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On Semitotal Domination in Graphs

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Abstract. A set S of vertices of a connected graph G is a semitotal dominating set if every vertex in $V(G)\setminus S$ is adjacent to a vertex in S, and every vertex in S is of distance at most 2 from another vertex in S. A semitotal dominating set S in G is a secure semitotal dominating set if for every $v \in V(G) \setminus S$, there is a vertex $x \in S$ such that x is adjacent to v and that $(S \setminus \{x\}) \cup \{v\}$ is a semitotal dominating set in G.
In this paper, we characterize the semitotal dominating sets and the secure semitotal dominating sets in the join, corona and lexicographic product of graphs and determine their corresponding semitotal domination and secure semitotal domination numbers.

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1. Introduction

The concept of semitotal domination was introduced by W. Goddard, M. Henning and C. McPil (see [13]) in 2014. It is further studied by M.Henning and A. Marcon (see [17, 18]) in 2014 and 2016, and by G. Hao and W. Zhuang (see [14]) in 2018. Accordingly, this parameter is a strengthening of domination but a relaxation of both total domination and weakly connected domination [13].

In this paper, we investigate semitotal domination in the join, corona and lexicographic product of graphs. We also introduce the secure semitotal domination and investigate the concept in these classes of graphs.

All graphs considered in this study are finite and undirected. We refer to [7] for the basic graph terminologies used here. The symbols $V(G)$ and $E(G)$ denote the *vertex set*

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and *edge set*, respectively, of G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ is the *order* of G.

Given two graphs G and H with disjoint vertex sets, the *join* of G and H is the graph $G+H$ with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. The *lexicographic product or composition* $G[H]$ of G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. In any of these graphs, G and H are referred to as their basic component graphs.

For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v refers to the set of all vertices of G that are adjacent to v. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G(S) = \cup_{v \in S} N_G(v)$ and $N_G[S] = S \cup N_G(S)$. A set $S \subseteq V(G)$ is a *dominating set* in G if $N_G[S] = V(G)$. Thus, S is a dominating set in G if and only if for each $v \in V(G) \setminus S$ there exists $u \in S$ such that $uv \in E(G)$. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is the *domination number* of G. Provided that G has no isolated vertices, a set $S \subseteq V(G)$ is a *total dominating set* in G if for every $v \in V(G)$ there exists $u \in S$ such that $uv \in E(G)$. The minimum cardinality of a total dominating set in G, denoted by $\gamma_t(G)$, is the *total domination number* of G. We refer to [1–3, 6, 8, 9, 11, 15, 16] for the fundamentals and recent developments and applications of domination theory in graphs.

A set $S \subseteq V(G)$ is said to be *nearly dominating* in G if for every $v \in V(G) \setminus N_G[S]$, $S \cup \{v\}$ is a dominating set in G. The symbol $\gamma_\eta(G)$ denotes the minimum cardinality of a nearly dominating set in G. Clearly, $\gamma_\eta(G) = 0$ if and only if G is a complete graph, and since dominating sets are nearly dominating sets, $\gamma_\eta(G) \leq \gamma(G)$.

A *secure (total) dominating set* is a (total) dominating set S having the property that for each $v \in V(G) \setminus S$, there exists $u \in S \cap N_G(v)$ such that $(S \setminus \{u\}) \cup \{v\}$ is a (total) dominating set in G. The minimum cardinality $\gamma_s(G)$ (resp. $\gamma_{st}(G)$) of a secure dominating set (resp. secure total dominating set) in G is the *secure domination number* (resp. *secure total domination number*) of G. A secure dominating set of cardinality $\gamma_s(G)$ is called a *γ_s-set*. Secure domination and secure total domination in graphs have been studied in [4, 5, 10, 12, 19].

Let G be a graph without isolated vertices. A set $S \subseteq V(G)$ is a *semitotal dominating set* in G if S is a dominating set in G such that for every $x \in S$ there exists $y \in S \setminus \{x\}$ such that $d_G(x, y) \leq 2$. The smallest cardinality of a semitotal dominating set in G, denoted by $\gamma_{t2}(G)$, is called the *semitotal domination number* of G. A semitotal dominating set in G with cardinality $\gamma_{t2}(G)$ is called a *γ_{t2}-set*. It is worth noting that since a semitotal dominating set is a dominating set and total dominating sets are semitotal dominating sets, $\max\{2, \gamma(G)\} \leq \gamma_{t2}(G) \leq \gamma_t(G)$ for graphs G without isolated vertices. For all connected graphs G on $n \geq 4$ vertices, $\gamma_{t2}(G) \leq \frac{n}{2}$ [13]. In the referred paper, the authors characterized those trees and graphs of minimum degree 2 achieving this bound. Other excellent exposition on semitotal domination are found in [17] and in [18].

2. Secure semitotal domination

A semitotal dominating set $S \subseteq V(G)$ is a *secure semitotal dominating set* if for each $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a semitotal dominating set in G. The smallest cardinality of a secure semitotal dominating set in G is called the *secure semitotal domination number* of G and is denoted by $\gamma_{st2}(G)$. A secure semitotal dominating set with cardinality $\gamma_{st2}(G)$ is called a γ_{st2}*-set*. Secure semitotal dominating sets are both semitotal dominating sets and secure dominating sets. On the other hand, secure total dominating sets are secure semitotal dominating sets. Thus, $\max\{\gamma_{t2}(G), \gamma_s(G)\} \leq \gamma_{st2}(G) \leq \gamma_{st}(G)$ for all graphs G without isolated vertices.

Example 1. (1) *For* $n \geq 2$, $\gamma_{st2}(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \neq 2, 6 \\ 2, & n = 2 \\ 4, & n = 6. \end{cases}$

(2) *For* $n \geq 3$, $\gamma_{st2}(C_n) = \lceil \frac{n}{2} \rceil$.

(3) *For* $m, n \geq 2$, $\gamma_{st2}(K_{m,n}) = \min\{m, n, 4\}$.

For $v \in V(G)$, we write $N_G^2(v) = \{u \in V(G) \setminus \{v\} : d_G(u, v) \leq 2\}$, and for $S \subseteq V(G)$, we write $N_G^2(S) = \cup_{v \in S} N_G^2(v)$. Precisely, S is a semitotal dominating set if and only if $V(G) \setminus S \subseteq N_G(S)$ and $S \subseteq N_G^2(S)$.

Theorem 1. *Let* G *be a connected graph of order* $n \geq 2$*. Then* $\gamma_{st2}(G) = 2$ *if and only if there exists a dominating set* $\{x, y\}$ *in* G *satisfying the following properties:*

(*i*) $d_G(x, y) \leq 2$;

(*ii*) $N_G^2(x) = V(G) \setminus \{x\}$ *and* $N_G^2(y) = V(G) \setminus \{y\}$; *and*

(*iii*) $\{x, z\}$ *and* $\{u, y\}$ *are dominating sets in* G *for all* $z \in N_G(y) \setminus \{x\}$ *and for all* $u \in N_G(x) \setminus \{y\}$.

Proof. Suppose that $\gamma_{st2}(G) = 2$, and let $S = \{x, y\}$ be a γ_{st2}-set of G. Then S is a dominating set in G and $d_G(x, y) \leq 2$. Suppose that, in the contrary, $N_G^2(x) \neq V(G) \setminus \{x\}$, and let $z \in V(G) \setminus N_G^2(x)$ with $z \neq x$. Then $z \notin S$. Since S is a secure semitotal dominating set in G and $z \notin N_G^2(x)$, $y \in N_G(z)$ and $(S \setminus \{y\}) \cup \{z\} = \{x, z\}$ is a semitotal dominating set in G, a contradiction since $d_G(x, z) > 2$. Thus, $N_G^2(x) = V(G) \setminus \{x\}$. Similarly, $N_G^2(y) \setminus \{y\} = V(G) \setminus \{y\}$. Now let $z \in V(G) \setminus S$. Since S is a secure semitotal dominating set, there exists $w \in S \cap N_G(z)$ such that $T = (S \setminus \{w\}) \cup \{z\}$ is a semitotal dominating set, and hence a dominating set in G. If $w = y$, then $T = \{x, z\}$ and if $x = w$, then $T = \{y, z\}$.

Conversely, let $S = \{x, y\}$ be a dominating set in G satisfying the properties (i), (ii) and (iii). By property (i), S is a semitotal dominating set in G. Let $z \in V(G) \setminus S$. Then $x \in S \cap N_G(z)$ or $y \in S \cap N_G(z)$. Assume that $x \in S \cap N_G(z)$. Note that, by properties (ii) and (iii), $(S \setminus \{x\}) \cup \{z\} = \{y, z\}$ is a semitotal dominating set in G. Thus, $\gamma_{st2}(G) = |S| = 2$. □

3. In the join of graphs

For any graph G, $\gamma_{t2}(G+K_1)=2$. More specifically, a semitotal dominating set in $G+K_1$ is either of the form $V(K_1)\cup S$ for some nonempty $S\subseteq V(G)$, or a nonsingleton dominating set in G in case G is nontrivial.

Theorem 2. *Let G and H be nontrivial graphs, and $S\subseteq V(G+H)$. Then S is a semitotal dominating set in $G+H$ if and only if one of the following holds:*

(i) *$S\subseteq V(G)$ is a nonsingleton dominating set in G;*

(ii) *$S\subseteq V(H)$ is a nonsingleton dominating set in H;*

(iii) *$S\cap V(G)\neq\varnothing$ and $S\cap V(H)\neq\varnothing$.*

Proof. Suppose that $S\subseteq V(G)$ is a nonsingleton dominating set in G. Then S is a dominating set in $G+H$. Let $v\in S$. Since S is nonsingleton, we may take $u\in S$ with $u\neq v$. Note that $d_{G+H}(u,v)\leq 2$. Since v is arbitrary, S is a semitotal dominating set in $G+H$. Similarly, if $S\subseteq V(H)$ is a nonsingleton dominating set in H, then S is a semitotal dominating set in $G+H$. Suppose that S intersects both $V(G)$ and $V(H)$. Then S is a total dominating set, hence a semitotal dominating set, in $G+H$.

Conversely, suppose that S is a semitotal dominating set in $G+H$. Then S is a dominating set in $G+H$ and $|S|\geq 2$. If $S\subseteq V(G)$ (resp. $S\subseteq V(H)$), then (i) (resp. (ii)) holds. Otherwise, property (iii) holds. □

Corollary 1. *For all graphs G and H, $\gamma_{t2}(G+H)=2$.*

Let G be any graph and K_p the complete graph of order $p\geq 2$. Note that for any $x,y\in V(K_p)$, $\{x,y\}$ is a dominating set in $G+K_p$ satisfying the properties (i), (ii) and (iii) of Theorem 1. Thus, $\gamma_{st2}(G+K_p)=2$.

Proposition 1. *For noncomplete graphs G and H,*

$$2\leq\gamma_{st2}(G+H)\leq 4.$$

Proof. Let $S=\{x,y,u,v\}$, where $x,y\in V(G)$ and $u,v\in V(H)$. Then S is a semitotal dominating set in $G+H$ by Theorem 2. For each $w\in V(G)\setminus S$ (resp. each $w\in V(H)\setminus S$), $wv\in E(G+H)$ (resp. $wx\in E(G+H)$) and, by Theorem 2, $(S\setminus\{v\})\cup\{w\}$ (resp. $(S\setminus\{x\})\cup\{w\}$) is a semitotal dominating set in $G+H$. Thus, S is a secure semitotal dominating set in $G+H$. Consequently, $2\leq\gamma_{st2}(G+H)\leq|S|=4$. □

Corollary 2. *Let G and H be noncomplete graphs of orders m and n, respectively. Then $\gamma_{st2}(G+H)=2$ if and only if at least one of the following is true:*

(i) *$\gamma_s(G)=2$;*

(ii) *$\gamma_s(H)=2$;*

(*iii*) *there exist* $x \in V(G)$ *and* $y \in V(H)$ *such that* $\{x\}$ *and* $\{y\}$ *are nearly dominating sets in* G *and* H*, respectively.*

Proof. Suppose that $\gamma_{st2}(G+H) = 2$, and let $S = \{x, y\}$ be a dominating set in $G+H$ satisfying properties (i), (ii) and (iii) in Theorem 1. First, suppose that $S \subseteq V(G)$. Then S is a dominating set in G. Let $z \in V(G) \setminus S$. Assume $xz \in E(G)$. By Theorem 1(iii), $\{z, y\} = (S \setminus \{x\}) \cup \{z\}$ is a dominating set in $G + H$, hence in G. Thus, S is a secure dominating set in G. Since G in not complete, $\gamma_s(G) = 2$. Similarly, if $S \subseteq V(H)$, then $\gamma_s(H) = 2$. Suppose that $x \in V(G)$ and $y \in V(H)$. Let $z \in V(G) \setminus N_G[x]$. Then $z \in N_{G+H}(y) \setminus \{x\}$. By Theorem 1$(iii)$, $\{x, z\}$ is a dominating set in $G + H$, hence a dominating set in G. Similarly, $\{w, y\}$ is a dominating set in H for all $w \in V(H) \setminus N_G[y]$.

Conversely, suppose that $\gamma_s(G) = 2$, and let $S = \{x, y\}$ be a γ_s-set of G. By Theorem 2, S is a semitotal dominating set in $G + H$. Let $z \in V(G + H) \setminus S$. Suppose that $z \in V(H)$. In particular, $xz \in E(G + H)$ and $(S \setminus \{x\}) \cup \{z\} = \{z, y\}$, which is a semitotal dominating set in $G + H$ by Theorem 2. Suppose that $z \in V(G)$. Since S is a secure dominating set, either $xz \in E(G)$ and $\{y, z\}$ is a dominating set in G or $zy \in E(G)$ and $\{x, z\}$ is a dominating set in G. In either case, S is a secure semitotal dominating set in $G + H$ by Theorem 2, so that $\gamma_{st2}(G + H) = |S| = 2$. Similarly, if $\gamma_s(H) = 2$, then $\gamma_{st2}(G+H) = 2$. Finally, suppose that x and y satisfy property (iii), and put $S = \{x, y\}$. By Theorem 2, S is a semitotal dominating set in $G+H$. Let $z \in V(G)\setminus S$. Suppose that $xz \in E(G)$. Then $x \in S \cap N_{G+H}(z)$ and $(S \setminus \{x\}) \cup \{z\} = \{z, y\}$, which by Theorem 2, is a semitotal dominating set in $G + H$. Suppose that $z \notin N_G[x]$. By property (iii), $\{x, z\}$ is a dominating set in G, and hence a semitotal dominating set in $G + H$ by Theorem 2. Now, $zy \in E(G + H)$ and $(S \setminus \{y\}) \cup \{z\} = \{x, z\}$. Similarly, if $z \in V(H) \setminus S$, then there exists $w \in S \cap N_{G+H}(z)$ such that $(S \setminus \{w\}) \cup \{z\}$ is a semitotal dominating set in $G+H$. Thus, S is a secure semitotal dominating set in $G + H$. Therefore, $\gamma_{st2}(G + H) = 2$. □

In view of Corollary 2(iii), $\gamma_{st2}(K_{1,n} + K_{1,m}) = 2 = \gamma_{st2}(C_4 + C_4)$.

Corollary 3. *Let* G *and* H *be noncomplete graphs of orders* m *and* n*, respectively, and suppose that* $\gamma_{st2}(G + H) \neq 2$*. Then* $\gamma_{st2}(G + H) = 3$ *if and only if at least one of the following is true:*

(*i*) $\gamma_s(G) = 3$;

(*ii*) $\gamma_s(H) = 3$;

(*iii*) *there exist* $x, y \in V(G)$ *such that* $\{x, y\}$ *is a nearly dominating set in* G*;*

(*iv*) *there exist* $x, y \in V(H)$ *such that* $\{x, y\}$ *is a nearly dominating set in* H*.*

Proof. Suppose that $\gamma_{st2}(G + H) = 3$, and let $S \subseteq V(G + H)$ be a γ_{st2}-set of $G + H$. Suppose that $S \subseteq V(G)$. By Theorem 2, S is a dominating set in G. Let $w \in V(G) \setminus S$. There exists $x \in S \cap N_{G+H}(w)$ such that $T = (S \setminus \{x\}) \cup \{w\}$ is a semitotal dominating set in $G + H$. Since $T \subseteq V(G)$, T is a dominating set in G by Theorem 2, showing that

S is a secure dominating set in G. Hence, $\gamma_s(G) \le |S| = 3$. In view of Corollary 2, since $\gamma_{st2}(G+H) \neq 2$, $\gamma_s(G) = 3$. Similarly, if $S \subseteq V(H)$, then $\gamma_s(H) = 3$. Now, let $S = \{x, y, z\}$, and assume that $T = \{x, y\} \subseteq V(G)$ and $z \in V(H)$. If T is a dominating set in G, then property (iii) holds. Suppose not, and let $u \in V(G) \setminus N_G[T]$. Since S is a secure semitotal dominating set in $G+H$, $(S \setminus \{z\}) \cup \{u\} = \{x, y, u\}$ is a semitotal dominating set in $G+H$. Thus, $\{x, y, u\}$ is a dominating set in G. Accordingly, T is a nearly dominating set in G. Property (iv) is proved similarly.

Conversely, suppose that $\gamma_s(G) = 3$, and $S = \{x, y, z\} \subseteq V(G)$ is a γ_s-set of G. By Theorem 2, S is a semitotal dominating set in $G+H$. Let $w \in V(G+H)\setminus S$. If $w \in V(H)$, then in particular, $xw \in E(G+H)$ and $(S \setminus \{x\}) \cup \{w\} = \{y, z, w\}$, which is a semitotal dominating set in $G+H$ by Theorem 2. Suppose that $w \in V(G)$. Since S is a secure dominating set in G, there exists $t \in S \cap N_G(w)$ such that $T = (S \setminus \{t\}) \cup \{w\}$ is a dominating set in G. Hence T is a semitotal dominating in $G+H$. Thus, S is a secure semitotal dominating set in $G+H$. Since $\gamma_{st2}(G+H) \neq 2$, $\gamma_{st2}(G+H) = 3 = |S|$. Similarly, if $\gamma_s(H) = 3$, then $\gamma_{st2}(G+H) = 3$. Suppose that property (iii) holds, and let $\{x, y\}$ be a nearly dominating set in G. Pick any $z \in V(H)$, and put $S = \{x, y, z\}$. Then S is a semitotal dominating set in $G+H$. Let $w \in V(G+H) \setminus S$. If $w \in V(H)$, then $wx \in E(G+H)$ and $(S \setminus \{x\}) \cup \{w\} = \{y, z, w\}$ is a semitotal dominating set in $G+H$ by Theorem 2. Suppose that $w \in V(G)$, and suppose that $w \in N_G[\{x, y\}]$, say $wx \in E(G)$. Then $x \in S \cap N_{G+H}(w)$ and $(S \setminus \{x\}) \cup \{w\} = \{w, y, z\}$ is a semitotal dominating set in $G+H$. Suppose that $w \notin N_G[\{x, y\}]$. Here we note that $wz \in E(G+H)$ and $(S \setminus \{z\}) \cup \{w\} = \{x, y, w\}$. Since $\{x, y\}$ is a nearly dominating set, $\{x, y, w\}$ is a dominating set in G, and therefore, is a semitotal dominating set in $G+H$ by Theorem 2. All these imply that S is a secure semitotal dominating set, and since $\gamma_{st2}(G+H) \neq 2$, $\gamma_{st2}(G+H) = |S| = 3$. Similarly, if property (iv) holds, then $\gamma_{st2}(G+H) = 3$. □

4. In the corona of graphs

The following lemma is used in the succeeding proposition.

Lemma 1. [8] *Let G be any connected graph and H any graph. Then $S \subseteq V(G \circ H)$ is a dominating set in $G \circ H$ if and only if $S \cap V(H^v + v)$ is a dominating set in $H^v + v$ for all $v \in V(G)$.*

Proposition 2. *Let G be a nontrivial connected graph and H any nontrivial graph, and $S \subseteq V(G \circ H)$. Then S is a semitotal dominating set in $G \circ H$ if and only if the following hold:*

(i) $S \cap V(H^v + v)$ *is a dominating set in* $H^v + v$ *for all* $v \in V(G)$*; and*

(ii) $|S \cap V(H^v)| \ge 2$ *for each* $v \in V(G) \setminus S$ *with* $N_G(v) \cap S = \varnothing$*;*

Proof. Suppose that S is a semitotal dominating set in $G \circ H$. Then S is a dominating set in $G \circ H$ so that property (i) follows immediately from Lemma 1. Let $v \in V(G) \setminus S$

such that $N_G(v) \cap S = \varnothing$. Then $S \cap V(H^v + v) = S \cap V(H^v)$. By property (i), $S \cap V(H^v)$ is a dominating set in $H^v + v$, and consequently in H^v. Let $u \in S \cap V(H^v)$. Since S is a semitotal dominating set in $G \circ H$, there exists $w \in S \setminus \{u\}$ such that $d_{G \circ H}(u, w) \leq 2$. Since $S \cap N_G[v] = \varnothing$, $w \in V(H^v)$. This proves that property (ii) holds.

Conversely, suppose that all properties hold for S. By property (i), S is a dominating set in $G \circ H$. Let $u \in S$, and let $v \in V(G)$ such that $u \in V(H^v + v)$. Suppose that $u \in V(H^v)$. If $v \in S$, then v is the required vertex in S for which $d_{G \circ H}(u, v) \leq 2$. Suppose that $v \notin S$. If $N_G(v) \cap S \neq \varnothing$, say $w \in N_G(v) \cap S$, then $d_{G \circ H}(u, w) = 2$. Suppose that $N_G(v) \cap S = \varnothing$. By property (ii), we may pick $w \in S \cap V(H^v) \setminus \{u\}$. Then $d_{G \circ H}(u, w) \leq 2$. Finally, suppose that $u = v$. Since G is a nontrivial connected graph, we may pick $w \in V(G)$ such that $vw \in E(G)$. Since $S \cap V(H^w + w)$ is a dominating set in $H^w + w$, $S \cap V(H^w + w) \neq \varnothing$. For any $z \in S \cap V(H^w + w)$, $d_{G \circ H}(v, z) \leq 2$. Accordingly, S is a semitotal dominating set in $G \circ H$. □

Corollary 4. *Let G be a nontrivial connected graph and H any nontrivial graph, and $S \subseteq V(G \circ H)$. Then S is a semitotal dominating set in $G \circ H$ if and only if*

$$S = A \cup [\cup_{v \in A} S_v] \cup [\cup_{u \in V(G) \setminus A} D_u],$$

where

- (i) *$A \subseteq V(G)$;*
- (ii) *$S_v \subseteq V(H^v)$ for each $v \in A$;*
- (iii) *D_u is a dominating set in H^u for each $u \in V(G) \setminus A$; and*
- (iv) *$|D_u| \geq 2$ for each $u \in V(G) \setminus A$ with $N_G(u) \cap A = \varnothing$.*

Corollary 5. *For all nontrivial connected graphs G and any graph H,*

$$\gamma_{t2}(G \circ H) = |V(G)|.$$

For nontrivial connected graphs G, $V(G)$ is a secure semitotal dominating set in the corona $G \circ K_p$ for any integer $p \geq 1$. This, together with Corollary 5, yields $\gamma_{st2}(G \circ K_p) = |V(G)|$.

In what follows, we consider $G \circ H$, where H is noncomplete.

Theorem 3. *Let G be a nontrivial connected graph and H be any noncomplete graph without isolated vertices, and let $S \subseteq V(G \circ H)$. Then S is a secure semitotal dominating set if and only if S is a semitotal dominating set in $G \circ H$ satisfying the following properties:*

- (i) *$S \cap V(H^v)$ is a secure dominating set in H^v for each $v \in V(G) \setminus S$; and*
- (ii) *$S \cap V(H^v)$ is a nearly dominating set in H^v for all $v \in S \cap V(G)$.*

Proof. For each $v \in V(G)$, we write $S_v = S \cap V(H^v)$. Suppose that S is a secure semitotal dominating set in $G \circ H$. Then S is a semitotal dominating set in $G \circ H$. Let $v \in V(G) \setminus S$. By Proposition 2, S_v is a dominating set in H^v. Let $x \in V(H^v) \setminus S_v$. Since S is a secure semitotal dominating set in $G \circ H$, there exists $y \in S \cap N_{G \circ H}(x)$ such that $S^* = (S \setminus \{y\}) \cup \{x\}$ is a semitotal dominating set in $G \circ H$. Clearly, $y \in S_v \cap N_{H^v}(x)$. Write

$$S^* = \big(\cup_{u \in V(G)\setminus\{v\}} S \cap V(H^u + u)\big) \cup (S_v \setminus \{y\}) \cup \{x\}. \tag{1}$$

Since S^* is a dominating set in $G \circ H$, $(S_v \setminus \{y\}) \cup \{x\}$ is a dominating set in H^v by Lemma 1. Thus, S_v is a secure dominating set in H^v. This proves property (i). To prove (ii), let $v \in S \cap V(G)$. If S_v is a dominating set in H^v, then we are done. Suppose that S_v is not a dominating set in H^v, and let $x \in V(H^v) \setminus N_{H^v}[S_v]$. Since S is a secure semitotal dominating set in $G \circ H$ and $x \in V(G \circ H) \setminus S$, there exists $u \in S \cap N_{G \circ H}(x)$ such that $(S \setminus \{u\}) \cup \{x\}$ is a semitotal dominating set in $G \circ H$. Necessarily, $u = v$ so that $S_v \cup \{x\}$ is a semitotal dominating set in $H^v + v$. By Theorem 2, $S_v \cup \{x\}$ is a dominating set in H^v. Thus, S_v is nearly dominating in H^v.

Conversely, suppose that all the properties hold for a semitotal dominating set S in $G \circ H$. Let $x \in V(G \circ H) \setminus S$, and let $v \in V(G)$ such that $x \in V(H^v + v)$. We consider two cases:

Case 1: $x = v$.

Pick any $y \in S_x$. Since $x \in V(G)$, $(S_x \setminus \{y\}) \cup \{x\}$ is a dominating set in $H^x + x$. Put $S^* = (S \setminus \{y\}) \cup \{x\}$. Then $S \cap V(H^u + u) = S^* \cap V(H^u + u)$ for all $u \in V(G) \setminus \{x\}$, and $S^* \cap V(H^x + x) = (S_x \setminus \{y\}) \cup \{x\}$. Thus, S^* satisfies property (i) of Proposition 2. Let $u \in V(G) \setminus S^*$ with $N_G(u) \cap S^* = \varnothing$. Then $u \neq x$ so that $u \in V(G) \setminus S$ and $N_G(u) \cap S = \varnothing$. Since S is a semitotal dominating set in $G \circ H$, $|S^*_u| = |S_u| \geq 2$. Since u is arbitrary, property (ii) of Proposition 2 holds for S^*. Thus, S^* is a semitotal dominating set in $G \circ H$.

Case 2: $x \neq v$.

In this case, $x \in V(H^v) \setminus S_v$. First, suppose that $v \notin S$. By property (i), S_v is a secure dominating set in H^v. Thus, there exists $y \in S_v \cap N_{H^v}(x)$ for which $(S_v \setminus \{y\}) \cup \{x\}$ is a dominating set in H^v, and consequently in $H^v + v$ as well. Put $S^* = (S \setminus \{y\}) \cup \{x\}$. Then $S \cap V(H^u + u) = S^* \cap V(H^u + u)$ for all $u \in V(G) \setminus \{v\}$, and $S^* \cap V(H^v + v) = (S_v \setminus \{y\}) \cup \{x\}$. Thus, S^* satisfies property (i) of Proposition 2. Let $u \in V(G) \setminus S^*$ with $N_G(u) \cap S^* = \varnothing$. Since S and S^* differ only by their respective S_v and S^*_v, $u \in V(G) \setminus S$ and $N_G(u) \cap S = \varnothing$. If $u \neq v$, then $|S^*_u| = |S_u| \geq 2$. If $u = v$, then $|S_u \setminus \{y\}| \geq 1$ so that $|S^*_u| = |\,(S_u \setminus \{y\}) \cup \{x\}| \geq 2$. This shows that S^* satisfies property (ii) of Proposition 2. Thus, S^* is a semitotal dominating set in $G \circ H$.

Next, suppose that $v \in S$. By property (ii), S_v is a nearly dominating set in H^v. Suppose that $x \in N_{H^v}[S_v]$, and $y \in S_v$ for which $xy \in E(H^v)$. Put $S^* = (S \setminus \{y\}) \cup \{x\}$. Since $v \in S^* \cap V(H^v + v)$, $S^* \cap V(H^v + v)$ is a dominating set in $H^v + v$, and S^* satisfies property (i) of Proposition 2. Let $u \in V(G) \setminus S^*$ with $N_G(u) \cap S^* = \varnothing$. Note also in here that S^* and S differ only by their respective S^*_v and S_v. Thus $u \in V(G) \setminus S$

and $N_G(u) \cap S = \varnothing$. Since $u \neq v$, $|S^*_u| = |S_u| \geq 2$, and S^* satisfies property (ii) of Proposition 2. Accordingly, S^* is a semitotal dominating set in $G \circ H$. Finally, suppose that $x \notin N_{H^v}[S_v]$. Put $S^* = (S \setminus \{v\}) \cup \{x\}$. Note in here that $S^* \cap V(H^v + v) = S_v \cup \{x\}$, which is a dominating set in H^v because S_v is a nearly dominating set in H^v. As argued previously, S^* satisfies property (i) of Proposition 2. By Lemma 1, S^* is a dominating set in $G \circ H$. Since H^v is a noncomplete graph and S_v is a nearly dominating set in H^v, $S_v \neq \varnothing$. Clearly, $d_{G \circ H}(x, y) \leq 2$ for all $y \in S^*_v \setminus \{x\$.Supposethatthereexistsz \in S^*$ such that $d_{G \circ H}(z, y) > 2$ for all $y \in S^* \setminus \{z\}$. In view of the preceding arguments, $z \in S \setminus V(H^v + v)$. Suppose that $z \in V(G)$. By property (ii), S_z is nearly dominating in H^z. But by the definition of z, $S_z = \varnothing$, which is impossible. Suppose that $z \in V(H^w)$ for some $w \in V(G)$. Then $w \notin S$. By property (i), $S \cap V(H^w)$ is a secure dominating set. Since H is a noncomplete graph, $|S \cap V(H^w)| \geq 2$, which is impossible. This shows that S^* is a semitotal dominating set in $G \circ H$. Therefore, S is a secure semitotal dominating set. □

Corollary 6. *Let G be a nontrivial connected graph and H be any noncomplete graph without isolated vertices, and let $S \subseteq V(G \circ H)$. Then S is a secure semitotal dominating set if and only if*

$$S = A \cup [\cup_{v \in A} S_v] \cup [\cup_{u \in V(G) \setminus A} D_u],$$

satisfying the following properties:

- (i) *$A \subseteq V(G)$;*
- (ii) *S_v is a nearly dominating set in H^v for each $v \in A$;*
- $(iii$ *D_u is a secure dominating set in H^v for each $u \in V(G) \setminus A$; and*
- (iv) *$|D_u| \geq 2$ for each $u \in V(G) \setminus A$ with $N_G(u) \cap A = \varnothing$.*

Corollary 7. *Let G be a nontrivial connected graph and H be any noncomplete graph without isolated vertices.*

- (i) *If $\gamma_\eta(H) = \gamma_s(H)$, then $\gamma_{st2}(G \circ H) = |V(G)|\gamma_\eta(H)$.*
- (ii) *If $\gamma_\eta(H) < \gamma_s(H)$, then $\gamma_{st2}(G \circ H) = |V(G)|\,(1 + \gamma_\eta(H))$.*

5. In the lexicographic product of grahs

Theorem 4. [2] *Let G and H be nontrivial connected graphs. Then $C = \cup_{x \in S} (\{x\} \times T_x)$ is a dominating set if and only if one of the following holds:*

- (i) *S is a total dominating set in G;*
- (ii) *S is a dominating set in G and for each $x \in S \setminus N_G(S)$, T_x is a dominating set in H.*

The next theorem follows immediately from Theorem 4.

Theorem 5. [3] *Let G and H be nontrivial connected graphs. Then $C = \cup_{x\in S}(\{x\} \times T_x)$ is a total dominating set if and only if one of the following holds:*

- (i) *S is a total dominating set in G;*
- (ii) *S is a dominating set in G and for each $x \in S \setminus N_G(S)$, T_x is a total dominating set in H.*

Theorem 6. *Let G and H be nontrivial connected graphs, and let $C = \cup_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$. Then C is a semitotal dominating set in $G[H]$ if and only if one of the following holds:*

- (i) *S is a total dominating set in G;*
- (ii) *S is semitotal dominating set in G and for each $x \in S \setminus N_G(S)$, T_x is a dominating set in H;*
- (iii) *S is a dominating set in G such that T_x is a dominating set in H for each $x \in S \setminus N_G(S)$, and $|T_x| \geq 2$ for each $x \in S \setminus N^2_G(S)$.*

Proof. By Theorem 4, each of the conditions (i), (ii) and (iii) implies that C is a dominating set in $G[H]$. If condition (i) holds, then by Theorem 5, C is a total dominating set, hence a semitotal dominating set in $G[H]$. Suppose that condition (ii) holds, and let $(x, y) \in C$. Since S is a semitotal dominating set in G, there exists $u \in S$ such that $1 \leq d_G(x, u) \leq 2$. Pick $v \in T_u$. Then $(u, v) \in C$ and $1 \leq d_{G[H]}((x, y),(u, v)) \leq 2$. Thus, C is a semitotal dominating set in $G[H]$. Suppose that condition (iii) holds, and let $(x, y) \in C$. If $x \in N^2_G(S)$, then there exists $u \in S$ such that $1 \leq d_G(x, u) \leq 2$. Pick $v \in T_u$. Then $1 \leq d_{G[H]}((x, y),(u, v)) \leq 2$. Suppose that $x \notin N^2_G(S)$. Pick $z \in T_x \setminus \{y\}$. Then $(x, z) \in C$ and $d_{G[H]}((x, y),(x, z)) \leq 2$.

Suppose that C is a semitotal dominating set in $G[H]$. Then S is a dominating set in G by Theorem 4. If S is a total dominating set in G, then (i) holds. Suppose that S is not a total dominating set in G. By Theorem 4, T_x is a dominating set in H for each $x \in S \setminus N_G(S)$. If S is a semitotal dominating set in G, then (ii) holds. Suppose that S is not a semitotal dominating set in G. Let $x \in S \setminus N^2_G(S)$, and let $u \in T_x$. Since C is a semitotal dominating set in $G[H]$, there exists $(a, b) \in C$ such such that $1 \leq d_{G[H]}((x, u),(a, b)) \leq 2$. Since $x \in S \setminus N^2_G(S)$, $a = x$ and $|T_x| \geq 2$. □

The following well-known lemma is essential for the desired results.

Lemma 2. [3] *Let G be a nontrivial connected graph and $S \subseteq V(G)$ a dominating set in G. Then*

$$\gamma_t(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|.$$

Following the usual proof also establishes the next lemma.

Lemma 3. *If G is a nontrival connected graph and $S \subseteq V(G)$ is a dominating set in G, then*

$$\gamma_{t2}(G) \leq 2|S \setminus N_G^2(S)| + |S \cap N_G^2(S)|.$$

Corollary 8. *If G and H are nontrivial connected graphs with $\gamma(H) = 1$, then $\gamma_{t2}(G[H]) = \gamma_{t2}(G)$.*

Proof. Let $v \in V(H)$ be such that $\{v\}$ is a dominating set in H. Let $S \subseteq V(G)$ be a semitotal dominating set in G. By Theorem 6, $S \times \{v\}$ is a semitotal dominating set in $G[H]$. Thus, $\gamma_{t2}(G[H]) \leq |S|$. Since S is arbitrary, $\gamma_{t2}(G[H]) \leq \gamma_{t2}(G)$.

Let $C = \cup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ be a semitotal dominating set in $G[H]$. By Theorem 6, S is a dominating set in G. If S satisfies (i) or (ii) in Theorem 6, then S is a semitotal dominating set in G, and

$$\gamma_{t2}(G) \leq |S| \leq \sum_{x \in S} |T_x| = |C|.$$

Suppose that S is not a semitotal dominating set in G. Let $S_1 = S \setminus N_G^2(S)$, $S_2 = S \cap N_G^2(S)$. By Theorem 6,

$$C = (\cup_{x \in S_1} (\{x\} \times T_x)) \cup (\cup_{x \in S_2} (\{x\} \times T_x)),$$

where $|T_x| \geq 2$ for all $x \in S_1$. Thus,

$$\begin{aligned} |C| &= \sum_{x \in S_1} |T_x| + \sum_{x \in S_2} |T_x| \\ &\geq 2|S_1| + |S_2| \\ &= 2|S \setminus N_G^2(S)| + |S \cap N_G^2(S)|. \end{aligned}$$

By Lemma 3, $\gamma_{t2}(G) \leq |C|$. Since C is arbitrary, $\gamma_{t2}(G) \leq \gamma_{t2}(G[H])$. □

Corollary 9. *Let G and H be nontrivial connected graphs with $\gamma(H) = 2$, and let $C = \cup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$. Then C is a γ_{t2}-set of $G[H]$ if and only if one of the following holds:*

- (i) *S is a γ_t-set of G and $|T_x| = 1$ for each $x \in S$;*
- (ii) *S is a semitotal dominating set in G such that $\gamma_t(G) = 2|S \setminus N_G(S)| + |S \cap N_G(S)|$. Further, $|T_x| = 1$ for each $x \in S \cap N_G(S)$ and T_x is a γ-set of H for each $x \in S \setminus N_G(S)$;*
- (iii) *S is a dominating set in G such that $\gamma_t(G) = 2|S \setminus N_G(S)| + |S \cap N_G(S)|$ and where $|T_x| = 1$ for each $x \in S \cap N_G(S)$, T_x is a γ-set of H (thus, $|T_x| = 2$) for each $x \in S \cap \left(N_G^2(S) \setminus N_G(S)\right)$, and $|T_x| = 2$ for each $x \in S \setminus N_G^2(S)$.*

Proof. Let C be a γ_{t2}-set of $G[H]$. First, suppose that S is a total dominating set in G. We claim that $|T_x| = 1$ for each $x \in S$. Suppose that $|T_x| \geq 2$ for some $x \in S$. Let

$C^* = S \times \{v\} = \cup_{x\in S} (\{x\} \times \{v\})$, where $v \in V(H)$. Then C^* is a semitotal dominating set in $G[H]$ by Theorem 6, and $|C^*| = |S| < |C|$, which is impossible since C is a γ_{t2}-set. Necessarily, S is a γ_t-set of G. In this case, (i) holds.

Next, suppose that S is a semitotal dominating set in G and T_x is a dominating set for each $x \in S \setminus N_G(S)$. Invoking Lemma 2,

$$\begin{aligned} |C| &= \sum_{x\in S\cap N_G(S)} |T_x| + \sum_{x\in S\setminus N_G(S)} |T_x| \\ &\geq 2|S \setminus N_G(S)| + |S \cap N_G(S)| \\ &\geq \gamma_t(G). \end{aligned}$$

Suppose that $\gamma_t(G) < 2|S \setminus N_G(S)| + |S \cap N_G(S)|$. Take a γ_t-set $S^* \subseteq V(G)$ of G, and let $u \in V(H)$. Then $C^* = S^* \times \{u\}$ is a semitotal dominating set in $G[H]$ with $|C^*| = \gamma_t(G) < |C|$, a contradiction. Thus

$$\gamma_t(G) = 2|S \setminus N_G(S)| + |S \cap N_G(S)|.$$

Suppose that $|T_x| \geq 2$ for some $x \in S \cap N_G(S)$ or $|T_x| \geq 3$ for some $x \in S \setminus N_G(S)$. Let $\{u, v\}$ be a γ-set of H. By Theorem 6, $C^* = ((S \setminus N_G(S)) \times \{u, v\}) \cup ((S \cap N_G(S)) \times \{u\})$ is a semitotal dominating set in $G[H]$. Moreover,

$$|C^*| = 2|S \setminus N_G(S)| + |S \cap N_G(S)| < \sum_{x\in S\setminus N_G(S)} |T_x| + \sum_{x\in S\cap N_G(S)} |T_x| = |C|,$$

a contradiction. Thus, $|T_x| = 1$ for all $x \in S \cap N_G(S)$ and $|T_x| = 2$, hence T_x is a γ-set of H, for all $x \in S \setminus N_G(S)$. In this case, (ii) holds.

Now, suppose that S is not a semitotal dominating set in G. By Theorem 6, S is a dominating set in G such that T_x is a dominating set in H for each $x \in S \setminus N_G(S)$, and $|T_x| \geq 2$ for each $x \in S \setminus N_G^2(S)$. Invoking Lemma 2 and the assumptions that $\gamma(H) = 2$,

$$\begin{aligned} |C| &= \sum_{x\in S\setminus N_G^2(S)} |T_x| + \sum_{x\in S\cap\left(N_G^2(S)\setminus N_G(S)\right)} |T_x| + \sum_{x\in S\cap N_G(S)} |T_x| \\ &\geq 2|S \setminus N_G^2(S)| + 2|S \cap \left(N_G^2(S) \setminus N_G(S)\right)| + |S \cap N_G(S)| \\ &= 2|S \setminus N_G(S)| + |S \cap N_G(S)| \\ &\geq \gamma_t(G). \end{aligned}$$

As done previously, $\gamma_t(G) = 2|S \setminus N_G(S)| + |S \cap N_G(S)|$. Suppose that $|T_x| \geq 2$ for some $x \in S \cap N_G(S)$, $|T_x| > 2$ for some $x \in S \cap \left(N_G^2(S) \setminus N_G(S)\right)$, or $|T_x| > 2$ for some $x \in S \setminus N_G^2(S)$. Let $T = \{u, v\} \subseteq V(H)$ be a γ-set of H, and put

$$C^* = ((S \setminus N_G(S)) \times T) \cup ((S \cap N_G(S)) \times \{u\}).$$

Then C^* is a semitotal dominating set in $G[H]$ by Theorem 6. However,

$$|C| > |C^*| = 2|S \setminus N_G(S)| + |S \cap N_G(S)| = \gamma_t(G),$$

a contradiction. In this case, (iii) holds.

To prove the converse, we only show the case where S satisfies (i). The other two cases may follow similar arguments. By Theorem 6, C is a semitotal dominating set in $G[H]$. Let $C^* = \cup_{x\in S^*}(\{x\}\times T_x) \subseteq V(G[H])$ be a γ_{t2}-set of $G[H]$. Then as shown in the necessity part of the proof,

$$2|S^* \setminus N_G(S^*)| + |S^* \cap N_G(S^*)| = \gamma_t(G).$$

Thus, $|C^*| \geq \gamma_t(G) = |S| = |C|$, and C is a γ_{t2}-set of $G[H]$. □

Corollary 10. *Let G and H be nontrivial connected graphs with $\gamma(H) \geq 2$, and let $C = \cup_{x\in S}(\{x\}\times T_x) \subseteq V(G[H])$. Then C is a γ_{t2}-set of $G[H])$ if and only if S is a γ_t-set of G and $|T_x| = 1$ for all $x \in S$.*

Proof. Suppose that C is a γ_{t2}-set of $G[H]$. Suppose that S is not a total dominating set in G. By Theorem 6 and Lemma 2

$$\begin{aligned}|C| &\geq 2|S\setminus N_G^2(S)| + \gamma(H)|S\cap\left(N_G^2(S)\setminus N_G(S)\right)| + |S\cap N_G(S)|\\ &> 2|S\setminus N_G(S)| + |S\cap N_G(S)|\\ &\geq \gamma_t(G).\end{aligned}$$

Now, take any γ_t-set $S^* \subseteq V(G)$ of G and any $u \in V(H)$. Then $C^* = S^*\times\{u\}$ is a semitotal dominating set in $G[H]$ by Theorem 6. Further, $|C^*| = \gamma_t(G) < |C|$, a contradiction. Therefore, S is a total dominating set in G. In view of Theorem 6, since C is a γ_{t2}-set, $|S| = \gamma_t(G)$ and $|T_x| = 1$ for all $x \in S$.

At this point, the converse is a routine. □

Combining Corollary 9 and Corollary 10 yields the following:

Corollary 11. *Let G and H be nontrivial connected graphs with $\gamma(H) \geq 2$. Then $\gamma_{t2}(G[H]) = \gamma_t(G)$.*

Theorem 7. *Let G be a nontrivial connected graph, and let $n \geq 2$. Then $C = \cup_{x\in S}(\{x\}\times T_x) \subseteq V(G[K_n])$ is a secure semitotal dominating set in $G[H]$ if and only if one of the following holds:*

(i) S is a secure semitotal dominating set in G.

(ii) S is a semitotal dominating set in G or S is a dominating set in G with $|T_u| \geq 2$ for all $u \in S\setminus N_G^2(S)$. In any case, for each $u \in V(G)\setminus S$, there exists $x \in S\cap N_G(u)$ such that either

(a) $|T_x| \geq 2$ or

(b) $|T_x| = 1$, $(S\setminus\{x\})\cup\{u\}$ is a dominating set in G, $u \in N_G^2(S\setminus\{x\})$ and $|T_z| \geq 2$ for all $z \in (S\setminus\{x\})\setminus N_G^2(S\setminus\{x\})\cup\{u\})$.

Proof. Suppose that C is a secure semitotal dominating set in $G[K_n]$. Then C is a semitotal dominating set in $G[K_n]$. If S is a secure semitotal dominating set in G, then (i) holds. Suppose that S is a not a secure semitotal dominating set in G. By Theorem 6, S is a semitotal dominating set in G or S is a dominating set in G with $|T_u| \geq 2$ for all $u \in S \setminus N_G^2(S)$. Let $u \in V(G) \setminus S$. Pick any $v \in V(K_n)$. Then there exists $(x, y) \in C$ such that $(x, y)(u, v) \in E(G[K_n])$ and $C^* = (C \setminus \{(x, y)\}) \cup \{(u, v)\}$ is a semitotal dominating set in $G[K_n]$. Write $C^* = \cup_{a \in S^*}(\{a\} \times T_a^*)$. Then S^* is a dominating set in G. If $|T_x| \geq 2$, then $(ii)(a)$ holds. Suppose that $T_x = \{y\}$. Since C^* is a semitotal dominating set, $S^* = (S \setminus \{x\}) \cup \{u\}$ is a dominating set in G, and since $|T_u^*| = 1$, $u \in N_G^2(S \setminus \{x\})$. Let $z \in (S \setminus \{x\}) \setminus N_G^2(S^*)$. Suppose that $|T_z| = 1$, say $T_z = \{w\}$. Then $d_{G[K_n]}((z, w), (a, b)) > 2$ for all $(a, b) \in C^* \setminus \{(z, w)\}$, a contradiction. This shows that $|T_z| \geq 2$, and $(ii)(b)$ holds.

Conversely, suppose that (i) holds. Since S is a semitotal dominating set, C is a semitotal dominating set in $G[K_n]$ by Theorem 6. Let $(u, v) \in V(G[K_n]) \setminus C$. Suppose that $u \in S$. Pick $a \in T_u$. Then $(u, a) \in C$ and $(u, v)(u, a) \in E(G[K_n])$. Write $(C \setminus \{(u, a)\}) \cup \{(u, v)\} = \cup_{a \in S^*}(\{a\} \times T_a^*)$. Then $S^* = S$ and, therefore, S^* is a semitotal dominating set in G. Consequently, $(C \setminus \{(u, a)\}) \cup \{(u, v)\}$ is a semitotal dominating set in $G[K_n]$. Suppose that $u \notin S$. Since S is a secure semitotal dominating set, there exists $x \in S \cap N_G(u)$ such that $(S \setminus \{x\}) \cup \{u\}$ is a semitotal dominating set in G. Pick $y \in T_x$. Then $(x, y) \in C$ and $(x, y)(u, v) \in E(G[K_n])$. Write $(C \setminus \{(x, y)\}) \cup \{(u, v)\} = \cup_{a \in S^*}(\{a\} \times T_a^*)$. Either $S^* = S \cup \{u\}$ or $S^* = (S \setminus \{x\}) \cup \{u\}$. In either case, S^* is a semitotal dominating set in G. Thus, $(C \setminus \{(x, y)\}) \cup \{(u, v)\}$ is a dominating set in $G[K_n]$. This shows that C is a secure semitotal dominating set in $G[K_n]$. Suppose that (ii) holds. By Theorem 6, C is a semitotal dominating set in $G[K_n]$. Let $(u, v) \in V(G[K_n]) \setminus C$, and let $x \in S \cap N_G(u)$ be such that $|T_x| \geq 2$. Pick $y \in T_x$. Then $(x, y) \in C$ and $(x, y)(u, v) \in E(G[K_n])$. Write $C^* = (C \setminus \{(x, y)\}) \cup \{(u, v)\} = \cup_{a \in S^*}(\{a\} \times T_a^*)$. Then $S^* = S \cup \{u\}$. If S is a semitotal dominating set in G, then so is S^* and consequently, C^* is a semitotal dominating set in $G[K_n]$. Suppose, on the other hand that S is a dominating set in G with $|T_z| \geq 2$ for all $z \in S \setminus N_G^2(S)$. Since $u \in N_G^2(S^*)$ and $S \setminus N_G^2(S^*) \subseteq S \setminus N_G^2(S)$, $|T_z^*| \geq 2$ for all $z \in S^* \setminus N_G^2(S^*)$. Thus, C^* is a semitotal dominating set in $G[K_n]$. Now, let $x \in S \cap N_G(u)$ be such that $|T_x| = 1$, $S^* = (S \setminus \{x\}) \cup \{u\}$ is a dominating set in G, and $|T_z| \geq 2$ for all $z \in (S \setminus \{x\}) \setminus N_G^2(S^*)$. Pick $y \in T_x$. Then $(x, y)(u, v) \in E(G[K_n])$ and $C^* = (C \setminus \{(x, y)\} \cup \{(u, v)\} = \cup_{a \in S^*}(\{a\} \times T_a^*)$. Since $|T_z^*| \geq 2$ for all $z \in S^* \setminus N_G^2(S^*)$, C^* is a dominating set by Theorem 6. Let $(z, w) \in C^*$. If $z \in N_G^2(S^*)$, then pick $a \in S^* \setminus \{z\}$ such that $d_G(z, a) \leq 2$. For any $b \in T_a^*$, $(a, b) \in C^*$ and $d_{G[K_n]}((z, w), (a, b)) \leq 2$. Suppose that $z \notin N_G^2(S^*)$. Then $z \neq u$ and $|T_z| \geq 2$, say $w, t \in T_z$. Then $(z, t), (z, w) \in C^*$ and $d_{G[K_n]}((z, w), (z, t)) \leq 2$. Thus C^* is a semitotal dominating set in $G[K_n]$. □

Corollary 12. *Let G be a nontrivial connected graph, and let $n \geq 2$. Then* $\gamma_{st2}(G[K_n]) \leq \min\{\gamma_{st2}(G), 2\gamma_{t2}(G)\}$.

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