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On fuzzy sets in hypersemigroups

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Abstract. The concept of a fuzzy set, introduced by Zadeh, was first applied by Rosenfeld to groups, then by Kuroki to semigroups. In the present paper it is applied to hypersemigroups, and some properties of fuzzy hypersemigroups are discussed.

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1. Introduction

This note is based on the papers by Kuroki [7–8] and its aim is to show the way we pass from fuzzy semigroups to fuzzy hypersemigroups. We show, among others, that a regular hypersemigroup is left (resp. right) duo if and only if it is fuzzy left (resp. fuzzy right) duo. In a regular hypersemigroup, every bi-ideal is a right (resp. left) ideal if and only if every fuzzy bi-ideal is a fuzzy right (resp. fuzzy left) ideal. In an intra-regular hypersemigroup S for every fuzzy ideal f of S and any $a \in S$ there exists $u \in a \circ a$ such that $f(a) = f(u)$. "Conversely" if S is an hypersemigroup such that for every fuzzy ideal f of S, any $a \in S$ and any $u \in a \circ a$, we have $f(a) = f(u)$, then S is intra-regular. If S is a left (resp. right) regular hypersemigroup, then for every fuzzy left (resp. fuzzy right) ideal f of S and any $a \in S$ there exists $u \in a \circ a$ such that $f(a) = f(u)$. "Conversely" if S is an hypersemigroup such that for any fuzzy left (resp. fuzzy right) ideal f of S, any $a \in S$ and any $u \in a \circ a$, we have $f(a) = f(u)$, then S is left (resp. right) regular. An hypergroupoid is left (resp. right) simple if and only if it is fuzzy left (resp. fuzzy right) simple and so it is simple if and only if it is fuzzy simple. The simple hypersemigroups are regular and intra-regular. The left (resp. right) simple hypersemigroups are left (resp. right) regular. Finally, in left (resp. right) simple hypersemigroups, every fuzzy bi-ideal is a fuzzy right (resp. fuzzy left) ideal. As a consequence, in a left (resp. right) simple hypersemigroup, every bi-ideal is a right (resp. left) ideal.

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2. Necessary Definitions

For the sake of completeness, we give the following definitions:

$\mathcal{P}^*(S)$ denotes the set of (all) nonempty subsets of S.

An *hypergroupoid* is a nonempty set S with an "operation"

$$\circ : S \times S \to \mathcal{P}^*(S) \mid (a,b) \to a \circ b$$

on S called *hyperoperation* (as it assigns to each couple (a,b) a nonempty subset of S). The hyperoperation "$\circ$" induces an operation on $\mathcal{P}^*(S)$ defined by

$$A * B = \bigcup_{a \in A,\, b \in B} a \circ b.$$

Clearly $a \in A, b \in B$ implies $a \circ b \subseteq A * B$; and if $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B$. Also, for any nonempty subsets A, B, C of an hypergroupoid S, $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$. Recall that $\{x\} * \{y\} = x \circ y$ for every $x, y \in S$.

A nonempty subset A of an hypergroupoid $(S, \circ)$ is called a *subgroupoid* of S is $A * A \subseteq A$; that is if $a, b \in A$, then $a \circ b \subseteq A$. A nonempty subset A of an hypergroupoid $(S, \circ)$ is called a *left ideal* of S if $S * A \subseteq A$; that is if $u \in s \circ a$ for some $s \in S$, $a \in A$, then $u \in A$. It is called a *right ideal* of S if $A * S \subseteq A$. A subset of S that is a both a right and a left ideal of S is called an *ideal* of S. An hypergroupoid S is called *left* (resp. *right*) *duo* if every left (resp. right) ideal of S is an ideal of S. It is called *duo* if it is both left and right duo. An hypergroupoid S is called *hypersemigroup* if

$$\{x\} * (y \circ z) = (x \circ y) * \{z\} \text{ for all } x, y, z \in S.$$

If is convenient and no confusion is possible, the singleton $\{x\}$ can be identified by the element x and write, for short, $x * (y \circ z) = (x \circ y) * z$ for all $x, y, z \in S$ for the associativity relation, also expressions of the form $S * x$, $x * S * x$, $a * (x \circ a)$ etc.; we will freely use both of them in the present paper.

For any nonempty subsets A, B, C of an hypersemigroup S, we have $A * (B * C) = (A * B) * C$, that is the operation "$*$" on $\mathcal{P}^*(S)$ is associative and so $(\mathcal{P}^*(S), *)$ is a semigroup. This important result that allow us to write expressions of the form $A_1 * A_2 * .....A_n$ (n natural number) without using parentheses, has been first proved in [5]. Let us give here a very easy proof of this statement: Suppose $x \in A * (B * C)$. Then $x \in a \circ v$ for some $a \in A$, $v \in B * C$ and $v \in b \circ c$ for some $b \in B$, $c \in S$. Then we have

$$x \in a \circ v \subseteq \{a\} * (b \circ c) = (a \circ b) * \{c\} \subseteq (A * B) * C$$

and so $A * (B * C) \subseteq (A * B) * C$. Similarly we get $(A * B) * C \subseteq A * (B * C)$.

A nonempty subset A of an hypersemigroup S is called a *bi-ideal* of S if $A * S * A \subseteq A$; that is, if $u \in v \circ a$ and $v \in w \circ s$ for some $a, w \in A$, $s \in S$, then $u \in A$. Because of the associativity relation of the operation "$*$", this can be also expressed as follows: if $u \in a \circ v$ and $v \in s \circ w$ for some $a, w \in A$, $s \in S$, then $u \in A$.

An hypersemigroup S is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \in (a \circ x) * \{a\}$. That is, for every $a \in S$ there exist $x \in S$ and $u \in a \circ x$ such that $a \in u \circ a$. This is equivalent to saying that $a \in a * S * a$ for every $a \in S$ or $A \subseteq A * S * A$ for every nonempty subset A of S. It is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \in (x \circ a) * (a \circ y)$ $(= \{x\} * (a \circ a) * \{y\})$. That is, for any $a \in S$ there exist $x, y \in S$, $u \in x \circ a$ and $v \in a \circ y$ such that $a \in u \circ v$. This is equivalent to saying that $a \in S * (a \circ a) * S$ for every $a \in S$ or $A \subseteq S * A * A * S$ for every nonempty subset A of S. An hypersemigroup S is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \in \{x\} * (a \circ a)$. That is, for every $a \in S$ there exist $x \in S$ and $u \in a \circ a$ such that $a \in x \circ u$. Equivalently, if $a \in S * (a \circ a)$ for every $a \in S$ or $A \subseteq S * A * A$ for any nonempty subset A of S. It is called right regular if for every $a \in S$ there exists $x \in S$ such that $a \in (a \circ a) * \{x\}$. That is, for every $a \in S$ there exist $x \in S$ and $u \in a \circ a$ such that $a \in u \circ x$. Equivalently, if $a \in (a \circ a) * S$ for any $a \in S$ or $A \subseteq A * A * S$ for any nonempty subset A of S. As $(a \circ x) * \{a\} = \{a\} * (x \circ a)$, the regularity can be also defined as follows: For every $a \in S$ there exist $x \in S$ and $u \in x \circ a$ such that $a \in a \circ u$. Of course, similar arguments for intra-regularity and for left and right regularity also hold.

Following Zadeh, if $(S, \circ)$ is an hypergroupoid, then every mapping f of S into the real closed interval $[0, 1]$ is called a *fuzzy subset* of S or a *fuzzy set* in S. For any nonempty subset A of S, the characteristic function f_A is the fuzzy subset of S defined by

$$f_A : S \to \{0,1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

A fuzzy subset f of S is called a *fuzzy subgroupoid* of S if $f(x \circ y) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$, in the sense that if $u \in x \circ y$, then $f(u) \geq \min\{f(x), f(y)\}$. A fuzzy subset f of S is called a *fuzzy left ideal* of S if $f(x \circ y) \geq f(y)$ for every $x, y \in S$, in the sense that if $u \in x \circ y$, then $f(u) \geq f(y)$; it is called a *fuzzy right ideal* of S if $f(x \circ y) \geq f(x)$ for every $x, y \in S$, in the sense that if $u \in x \circ y$, then $f(u) \geq f(x)$ [2]. By a *fuzzy ideal* of S we mean an ideal of S which is both a fuzzy left ideal and a fuzzy right ideal of S. An hypergroupoid S is called *fuzzy left* (resp. *fuzzy right*) *duo* if the fuzzy left (resp. right) ideals of S are at the same time fuzzy right (resp. left) ideals of S (that is, ideals of S). A fuzzy subset f of an hypersemigroup S is called a *fuzzy bi-ideal of S* if $f\Big((x \circ y) * \{z\}\Big) \geq \min\{f(x), f(z)\}$ for every $x, y, z \in S$, in the sense that if $u \in (x \circ y) * \{z\}$, then $f(u) \geq \min\{f(x), f(z)\}$. As $(x \circ y) * \{z\} = \{x\} * (y \circ z)$, the fuzzy bi-ideal can be also defined by $f\Big(\{x\} * (y \circ z)\Big) \geq \min\{f(x), f(z)\}$ for every $x, y, z \in S$, meaning that if $u \in \{x\} * (y \circ z)$, then $f(u) \geq \min\{f(x), f(z)\}$. One can find some further results related to this subject in [1–6].

3. Fuzzy ideals and regular fuzzy hypersemigroups

Lemma 3.1. *Let $(S, \circ)$ be an hypergroupoid. If A is a left (resp. right) ideal of S, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of S. "Conversely",* *if A is a nonempty subset of S and f_A is a fuzzy left (resp. fuzzy right) ideal of S, then A is a left (resp. right) ideal of S.*

Proof. $\Longrightarrow$. Let A be a left ideal of S and $x, y \in S$. Then $f_A(x \circ y) \geq f_A(y)$. Indeed: Let $u \in x \circ y$. If $y \in A$, then $f_A(y) = 1$, $x \circ y \subseteq S * A \subseteq A$, $f_A(u) = 1$ and so $f_A(u) \geq f_A(y)$. If $y \notin A$, then $f_A(y) = 0 \leq f_A(u)$. Thus f_A is a fuzzy left ideal of S.
$\Longleftarrow$. Let A be a nonempty subset of S such that f_A is a fuzzy left ideal of S. Then $S * A \subseteq A$. Indeed: Let $u \in S * A$. Then $u \in s \circ a$ for some $s \in S$, $a \in A$, $f_A(s \circ a) \geq f_A(a) = 1$, $f_A(u) = 1$ and so $u \in A$. Thus A is a left ideal of S. □

By Lemma 3.1, we clearly have the following lemma.

Lemma 3.2. *Let $(S, \circ)$ be an hypergroupoid. If A is an ideal of S, then the characteristic function f_A is a fuzzy ideal of S. "Conversely", if A is a nonempty subset of S such that f_A is a fuzzy ideal of S, then A is an ideal of S.*

Thus we have the following corollary.

Corollary 3.3. *A nonempty subset A of an hypergroupoid S is an ideal of S if and only if the characteristic function f_A is a fuzzy ideal of S.*

Corollary 3.3 also holds if we replace the word "ideal" by "subgroupoid" and we have the following proposition.

Proposition 3.4. *Let $(S, \circ)$ be an hypergroupoid. If A is a subgroupoid of S, then the characteristic function f_A is a fuzzy subgroupoid of S. "Conversely", if A is a nonempty subset of S and f_A is a fuzzy subgroupoid of S, then A is a subgroupoid of S.*

Proof. Let A be a subgroupoid of S and $x, y \in A$. Then $f_A(x \circ y) \geq \min\{f_A(x), f_A(y)\}$. Indeed: Let $u \in x \circ y$. Since $x \circ y \subseteq A * A \subseteq A$, we have $u \in A$, then $f_A(u) = 1$. Since $x, y \in A$, we have $f_A(x) = f_A(y) = 1$. Thus we have

$$f_A(u) = 1 \geq \min\{f_A(x), f_A(y)\}.$$

For the converse statement, let A be a nonempty subset of S such that f_A is a fuzzy subgroupoid of S. Then $A * A \subseteq A$. Indeed: Let $u \in A * A$. Then $u \in a \circ b$ for some $a, b \in A$. Since f_A is a fuzzy subgroupoid of S, we have $f_A(a \circ b) \geq \min\{f_A(a), f_A(b)\}$ and, since $u \in a \circ b$, we have $f_A(u) \geq \min\{f_A(a), f_A(b)\}$. On the other hand, since $a, b \in A$, we have $f_A(a) = f_A(b) = 1$, and then $f_A(u) \geq 1$. Since $u \in S$, we have $f_A(u) \leq 1$. Thus we have $f_A(u) = 1$, and $u \in A$. Therefore $A * A \subseteq A$ and the proof is complete. □

Corollary 3.5. *A nonempty subset A of an hypergroupoid S is a subgroupoid of S if and only if it is a fuzzy subgroupoid of S.*

Definition 3.6. An hypergroupoid S is called *left* (resp. *right*) *zero* if, for every $x, y \in S$, we have

$$x \in x \circ y \ \ (\text{resp. } y \in x \circ y).$$

Following Kuroki, a fuzzy subset f of an hypergroupoid S is said to be a *constant function* if, for any $a, b \in S$, we have $f(a) = f(b)$.

Proposition 3.7. *If an hypergroupoid S is left (resp. right) zero, then every fuzzy left (resp. fuzzy right) ideal f of S is a constant function.*

Proof. Let S be left zero, f a fuzzy left ideal of S and $a, b \in S$. Since S is left zero, for every $x, y \in S$, we have $x \in x \circ y$. So, for the elements a, b of S, we have $a \in a \circ b$ and $b \in b \circ a$. Since f is a fuzzy left ideal of S, we have $f(a \circ b) \geq f(b)$ and $f(b \circ a) \geq f(a)$. Since $a \in a \circ b$, we have $f(a) \geq f(b)$ and since $b \in b \circ a$, we have $f(b) \geq f(a)$; thus we have $f(a) = f(b)$.

Let now S be right zero, f a fuzzy right ideal of S and $a, b \in S$. Then $a \in b \circ a$, $b \in a \circ b$, $f(b \circ a) \geq f(b)$ and $f(a \circ b) \geq f(a)$. Thus we get $f(a) \geq f(b)$ and $f(b) \geq f(a)$ and so $f(a) = f(b)$. □

Proposition 3.8. *If $(S, \circ)$ is a fuzzy left (resp. fuzzy right) duo hypergroupoid, then it is left (resp. right) duo.*

Proof. Let A be a left ideal of S. Then, by Lemma 3.1, the characteristic function f_A is a fuzzy left ideal of S. By assumption, f_A is a fuzzy right ideal of S as well. Since A is a nonempty set and f_A is a fuzzy right ideal of S, again by Lemma 3.1, A is a right ideal of S and so S is left duo. For fuzzy right duo hypergroupoids the proof is analogous. □

Proposition 3.9. *Let $(S, \circ)$ be a regular hypersemigroup. If S is left (resp. right) duo, then it is fuzzy left (resp. fuzzy right) duo.*

Proof. Let S be left duo and f be a fuzzy left ideal of S. Then f is a fuzzy right ideal of S, that is $f(x \circ y) \geq f(x)$ for every $x, y \in S$. Indeed: Let $x, y \in S$ and $u \in x \circ y$. Since S is regular, we have $x \circ y \subseteq (x * S * x) * \{y\} \subseteq (S * x) * S$. Since $S * x$ is a left ideal of S, by hypothesis, it is a right ideal of S as well and so $(S * x) * S \subseteq S * x$, hence $x \circ y \subseteq S * x$ and $u \in S * x$. Then there exists $v \in S$ such that $u \in v \circ x$. Since f is a fuzzy left ideal of S, we have $f(v \circ x) \geq f(x)$ and, since $u \in v \circ x$, we have $f(u) \geq f(x)$. Thus f is a fuzzy right ideal of S and f is fuzzy left duo.

Let now S be right duo, f a fuzzy right ideal of S, $x, y \in S$ and $u \in x \circ y$. Since S is regular, we have $x \circ y \subseteq x * (y * S * y) \subseteq S * (y * S) \subseteq y * S$. Then $u \in y \circ s$ for some $s \in S$, $f(y \circ s) \geq f(y)$ and $f(u) \geq f(y)$. Thus f is a fuzzy left ideal of S. □

By Propositions 3.8 and 3.9, we have the following theorem.

Theorem 3.10. *Let S be a regular hypersemigroup. Then we have the following:*
(1) *S is left duo if and only if it is fuzzy left duo.*
(2) *S is right duo if and only if it is fuzzy right duo.*

As a consequence, a regular hypersemigroup is duo if and only if it is fuzzy duo.

Lemma 3.11. *Let $(S, \circ)$ be an hypersemigroup. If A is a bi-ideal of S, then the characteristic function f_A is a fuzzy bi-ideal of S. "Conversely", if A is a nonempty subset of S such that f_A is a fuzzy bi-ideal of S, then A is a bi-ideal of S.*

Proof. $\Longrightarrow$. Let A be a bi-ideal of S and $x, y, z \in S$. Then

$$f_A\Big((x \circ y) * z\Big) \geq \min\{f_A(x), f_A(z)\} \tag{1}$$

In fact: Let $u \in (x \circ y) * z$. If $x \in A$ and $z \in A$, then $f_A(x) = f_A(z) = 1$, $\min\{f_A(x), f_A(z)\} = 1$ and $u \in A * S * A \subseteq A$, so $u \in A$ and $f_A(u) = 1$. Thus we have $f_A(u) = 1 \geq \min\{f_A(x), f_A(z)\}$ and property (1) is satisfied. If $x \notin A$ or $z \notin A$, then $f_A(x) = 0$ or $f_A(z) = 0$, so $\min\{f_A(x), f_A(z)\} = 0$. Since $u \in S$, we have $f_A(u) \geq 0$, then $f_A(u) = 0 \geq \min\{f_A(x), f_A(z)\}$ and again property (1) holds.

$\Longleftarrow$. Let A be a nonempty subset of S and f_A be a fuzzy bi-ideal of S. Then A is a bi-ideal of S, that is $A * S * A \subseteq A$. Indeed: Let $x \in A * S * A$. Then $x \in u \circ a$ for some $u \in A * S$, $a \in A$ and $u \in v \circ s$ for some $v \in A$, $s \in S$. Then we have $x \in u \circ a \subseteq (v \circ s) * a$. Since f_A is a fuzzy bi-ideal of S, we have

$$f_A\Big((v \circ s) * a\Big) \geq \min\{f_A(v), f_A(a)\}$$

and, since $x \in (v * s) * a$, we have $f_A(x) \geq \min\{f_A(v), f_A(a)\}$. Since $v, a \in A$, we have $f_A(v) = f_A(a) = 1$. Then $f_A(x) \geq 1$, hence $f_A(x) = 1$ and so $x \in A$. Hence we obtain $A * S * A \subseteq A$ and the proof is complete. □

Proposition 3.12. *Let S be an hypersemigroup in which the fuzzy bi-ideals are fuzzy right (resp. fuzzy left) ideals. Then every bi-ideal of S is a right (resp. left) ideal of S.*

Proof. Assuming the fuzzy bi-ideals of S are fuzzy right ideals, let A be a bi-ideal of S. By Lemma 3.11, the characteristic function f_A is a fuzzy bi-ideal of S. By the assumption, f_A is a fuzzy right ideal of S. Since A is a nonempty set and f_A is a fuzzy right ideal of S, by Lemma 3.1, A is a right ideal of S. The other case can be proved similarly. □

Proposition 3.13. *Let $(S, \circ)$ be a regular hypersemigroup in which every bi-ideal is a right (resp. left) ideal. Then every fuzzy bi-ideal of S is a fuzzy right (resp. fuzzy left) ideal of S.*

Proof. Assuming the bi-ideals are right ideals, let f be a fuzzy bi-ideal of S and $x, y \in S$. Then $f(x \circ y) \geq f(x)$. In fact: Let $u \in x \circ y$. Since S is regular, we have

$$x \circ y \subseteq (x * S * x) * \{y\} \subseteq (x * S * x) * S.$$

The set $x * S * x$ is a bi-ideal of S. This is because $(x * S * x) * S * (x * S * x) \subseteq x * S * x$. By the assumption, $x * S * x$ is a right ideal of S, that is $(x * S * x) * S \subseteq x * S * x$. Then we have $u \in (x * S) * x$, so $u \in v \circ x$ for some $v \in x * S$ and $v \in x \circ w$ for some $w \in S$. Since $v \circ x \subseteq (x \circ w) * x$, we have $u \in (x \circ w) * x$. Since f is a fuzzy bi-ideal of S, we have $f\Big((x \circ w) * x\Big) \geq \min\{f(x), f(x)\} = f(x)$ and, since $u \in (x \circ w) * x$, we have $f(u) \geq f(x)$. Thus f is a fuzzy right ideal of S.

Suppose now that every bi-ideal of S is a left ideal of S and let f be a fuzzy bi-ideal of S, $x, y \in S$ and $u \in x \circ y$. Then $x \circ y \subseteq \{x\} * (y * S * y) \subseteq S * (y * S * y) \subseteq y * S * y$ and so $u \in y \circ v$ for some $v \in S * y$ and $v \in w \circ y$ for some $w \in S$. Thus we have $u \in y * (w \circ y) = (y \circ w) * y$. Since f is a fuzzy bi-ideal of S, we have $f\Big((y \circ w) * y\Big) \geq \min\{f(y), f(y)\} = f(y)$ and, since $u \in (y \circ w) * y$, we have $f(u) \geq f(y)$. Thus f is a fuzzy left ideal of S and the proof is complete. □

Combining Propositions 3.12 and 3.13 we have the following theorem.

Theorem 3.14. *Let S be a regular hypersemigroup. Then we have the following:*

(1) *Every bi-ideal of S is a right ideal of S if and only if every fuzzy bi-ideal of S is a fuzzy right ideal of S.*

(2) *Every bi-ideal of S is a left ideal of S if and only if every fuzzy bi-ideal of S is a fuzzy left ideal of S.*

As a consequence, in a regular hypersemigroup every bi-ideal is an ideal if and only if every fuzzy bi-ideal is a fuzzy ideal.

4. On intra-regular and left regular fuzzy hypersemigroups

Kuroki characterized the intra-regular semigroup as a semigroup S such that $f(a) = f(a^2)$ for every fuzzy ideal f of S and every $a \in S$ [8; Theorem 4.1]. He also characterized the left (right) regular semigroup as a semigroup S such that $f(a) = f(a^2)$ for every fuzzy left (right) ideal f and any $a \in S$ [8; Theorems 5.1 and 5.2]. In this section we examine these results in case of hypersemigroups.

Denote by $I(u)$ the ideal of S generated by the element u of S and by $L(u)$ the left ideal of S generated by u. For an hypersemigroup S and any $u \in S$, we have
$I(u) = u \cup (S * u) \cup (u * S) \cup S * u * S$ and
$L(u) = u \cup (S * u)$.

Every left regular and every right regular hypersemigroup is intra-regular. In fact, if S is left regular then, for any nonempty subset A of S, we have $A \subseteq S * A * A$, then we have $A \subseteq S * (S * A * A) * A \subseteq S * A * A * S$, and so S is intra-regular. In a similar way the right regular hypersemigroups are intra-regular.

Theorem 4.1. *Let $(S, \circ)$ be an hypersemigroup. If S is intra-regular then, for every fuzzy ideal f of S and every $a \in S$, we have $f(a) = f(a \circ a)$ in the sense that there exists $u \in a \circ a$ such that $f(a) = f(u)$. "Conversely", if for any fuzzy ideal f of S and any $a \in S$ we have $f(a) = \hat{f}(a \circ a)$ in the sense that for each $u \in a \circ a$, $f(a) = f(u)$, then S is intra-regular.*

Proof. Let S be intra-regular, f a fuzzy ideal of S and $a \in S$. Since S is intra-regular, there exist $x, y \in S$ such that $a \in \Big(x * (a \circ a)\Big) * \{y\}$. Then $a \in v \circ y$ for some $v \in x * (a \circ a)$ and $v \in x \circ u$ for some $u \in a \circ a$. Since f is a fuzzy right ideal of S, we have $f(v \circ y) \geq f(v)$ and, since $a \in v \circ y$, we have $f(a) \geq f(v)$. Since f is a fuzzy left ideal of S, we have $f(x \circ u) \geq f(u)$ and, since $v \in x \circ u$, we get $f(v) \geq f(u)$. Thus we have $f(a) \geq f(u)$. Besides, from the fact that f is a left (or right) ideal of S, we have $f(a \circ a) \geq f(a)$ and since $u \in a \circ a$, we have $f(u) \geq f(a)$. Thus we have $f(a) = f(u)$.

For the "converse" statement, let $a \in S$. Take an element $u \in a \circ a$ ($a \circ a \neq \emptyset$). Since $I(u)$ is an ideal of S, by Lemma 3.2, the characteristic function $f_{I(u)}$ is a fuzzy ideal of S. By hypothesis, we have $f_{I(u)}(a) = \hat{f}_{I(u)}(a \circ a)$ and, since $u \in a \circ a$, we have

$f_{I(u)}(a) = f_{I(u)}(u)$. Since $u \in I(u)$, we have $f_{I(u)}(u) = 1$. Then we $f_{I(u)}(a) = 1$, and then

$$a \in I(u) = u \cup S * u \cup u * S \cup S * u * S.$$

If $a = u$, then $a \in \{a\} * \{a\} \subseteq \{a\} * \{a\} * \{a\} * \{a\} \subseteq S * (a \circ a) * S$.
If $a \in S * u$, then

$$\begin{aligned} a \in S * \{a\} * \{a\} &\subseteq S * \Big(S * \{a\} * \{a\}\Big) * \{a\} \subseteq S * \Big(\{a\} * \{a\}\Big) * S \\ &= S * (a \circ a) * S. \end{aligned}$$

The case $a \in u * S$ is similar.
If $a \in S * u * S$, then $a \in S * (a \circ a) * S$.
In any case, we have $a \in S * (a \circ a) * S$ and so S is intra-regular. □

Theorem 4.2. *Let $(S, \circ)$ be an hypersemigroup. If S is left regular, then for every fuzzy left ideal f of S and every $a \in S$, we have $f(a) = f(a \circ a)$ in the sense that there exists $u \in a \circ a$ such that $f(a) = f(u)$. "Conversely" if, for any fuzzy left ideal f of S and any $a \in S$ we have $f(a) = \hat{f}(a \circ a)$ in the sense that if $u \in a \circ a$, then $f(a) = f(u)$, then S is left regular.*

Proof. Let S be left regular, f a fuzzy left ideal of S and $a \in S$. Since S is left regular, there exists $x \in S$ such that $a \in x * (a \circ a)$. Then there exists $u \in a \circ a$ such that $a \in x \circ u$. Since f is a fuzzy left ideal of S, we have $f(x \circ u) \geq f(u)$ and, since $a \in x \circ u$, we have $f(a) \geq f(u)$. Again since f is a fuzzy left ideal of S, we have $f(a \circ a) \geq f(a)$ and, since $u \in a \circ a$, we have $f(u) \geq f(a)$. Thus we have $f(a) = f(u)$.

For the "converse" statement, let $a \in S$. Take an element $u \in a \circ a$ ($a \circ a \neq \emptyset$). Since $L(u)$ is a left ideal of S, by Lemma 3.1, $f_{L(u)}$ is a fuzzy left ideal of S. By hypothesis, we have $f_{L(u)}(a) = f_{L(u)}(a \circ a)$ and, since $u \in a \circ a$, we have $f_{L(u)}(a) = f_{L(u)}(u)$. Since $u \in L(u)$, we have $f_{L(u)}(u) = 1$. Then $f_{L(u)}(a) = 1$, and $a \in L(u) = u \cup S * u$. If $a = u$, then $a \in \{a\} * \{a\} \subseteq \{a\} * \{a\} * \{a\} \subseteq S * (a \circ a)$, so $a \in S * (a \circ a)$. If $a \in S * u$, then again $a \in S * (a \circ a)$. In any case, $a \in S * (a \circ a)$ holds and so S is left regular. □

In a similar way we can prove the following theorem.

Theorem 4.3. *Let S be an hypersemigroup. If S is right regular, then for every fuzzy right ideal f of S and every $a \in S$, we have $f(a) = f(a \circ a)$ in the sense that there exists $u \in a \circ a$ such that $f(a) = f(u)$. "Conversely" if, for any fuzzy right ideal f of S and any $a \in S$ we have $f(a) = \hat{f}(a \circ a)$ in the sense that if $u \in a \circ a$, then $f(a) = f(u)$, then S is right regular.*

5. On left simple hypergroupoids

An hypergroupoid $(S, \circ)$ is called *left* (resp. *right*) *simple* if S is the only left (resp. right) ideal of S, that is if M is a left (resp. right) ideal of S, then $M = S$. An hypergroupoid is called *simple* if it is both left and right simple.

Lemma 5.1. *Let S be an hypergroupoid. If $S * a = S$ for every $a \in S$, then S is left simple. "Conversely" if S is a left simple hypersemigroup, then $S * a = S$ for every $a \in S$.*

Proof. $\Longrightarrow$. Let A be a left ideal of S and $a \in S$. Take an element $b \in A$ ($A \neq \emptyset$). By hypothesis, we have $S = S * b \subseteq S * A \subseteq A$ so $A = S$, and S is left simple.
$\Longleftarrow$. Let $a \in S$. The set $S * a$ is a left ideal of S since $S * (S * a) = (S * S) * a \subseteq S * a$. Since S is left simple, we have $S * a = S$. □

Similarly the hypergroupoids in which $a * S = S$ for every $a \in S$ are right simple and in right simple hypersemigroups, for any $a \in S$, we have $a * S = S$.

Condition $S * a = S$ for every $a \in S$ is equivalent to $S * A = S$ for every nonempty subset A of S. Indeed, if $S * a = S$ for every $a \in S$ and A is a nonempty subset of S then, for an element $a \in A$, we have $S * A \supseteq S * a = S$ and so $S * A = S$. The $\Leftarrow$-part is obvious. Similarly, $a * S = S \ \forall\, a \in S$ is equivalent to $A * S = S \ \forall\, \emptyset \neq A \subseteq S$.

As a consequence, we immediately have the following corollary.

Corollary 5.2. *Let S be an hypersemigroup. The following are equivalent:*
(1) *S is left (resp. right) simple.*
(2) *$S * a = S$ (resp. $a * S = S$) for every $a \in S$.*
(3) *$S * A = S$ (resp. $A * S = S$) for every nonempty subset A of S.*

Remark 5.3. *The left (resp. right) simple hypersemigroups are left (resp. right) regular and intra-regular. The simple hypersemigroups are regular.*

Proof. Let A be a nonempty subset of S. If S is left simple then, by Corollary 5.2, we have $S * A = S$. Then $A \subseteq S = S * A = (S * A) * A = S * A * A$ and so S is left regular. Since S is left regular, it is intra-regular as well. If S is simple then, by Corollary 5.2, we have $S * A = A * S = S$. Then we have $A \subseteq S = A * S = A * (S * A) = A * S * A$ and so S is regular. □

Definition 5.4. An hypergroupoid $(S, \circ)$ is called *fuzzy left* (resp. *fuzzy right*) *simple* if every fuzzy left (resp. fuzzy right) ideal of S is a constant function. An hypergroupoid that is both left simple and right simple is called *simple*.

Theorem 5.5. *An hypergroupoid $(S, \circ)$ is left simple if and only if it is fuzzy left simple.*

Proof. $\Longrightarrow$. Let f be a fuzzy left ideal of S and $a, b \in S$. Then $f(a) = f(b)$. Indeed: Since S is left simple, we have $S * a = S$ and $S * b = S$. Then we have $b \in x \circ a$ and $a \in y \circ b$ for some $x, y \in S$. Since f is a fuzzy left ideal of S, we have $f(x \circ a) \geq f(a)$ and $f(y \circ b) \geq f(b)$. Since $b \in x \circ a$ and $a \in y \circ b$, we have $f(b) \geq f(a)$ and $f(a) \geq f(b)$ and so $f(a) = f(b)$.
$\Longleftarrow$. Let A be a left ideal of S and $b \in S$. Then $b \in A$. Indeed: By Lemma 3.1, f_A is a fuzzy left ideal of S. Since S is fuzzy left simple, f_A is a constant function, that is $f_A(x) = f_A(y)$ for every $x, y \in S$. Take an element $a \in A$. We have $f_A(a) = f_A(b)$ and $f_A(a) = 1$, so $f_A(b) = 1$ and $b \in A$. Thus we have $A = S$, and S is left simple. □

Theorem 5.5 is also true if we replace the word "left" by "right". As a consequence, the following theorem also holds.

Theorem 5.6. *An hypergroupoid $(S, \circ)$ is simple if and only if it is fuzzy simple.*

Proposition 5.7. *Let S be an hypersemigroup. Then every left ideal of S and every right ideal of S is a bi-ideal of S. In particular, if S is left (resp. right) simple, then every bi-ideal of S is a right (resp. left) ideal of S.*

Proof. Let A be a left ideal of S. Then $A*(S*A) \subseteq A*A \subseteq S*A \subseteq A$, so $A*S*A \subseteq A$ and A is a bi-ideal of S. If A is a right ideal of S, then $(A*S)*A \subseteq A*A \subseteq A*S \subseteq A$ and again A is a bi-ideal of S.

Let now S be left simple and A be a bi-ideal of S. Since A is a bi-ideal of S, we have $A*(S*A) \subseteq A$. Since S is left simple, we have $S*A = S$, then $A*S \subseteq A$ and so A is a right ideal of S. If S is right simple and A is a bi-ideal of S, then $(A*S)*A \subseteq A$ and $A*S = S$, thus $S*A \subseteq A$ and so A is a left ideal of S. $\square$

As a consequence, in an hypersemigroup, every ideal is a bi-ideal and in simple hypersemigroups the ideals and the bi-ideals coincide.

It is natural to ask if Proposition 5.7 remains true if we replace the word "left (right)" by "fuzzy left (fuzzy right)" and the word "bi-ideal" by "fuzzy bi-ideal". The answer is given in Proposition 5.10 below and Proposition 5.8 has been used in it.

Proposition 5.8. *Let $(S, \circ)$ be an hypergroupoid and A a nonempty subset of S. Then we have the following:*

(1) *If f is a fuzzy left ideal of S, then $f(A*y) \geq f(y)$ for any $y \in S$; in the sense that if $y \in S$ and $u \in A*y$, then $f(u) \geq f(y)$.*

(2) *If f is a fuzzy right ideal of S, then $f(x*A) \geq f(x)$ for any $x \in S$; in the sense that if $x \in S$ and $u \in x*A$, then $f(u) \geq f(x)$.*

Proof. (1) Let $y \in S$ and $u \in A*y$. Then $u \in a \circ y$ for some $a \in A$. Since f is a fuzzy left ideal of S, we have $f(a \circ y) \geq f(y)$ and, since $u \in a \circ y$, we have $f(u) \geq f(y)$.
(2) Let $x \in S$ and $u \in x*A$. Then $u \in x \circ a$ for some $a \in A$. Since f is a fuzzy right ideal of S, we have $f(x \circ a) \geq f(x)$ and, since $u \in x \circ a$, we have $f(u) \geq f(x)$. $\square$

Regarding the fuzzy bi-ideals, we have the following proposition.

Proposition 5.9. *Let S be an hypersemigroup. If f is a fuzzy bi-ideal of S then, for any nonempty subset B of S, we have*

$$f(a*B*a) \geq f(a)$$

in the sense that if $u \in a*B*a$, then $f(u) \geq f(a)$.

Proof. Let $u \in a*B*a$. Then $u \in v \circ a$ for some $v \in a*B$ and $v \in a \circ b$ for some $b \in B$. Then we have $u \in v \circ a \subseteq (a \circ b)*a$. Since f is a fuzzy bi-ideal of S, we have $f\Big((a \circ b)*a\Big) \geq \min\{f(a), f(a)\} = f(a)$; and since $u \in (a \circ b)*a$, we have $f(u) \geq f(a)$. $\square$

Proposition 5.10. *Let $(S, \circ)$ be an hypersemigroup. Then every fuzzy left ideal and every fuzzy right ideal of S is a fuzzy bi-ideal of S. In particular, if S is left (resp. right) simple, then every fuzzy bi-ideal of S is a fuzzy right (resp. fuzzy left) ideal of S.*

Proof. Let f be a fuzzy left ideal of S and $x, y, z \in S$. Since $x \circ y$ is a nonempty subset of S, by Proposition 5.8(1), we have $f\Big((x \circ y) * z\Big) \geq f(z) \geq \min\{f(x), f(z)\}$. Thus f is a fuzzy bi-ideal of S. If f is a fuzzy right ideal of S and $x, y, z \in S$ then, by Proposition 5.8(2), we have

$$f\Big((x \circ y) * z\Big) = f\Big(x * (y \circ z)\Big) \geq f(x) \geq \min\{f(x), f(z)\}$$

and so f is a fuzzy bi-ideal of S.

Let now S be left simple, f be a fuzzy bi-ideal of S and $a, b \in S$. Then $f(a \circ b) \geq f(a)$. Indeed: Let $u \in a \circ b$. Since S is left simple, we have $S * a = S$, then $b \in x \circ a$ for some $x \in S$ and $u \in a * (x \circ a) = (a \circ x) * a$. Since f is a fuzzy bi-ideal of S, we have $f\Big((a \circ x) * a\Big) \geq min\{f(a), f(a)\} = f(a)$ and, since $u \in (a \circ x) * a$, we have $f(u) \geq f(a)$; thus f is a fuzzy right ideal of S.

Finally, let S be right simple, f be a fuzzy bi-ideal of S and $a, b \in S$. Then $f(a \circ b) \geq f(b)$. Indeed: Let $u \in a \circ b$. Since S is right simple, we have $b * S = S$, then $a \in b \circ x$ for some $x \in S$ and $u \in (b \circ x) * b$. Since f is a fuzzy bi-ideal of S, we have $f\Big((b \circ x) * b\Big) \geq min\{f(b), f(b)\} = f(b)$ and, since $u \in (b \circ x) * b$, we have $f(u) \geq f(b)$; thus f is a fuzzy left ideal of S. □

Remark 5.11. Proposition 5.7 can be also obtained as a corollary to Proposition 5.10. In fact: If A is a left ideal of S then, by Lemma 3.1, f_A is a fuzzy left ideal of S, by Proposition 5.10, f_A is a fuzzy bi-ideal of S and, by Lemma 3.11, A is a bi-ideal of S. If S is left simple and A a bi-ideal of S, then f_A is a fuzzy bi-ideal of S then, by Proposition 5.10, f_A is a fuzzy right ideal of S and so A is a right ideal of S.

By Proposition 5.10, we have the following theorem.

Theorem 5.12. *In hypersemigroups, the fuzzy ideals are fuzzy bi-ideals and in simple hypersemigroups, the fuzzy ideals and the fuzzy bi-ideals coincide.*

Theorem 5.13. *Let S be an hypersemigroup that is both right (resp. left) regular and left (resp. right) simple. Then, for any fuzzy bi-ideal f of S and any $a \in S$, we have $f(a) = f(a \circ a)$ in the sense that there exists $u \in a \circ a$ such that $f(a) = f(u)$.*

Proof. Let S be right regular and left simple, f be a fuzzy bi-ideal of S and $a \in S$. Since S is left simple, by Proposition 5.10, f is a fuzzy right ideal of S. Since S is right regular and f is a fuzzy right ideal of S, by Theorem 4.3, we have $f(a) = f(a \circ a)$. For left regular and right simple hypersemigroups the proof is analogous. □

Problem 1. A semigroup S is called completely regular if for every $a \in S$ there exists $x \in S$ such that $a = axa$ and $ax = xa$ [9]. The completely regular semigroups are regular, left regular and right regular. In addition, a semigroup S is completely regular if and only if, for every $a \in S$, there exists $x \in S$ such that $a \in a^2xa^2$ [9; IV.1.2 Proposition]. The concept of completely regular semigroups can be naturally transferred to hypersemigroups as follows: An hypersemigroup S is called *completely regular* if for any $a \in S$ there exists $x \in S$ such that $a \in (a \circ x) * a$ and $a \circ x = x \circ a$. Kuroki has shown that a semigroup S is completely regular if and only if for every fuzzy bi-ideal f of S and every $a \in S$, we have $f(a) = f(a^2)$ [7; Theorem 4]. Is there an analogous result in case of hypersemigroups?

Problem 2. According to Kuroki [8; Theorem 3.9], for a regular semigroup S the following conditions are equivalent: (1) The set of all idempotent elements of S forms a left zero subsemigroup of S. (2) For every fuzzy left ideal f of S and any idempotent elements $a, b \in S$, we have $f(a) = f(b)$ (In fact, the implication (1) $\Rightarrow$ (2) holds in groupoids in general –this being so, the regularity does not play any role in it).

Is there something analogous if we replace the words "groupoid", "semigroup" by "hypergroupoid", "hypersemigroup"?

Problem 3. According to [8; Theorem 6.3], a semigroup S is a semilattice of left simple semigroups if and only if for any left ideal f of S and any $a, b \in S$, we have $f(a) = f(a^2)$ and $f(ab) = f(ba)$. Examine it in case of hypersemigroups.

Note. The results of the present paper in which the word "fuzzy" is not included, that is Lemma 5.1, Corollary 5.2, Remark 5.3 and Proposition 5.7 come directly from the *poe*-semigroups (that is, from ordered semigroups having a greatest element).

Analogous results to the results of the present paper hold for Γ-hypersemigroups and can be obtained by easy modification. Analogous results hold for ordered Γ-hypersemigroups as well.

References

[1] N. Kehayopulu. A characterization of regular, intra-regular, left quasi-regular and semisimple hypersemigroups in terms of fuzzy sets. *Pure Math. Appl. (PU.M.A.)* 26(1):46–56, 2017.

[2] N. Kehayopulu. Hypersemigroups and fuzzy hypersemigroups. *Eur. J. Pure Appl. Math.* 10(5):929–945, 2017.

[3] N. Kehayopulu. Left regular and intra-regular ordered hypersemigroups in terms of semiprime and fuzzy semiprime subsets. *Sci. Math. Jpn.* 80(3):295–305, 2017.

[4] N. Kehayopulu. Fuzzy sets in $\leq$-hypergroupoids. *Sci. Math. Jpn.* 80(3):307–314, 2017.

[5] N. Kehayopulu. How we pass from semigroups to hypersemigroups. *Lobachevskii J. Math.* 39(1):121–128, 2018.

[6] N. Kehayopulu. Fuzzy right (left) ideals in hypergroupoids and fuzzy bi-ideals in hypersemigroups. *Armen. J. Math.* 10, Paper no. 3 (2018), 10 pp.

[7] N. Kuroki. Fuzzy bi-ideals in semigroups. *Comment. Math. Univ. St. Pauli* XXVIII–1:17–21, 1979.

[8] N. Kuroki. On fuzzy ideals and fuzzy bi-ideals in semigroups. *Fuzzy Sets Systems* 5:203–215, 1981.

[9] M. Petrich. Introduction to semigroups. Merrill Research and Lecture Series. Charles E. Merrill Publishing Co., Columbus, Ohio, 1973. viii+198 pp.

[10] A. Rosenfeld. Fuzzy groups. *J. Math. Anal. Appl.* 35:512–517, 1971.