = 0$ for all $x, y \in R$. Suppose $[y, d_2(x^*)] = 0$ for all $x, y \in R$. Replacing x by x^* we get $[y, d_2(x)] = 0$ for all $x, y \in R$. Thus in view of Posner's result [18], R is commutative, which is a contradiction. Now suppose $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. This further implies that $0 = d_1(k^2) = 2d_1(k)k$. Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we have $d_1(k) = 0$ for all $k \in S(R) \cap Z(R)$. Now since every $z \in Z(R)$ can be represented as $2z = h + k$ where $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$, we get $d_1(Z(R)) = (0)$. Now in view of (7), we have

$$d_2(h)k[d_1(x), y^*] = 0$$

for all $x, y \in R, h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. Using primeness, we get either $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$ or $[d_1(x), y^*] = 0$ for all $x, y \in R$. Replacing y by y^*, we get $[d_1(x), y] = 0$ for all $x, y \in R$. Again using Posner's result [18], we get a contradiction. Now suppose $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. This intern implies that $d_2(Z(R)) = (0)$. Replacing y by $-ky$ in (5), we have

$$k([d_1(x), d_2(y^*)] - [d_1(y), d_2(x^*)]) = 0.$$

This further implies that

$$[d_1(x), d_2(y^*)] - [d_1(y), d_2(x^*)] = 0 \tag{9}$$

for all $x, y \in R$, since $S(R) \cap Z(R) \neq (0)$. On comparing (9) with (5), we get $2[d_1(x), d_2(y^*)] = 0$ for all $x, y \in R$. The last relation gives, $[d_1(x), d_2(y)] = 0$ for all $x, y \in R$. This implies that $[d_1(x), d_2(x)] = 0$ for all $x \in R$. Hence in view of [15, Theorem 4], we conclude that $d_1 = \lambda d_2$, where $\lambda \in C$. This completes the proof of the theorem. □

Corollary 1. Let R be a 2-torsion free noncommutative prime ring with involution $*$ of the second kind and d_1, d_2 be two nonzero derivations on R such that $[d_1(x), d_2(y^*)] = 0$ for all $x, y \in R$. Then $d_1 = \lambda d_2$, where $\lambda \in C$.

Theorem 1. *Let R be a 2-torsion free noncommutative prime ring with involution $*$ of the second kind and d_1, d_2 be two nonzero derivations on R. If one of the following conditions holds:*

(i) $[d_1(x), d_2(x^*)] = [x, x^*]$ *for all* $x \in R$,

(ii) $[d_1(x), d_2(x^*)] = -[x, x^*]$ *for all* $x \in R$,

then $d_1 = \lambda d_2$, *where* $\lambda \in C$.

Proof. By the given assumption, we have

$$[d_1(x), d_2(x^*)] = [x, x^*] \text{ for all } x \in R. \tag{10}$$

A linearization of (10) yields that

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] = [x, y^*] + [y, x^*] \text{ for all } x, y \in R. \tag{11}$$

Replace y by hy in (11), where $h \in H(R) \cap Z(R)$, we get

$$[d_1(x), d_2((hy)^*)] + [d_1(hy), d_2(x^*)] = [x, (hy)^*] + [hy, x^*] \tag{12}$$

for all $x, y \in R$ and $h \in H(R) \cap Z(R)$. On solving, we obtain

$$\begin{gathered}[d_1(x), y^*]d_2(h) + [y, d_2(x^*)]d_1(h)+ \\ h([d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]) = ([x, y^*] + [y, x^*])h\end{gathered} \tag{13}$$

for all $x, y \in R$ and $h \in H(R) \cap Z(R)$. Multiplying (11) by h and adding with (13), we arrive at

$$[d_1(x), y^*]d_2(h) + [y, d_2(x^*)]d_1(h) = 0 \tag{14}$$

for all $x, y \in R$ and $h \in H(R) \cap Z(R)$. Replacing y by ky, where $k \in S(R) \cap Z(R)$, we get

$$-[d_1(x), y^*]kd_2(h) + d_1(h)k[y, d_2(x^*)] = 0 \text{ for all } x, y \in R. \tag{15}$$

Multiplying (14) by k and adding with (15), we obtain

$$2d_1(h)k[y, d_2(x^*)] = 0 \text{ for all } x, y \in R.$$

This implies that

$$d_1(h)k[y, d_2(x^*)] = 0 \text{ for all } x, y \in R.$$

Using the primeness of the ring R and the fact that $S(R) \cap Z(R) \neq (0)$, we arrive at

$$\text{either } d_1(h) = 0 \text{ or } [y, d_2(x^*)] = 0 \text{ for all } x, y \in R. \tag{16}$$

$[y, d_2(x^*)] = 0$ for all $x, y \in R$ implies that R is commutative, a contradiction. Therefore we are left with $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. Using this in (15), we get

$$-[d_2(x), y^*]kd_2(h) = 0 \text{ for all } x, y \in R.$$

The primeness of R yields that

$$d_2(h) = 0 \text{ for all } h \in H(R) \cap Z(R). \tag{17}$$

or

$$[d_1(x), y^*] = 0 \text{ for all } x, y \in R.$$

Again if $[d_1(x), y^*] = 0$ for all $x, y \in R$, we get a contradiction. Therefore we are left with $d_2(h) = 0$. This implies that $d_2(k) = 0$ and hence $d_2(Z(R)) = (0)$. Similarly in view of (16) we get $d_1(Z(R)) = (0)$. Now on replacing y by ky in (11), where $k \in S(R) \cap Z(R)$, we get

$$[d_1(x), d_2((ky)^*)] + [d_1(ky), d_2(x^*)] = [x, (ky)^*] + [ky, x^*] \tag{18}$$

for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. On solving, we have

$$-[d_1(x), d_2(y^*)]k + k[d_1(y), d_2(x^*)] = -[x, y^*]k + k[y, x^*] \tag{19}$$

for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. Multiplying (11) by k and adding with (19), we obtain $2k[d_1(y), d_2(x^*)] = 2[y, x^*]$. Since $char(R) \neq 2$ and invoking primeness of R, we get $[d_1(y), d_2(x^*)] = [y, x^*]$ for all $x, y \in R$. Hence $[d_1(y), d_2(x)] = [y, x]$ for all $x, y \in R$. Taking y for x, we finally arrive at $[d_1(x), d_2(x)] = 0$ for all $x \in R$. Thus in view of [15, Theorem 4], we get $d_1 = \lambda d_2$, where $\lambda \in C$.
(ii) This can be proved by similar manner with necessary variations.

Theorem 2. *Let R be a 2-torsion free noncommutative prime ring with involution $*$ of the second kind and d_1, d_2 be two nonzero derivations on R such that*

$$[d_1(x), x^*d_2(x)] = 0, \quad \textit{for all} \ \ x \in R.$$

Then $d_1 = \lambda d_2$, where $\lambda \in C$.

Proof. By the assumption, we have

$$[d_1(x), x^*d_2(x)] = 0 \text{ for all } x \in R. \tag{20}$$

Linearization of (20) give us

$$[d_1(x), x^*d_2(x)] + [d_1(x), x^*d_2(y)] + [d_1(x), y^*d_2(x)] + [d_1(x), y^*d_2(y)]$$

$$+[d_1(y), x^*d_2(x)] + [d_1(y), x^*d_2(y)] + [d_1(y), y^*d_2(x)] + [d_1(y), y^*d_2(y)] = 0$$

for all $x, y \in R$. Using (20), we get

$$[d_1(x), x^*d_2(y)] + [d_1(x), y^*d_2(x)] + [d_1(x), y^*d_2(y)] \tag{21}$$

$$+[d_1(y), x^*d_2(x)] + [d_1(y), x^*d_2(y)] + [d_1(y), y^*d_2(x)] = 0 \text{ for all } x, y \in R.$$

Replacing y by h where $h \in H(R) \cap Z(R)$, we get

$$[d_1(x), x^*]d_2(h) + h[d_1(x), d_2(x)] = 0 \text{ for all } x \in R. \tag{22}$$

Substituting $x + y$ for x where $x, y \in R$ and combining it with (22), we have

$$([d_1(x), y^*] + [d_1(y), x^*])d_2(h) + h([d_1(x), d_2(y)] + [d_1(y), d_2(x)]) = 0 \tag{23}$$

for all $x, y \in R$. Now replacing y by hy where $y \in R$ and $h \in H(R) \cap Z(R)$, we obtain

$$[d_1(x), y^*]hd_2(h) + [d_1(y), x^*]hd_2(h) + [y, x^*]d_1(h)d_2(h) + h^2[d_1(x), d_2(y)] \tag{24}$$

$$+h[d_1(x), y]d_2(h) + h^2[d_1(y), d_2(x)] + hd_1(h)[y, d_2(x)] = 0 \text{ for all } x, y \in R.$$

Multiplying (23) by h where $h \in H(R) \cap Z(R)$ and using in (24) we get

$$[y, x^*]d_1(h)d_2(h) + h[d_1(x), y]d_2(h) + hd_1(h)[y, d_2(x)] = 0 \tag{25}$$

for all $x, y \in R$. Replacing x by kx where $x \in R$ and $k \in S(R) \cap Z(R)$, we arrive at

$$-[y, x^*]kd_1(h)d_2(h) + hk[d_1(x), y]d_2(h) + h[x, y]d_1(k)d_2(h) \tag{26}$$

$$+hd_1(h)[y, d_2(x)]k + hd_1(h)[y, x]d_2(k) = 0 \text{ for all } x, y \in R.$$

Multiplying (25) by k where $k \in S(R) \cap Z(R)$ and adding it with (26), we get

$$2hk([d_1(x), y]d_2(h) + [y, d_2(x)]d_1(h)) + h[x, y]d_1(k)d_2(h) + hd_1(h)[y, x]d_2(k) = 0$$

for all $x, y \in R$. Taking $y = x$, we obtain

$$2hk([d_1(x), x]d_2(h) + [x, d_2(x)]d_1(h)) = 0$$

for all $x \in R$. Since R is 2-torsion free prime ring and $S(R) \cap Z(R) \neq (0)$, the above relation implies that

$$[d_1(x), x]d_2(h) + [x, d_2(x)]d_1(h) = 0 \text{ for all } x \in R. \tag{27}$$

Replacing y by x in (25), we get

$$[x, x^*]d_1(h)d_2(h) + h[d_1(x), x]d_2(h) + hd_1(h)[x, d_2(x)] = 0 \tag{28}$$

for $x \in R$ and $h \in H(R) \cap Z(R)$. Using (27) in (28), we get $[x, x^*]d_1(h)d_2(h) = 0$ for all $x \in R$ and $h \in H(R) \cap Z(R)$. Now use the primeness condition we get either $[x, x^*] = 0$ for all $x \in R$ or $d_1(h)d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. If we consider $[x, x^*] = 0$, then in view of [16, Lemma 2.1] R is commutative, which is a contradiction to our assumption, now we have $d_1(h)d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. Using the primeness of the ring R we get either $d_1(h) = 0$ or $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. If consider $d_1(h) = 0$.

Then by (25) we get $h[d_1(x), y]d_2(h) = 0$ for all $x, y \in R$. Now using the primeness of the ring R, we obtain $[d_1(x), y]d_2(h) = 0$ for all $x, y \in R$. Again by the primeness of the ring R, we have either $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$ or $[d_1(x), y] = 0$ for all $x, y \in R$. If we consider $[d_1(x), y] = 0$ for all $x, y \in R$. This gives R is commutative by Posner's result [18], a contradiction. Therefore we are left with $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. Similarly in view of (25) we get $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replacing y by h where $h \in H(R) \cap Z(R)$ in (21) and using $d_1(h) = 0$ and $d_2(h) = 0$, we get $h[d_1(x), d_2(x)] = 0$ for all $x \in R$. Now using the primeness and $S(R) \cap Z(R) \neq (0)$ conditions, we get $[d_1(x), d_2(x)] = 0$ for all $x \in R$. Thus by the result of Lanski [15, Theorem 4], we get $d_1 = \lambda d_2$, where $\lambda \in C$.

Corollary 2. Let R be a 2-torsion free noncommutative prime ring with involution $*$ of the second kind and d_1, d_2 be two nonzero derivations on R such that $[d_1(x), y^* d_2(y)] = 0$ for all $x, y \in R$. Then $d_1 = \lambda d_2$, where $\lambda \in C$.

Theorem 3. *Let R be a 2-torsion free noncommutative prime ring with involution $*$ of the second kind and d_1, d_2 be two nonzero derivations on R. If one of the following conditions holds:*

(i) $[d_1(x), d_2(x^*)] = x \circ x^*$ *for all* $x \in R$,

(ii) $[d_1(x), d_2(x^*)] = -x \circ x^*$ *for all* $x \in R$,

then $d_1 = \lambda d_2$, where $\lambda \in C$.

Proof. By the given hypothesis, we have

$$[d_1(x), d_2(x^*)] = x \circ x^* \quad \text{for all } x \in R. \tag{29}$$

Replacing x by $x + y$ in (29), we get

$$[d_1(x), d_2(x^*)] + [d_1(y), d_2(y^*)] + [d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]$$
$$= x \circ x^* + y \circ y^* + xy^* + yx^* + x^*y + y^*x$$

for all $x, y \in R$. Using (29), we get

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] = xy^* + yx^* + x^*y + y^*x \tag{30}$$

for all $x, y \in R$. Substituting hy for y in (30) where $h \in H(R) \cap Z(R)$, we have

$$h([d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]) + [d_1(x), y^*]d_2(h) + d_1(h)[y, d_2(x^*)] \tag{31}$$
$$= h(xy^* + yx^* + x^*y + y^*x) \quad \text{for all } x, y \in R.$$

Using (30), (31) reduces to

$$[d_1(x), y^*]d_2(h) + d_1(h)[y, d_2(x^*)] = 0 \quad \text{for all } x, y \in R. \tag{32}$$

Now (32) is same as (14) and thus following the same techniques we get $d_1(Z(R)) = (0)$ and $d_2(Z(R)) = (0)$. Now replace y by ky in (30) where $k \in S(R) \cap Z(R)$, we get

$$-[d_1(x), d_2(y^*)]k - [d_1(x), y^*]d_2(k) + d_1(k)[y, d_2(x^*)] + k[d_1(y), d_2(x^*)] \tag{33}$$
$$= -xy^*k + kyx^* + x^*ky - ky^*x$$

for all $x, y \in R$. Now multiplying (30) by $k \in S(R) \cap Z(R)$ and adding with (33), we get

$$2[d_1(y), d_2(x^*)]k = 2k(yx^* + x^*y) \text{ for all } x, y \in R.$$

This implies that

$$k([d_1(y), d_2(x^*)] - (y \circ x^*)) = 0 \text{ for all } x, y \in R.$$

Invoking the primeness of R, we get

$$[d_1(y), d_2(x^*)] - (y \circ x^*) = 0 \text{ for all } x, y \in R.$$

Now replace x by x^*, we obtain

$$[d_1(y), d_2(x)] - (y \circ x) = 0 \text{ for all } x, y \in R.$$

Hence application of Lemma 1 gives that $d_1 = \lambda d_2$, where $\lambda \in C$.
(ii) Similarly we can prove the second part.

The following example shows that the primeness hypothesis in main theorem and Theorem 2 is not superfluous.

Example 1. Let $R = \left\{ \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} \;\middle|\; a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R} \right\}$, where $\mathbb{R}$ is a ring of real numbers. Of course, R with matrix addition and matrix multiplication is a noncommutative prime ring. Define mappings $*, d_1 : R \longrightarrow R$ such that

$$\begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix}^* = \begin{pmatrix} a_1 - ib_1 & a_3 - ib_3 \\ a_2 - ib_2 & a_4 - ib_4 \end{pmatrix} \text{ and,}$$

$$d_1 \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ a_3 + ib_3 & a_4 + ib_4 \end{pmatrix} = \begin{pmatrix} 0 & -(a_2 + ib_2) \\ (a_3 + ib_3) & 0 \end{pmatrix}.$$

It can be easily checked that $*$ and d_1 are respectively involution and derivation on R. Let $\mathbb{H}$ be a ring of real quaternions. Define involution $-$ and derivation $d_2 = d_i$ (where d_i is an inner derivation on $\mathbb{H}$ determined by $i \in \mathbb{H}$) as follows $\overline{q} = \alpha - i\beta - j\gamma - k\delta$ and $d_i(q) = [i, q]$ for all $q \in \mathbb{H}$.

Let $S = R \times \mathbb{H}$, where R is same as defined above with involution $*$ and derivation d_1 and $\mathbb{H}$ is the ring of real quaternions with involution $-$ and derivation d_2 as above. Clearly, S is a 2-torsion free noncommutative semiprime ring. Now define an involution α on S, as $(x, y)^\alpha = (x^*, \overline{y})$. Clearly, α is an involution of the second kind. Further, we define the mappings D_1 and D_2 from S to S such that $D_1(x, y) = (d_1(x), 0)$ and $D_2(x, y) = (0, d_2(x))$ for all $(x, y) \in S$. It can be easily checked that D_1, D_2 are derivations on S and satisfying $[D_1(X), D_2(X^\alpha)] = 0$ and $[D_1(X), X^\alpha D_2(X)] = 0$ for all $X \in S$, but D_1 and D_2 are linearly independent derivations. Hence, in Main Theorem and Theorem 2, the hypothesis of primeness is essential.

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