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The Smoothness of Schrödinger Operator With Electromagnetic Potential

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Abstract. In this paper, we prove that the Feynman-Kac Itô formula of the Schrödinger operator with electromagnetic $\Psi(t,x)$ in equation (1) in [8] which defined as

$$\Psi(t,x) = \int d\mu_x^t(\omega)\exp\left(-i\int_0^t b(\omega(s))d\omega - \frac{i}{2}\int_0^t divb\omega(s)ds - \int_0^t V(\omega(s)ds\right)\varphi(\omega(t))$$

is differentiable of the variable t, and so establish that the infinitely differentiable in a region, therefore, investigate smoothness of this function.

2010 Mathematics Subject Classifications: 35J10, 35B65

Key Words and Phrases: Schrödinger operator, electric potential, magnetic potential, smoothness.

1. Introduction

The problem of the self-adjoint operator is central in the quantum machine (the Diracvon Neumann formulation of quantum mechanics, in which physical observables such as position, momentum, angular momentum).

Kato [5] who showed on the basis of his elegant inequality that, if $V(x) \geq 0$ and $V \in L^2_{loc}$, then the Schrödinger operator is essentially self-adjoint on the set of infinitely differentiable finite functions. Nextly, Gaysinsky, Goldstein [4] they proved smoothness of the Schrödinger operator which is one important step to prove self-adjointness must be smoothness. After that, Adam Ward [1] investigated the essential self-adjointness of Schrödinger operator.

Many researchers studied self-adjoint operator were done, for example [2], [6], [7], [9].

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We consider the Schrödinger operator with electromagnetic potentials

$$H = \sum_{j=1}^{n} \frac{1}{2}(i\partial_j + b_j(x))^2 + V(x),$$

in $L^2(R^n)$ where, $b_j(X), j = 1, 2, ..., n$ and $V(x)$ are real-valued functions on R^n, $V \in L^1_{loc}(R^n)$, $b \in C^2(R^n)$, $\partial_j = \frac{\partial}{\partial x_j}$ and $i = \sqrt{-1}$.

We proved in [8] the Feynman-Kac Itô formula of the electromagnetic Schrödinger operator $\Psi(t, x)$ which define as the equation (1) in [8]

$$\Psi(t,x) = \int d\mu_x^t(\omega)\exp\left(-i\int_0^t b(\omega(s))d\omega - \frac{i}{2}\int_0^t divb\omega(s)ds - \int_0^t V(\omega(s)ds\right)\varphi(\omega(t))$$

converges and is an analytic function of the variable t.

In this work, we prove that the Feynman-Kac Itô formula of the Schrödinger operator with electromagnetic potentials $\Psi(t, x)$ in equation (1) in [8] is differentiable of the variable t, and we have $\frac{\partial}{\partial t}\Psi(t, x) = - < e^{-tH}, Hh >$. Then, we discuss the infinite differentiability of the function $\Psi(t, x)$ in $R^n \backslash A$ where the potential $V = +\infty$ on a set A. Finally, we investigate the smoothness of this function $\Psi(t, x)$.

2. Statement of the problem and the main result

In [8] we proved that $\Psi(t, x)$ converges and has an analytic extension for a variable t. Now, we prove that the smoothness to achieve this goal, we will follow the steps below.

Proposition 2.1. *If $H = \sum_{j=1}^{n} \frac{1}{2}(i\partial_j + b_j(x))^2 + V(x)$ is the Schrödinger operator defined on the interval $[\alpha, \beta]^n$ with zero boundary conditions ($V(x)$ is a continuous function defined on $[\alpha, \beta]^n$), $\phi, h \in C_0^\infty$, then $< e^{-tH}\phi, h >$ is a differentiable.*

$$\frac{\partial}{\partial t} < e^{-tH}\varphi, h >= - < e^{-tH}\varphi, Hh > . \tag{2.1}$$

Proof. Let $F(t, V)$ be the analytic extension defined in [8] as

$$F(t, V) = \int_{R^n} \Psi(t, x)h(x)dx, \tag{2.2}$$

and let

$$F_{\alpha,\beta}(t, V) = \int_{R^n} \Psi_{\alpha,\beta}(t, x)h(x)dx \tag{2.3}$$

be the same as in [8] , where

$$\Psi_{\alpha,\beta}(t, x) = \int dy \exp(-tH_{\alpha,\beta}) < x, y >, \tag{2.4}$$

we define the operator H on the interval $[\alpha,\beta]^n$ which denoted by $H_{\alpha,\beta}$ we have

$$\lim_{\substack{\alpha_n\to-\infty\\ \beta_n\to+\infty}} \|F_{\alpha_n,\beta_n}(t,V)-F(t,V)\|=0,$$

uniformly by $t\in G$, where G is compact subdomain of $\{t=\tau+i\theta,\tau\geq\tau_0>0\}$. By the Weierstrass theorem

$$\lim_{\substack{\alpha_n\to-\infty\\ \beta_n\to+\infty}} \left\|\frac{\partial F_{\alpha_n,\beta_n}}{\partial t}-\frac{\partial F}{\partial t}\right\|_{L^2(R^n,dV)}=0.$$

Let $H_{\alpha,\beta}$ as above then by equations (2.3), (2.4), we have

$$\left\langle e^{-tH_{\alpha,\beta}}\varphi,h\right\rangle=F_{\alpha,\beta}(t,V).$$

Therefore,

$$\frac{\partial F_{\alpha,\beta}(t,V)}{\partial t}=-\int \Psi_{\alpha,\beta}(t,x)H_{\alpha,\beta}h(x)dx, \tag{2.5}$$

provided $supp\varphi, supph\subset(\alpha,\beta)^n, h(x)\equiv0$ in the neighborhood of the center $x=0$. Since $H_{\alpha,\beta}h=Hh$, then the right side of (2.5) represents a value of form $F_{\alpha,\beta}(t,V)$, but only for function

$$Hh=\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j(x))^2h+V(x)h.$$

According to the estimates for such functions, we may pass to the limit as $\alpha\to-\infty,\beta\to+\infty$.

We observe that we can determine the functions $\Psi(t,x)$ if the potentials V,b are equal to $+\infty$ on a set A that might have a positive measure

$$\mu\{s:V(\omega(s))=+\infty,b(\omega(s))=+\infty\}>0,$$

we set

$$\exp\left(-\int_0^t V(\omega(s))ds\right)=0,\exp\left(-\int_0^t -ib(\omega(s))ds\right)=0,\exp\left(-\int_0^t \frac{-i}{2}divb(\omega(s))ds\right)=0.$$

Then the function $\Psi(t,x)$ satisfies the equation

$$\frac{\partial}{\partial t}\Psi(t,x)\varphi=\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j(x))^2\Psi(t,x)\varphi+V(x)\Psi(t,x)\varphi.$$

Since $\Psi(t,x)$ is analytical with respect to tha variable t, we prove that $\Psi(t,x)$ is a smooth function for almost every V,b where $x\in R^n\backslash A$.

Proposition 2.2. *Let* $V \in L^2(R^n \backslash A), \varphi, h \in C_0^\infty$ *where* A *is closed set,* $V(x) = +\infty, b_j(x) = +\infty, x \in A$ *such that* $supp\varphi \cap A = \emptyset$. *Then* $\Psi(t,x)$ *is an infinitely differentiable function of the variable* x *for almost every* V, b *for* $Ret \geq \tau_0 > 0$.

Proof. From equation (2.5) and definition of $F(t,V)$ in equation (2.1)

$$\frac{\partial}{\partial t}\int E(\Psi(t,x))h(x) = -\int E(\Psi(t,x))\left[\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j + b_j\right)^2 h(x) + V(x)h(x)\right]dx. \quad (2.6)$$

Let $\theta(V) \in L^2(R^n, dv)$, we put

$$f(t) = \int_{R^n} E\left(\Psi(t,x)\theta(V)\right)h(x)dx = E(F(t,V)\theta(V)).$$

depending on above that $f(t)$ is an analytic function and

$$\begin{aligned}\frac{\partial f}{\partial t} &= E\left(\frac{\partial F(t,x)}{\partial t}\theta(V)\right)\\ &= -\int_{R^n} E\left\{\Psi(t,x)\left[\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j + b_j\right)^2 h(x) + V(x)h(x)\right]\theta(V)\right\}dx\\ &= -\int_{R^n} E\left(\Psi(t,x)\theta(V)\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j + b_j\right)^2\right)h(x)dx - \int_{R^n} E\left(\Psi(t,x)V(x)\theta(V)\right)h(x)dx.\end{aligned}$$

On the other hand, $f(t) = \int_{R^n} f(t,x)h(x)dx$, where $f(t,x) = E(\Psi(t,x)\theta(V))$.
We have

$$\begin{aligned}|f(t,x)|^2 &\leq E(\Psi(t,x)^2)E(\theta(V)^2)\\ &= \|\theta\|^2_{L^2(R^n,dV)} \times E(\Psi(t,x)^2).\end{aligned}$$

Therefore,

$$\int_{R^n} f(t,x)^2 dx \leq const\,\|\theta\|^2_{L^2(dV)}$$

i.e. $f(t,x) \in L^2(R^n, dV)$. Further,

$$\begin{aligned}\left|\int_{R^n} E(\Psi(t,x)V(x)\theta(V))\right| h(x)dx &\leq \left(\int_{R^n} h(x)^2 dx\right)^{\frac{1}{2}}\left(\int_{R^n} E(\Psi(t,x)V(x)\theta(V))^2 dx\right)^{\frac{1}{2}}\\ &\leq \|h\|_{L^2(R^n,dx)}\left(\int_{R^n} E(\Psi^2(t,x)V^2(x))E(\theta^2(V(x)))dx\right)^{\frac{1}{2}}\\ &\leq \|h\|_{L^2(R^n,dx)}\,\|\theta\|_{L^2(R^n,dV)}\left(\int_{R^n} E(\Psi^2(t,x)V^2(x))dx\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq const \left\| h \right\|_{L^2(R^n,dx)} \left\| \theta \right\|_{L^2(R^n,dV)},$$

where we have used the estimates

$$\int_{R^n} E(\Psi(t,x)^2 \left| V \right|^m) dx \leq const, \tag{2.7}$$

where $m = 1, 2, ...$ and the constant depends on m. We estimate the value $\frac{\partial f}{\partial t}$with the help Cauchy Schwartz inequality for drivatives of analytic function:

$$\left| \frac{\partial f}{\partial t} \right| \leq const. \max_{|z-t|} \left| f(z) \right|.$$

Further,

$$\begin{aligned} |f(z)| &= |E(F(z,V)\theta(V))| \\ &\leq \left\| \theta \right\|_{L^2(R^n,dV)} E(|F(z,V)|^2)^{\frac{1}{2}} \\ &\leq const \left\| \theta \right\|_{L^2(R^n,dV)} \left\| h \right\|_{L^2(R^n,dx)}. \end{aligned}$$

Thus,

$$\left| \int_{R^n} f(t,x) \sum_{j=1}^{n} \frac{1}{2} \left(i\partial_j + b_j(x) \right)^2 h(x) dx \right| \leq const \left\| \theta \right\|_{L^2(R^n,dV)} \left\| h \right\|_{L^2(R^n,dx)}.$$

One may check in just same way that if $h_1(x), ..., h_p(x), \theta_1(V), ..., \theta_p(V)$ and constats $C_{k,l}, k, l = 1, ..., p$ are given, then

$$\left| E\left(\int_{R^n} \Psi(t,x) \sum_{k,l=1}^{p} C_{k,l} \sum_{j=1}^{n} \frac{1}{2} \left(i\partial_j + b_j(x) \right)^2 h_k(x)\theta_l(V) dx \right) \right| \leq const \left\| \sum_{k,l=1}^{p} C_{k,l} h_k(x)\theta_l(V) \right\|_{L^2(R^n,dx,dV)},$$

the left side equal to

$$\left| E\left(\int_{R^n} \Psi(t,x) \sum_{k,l=1}^{p} C_{k,l} \sum_{j=1}^{n} \frac{1}{2} \left(\partial_j^2 + i\partial_j b_j(x) + ib_j(x)\partial_j + b_j^2(x) \right) h_k(x)\theta_l(V) dx \right) \right|,$$

we pass at the left side to Fourier transformation by the variable x.

We get the following expression:-

$$\left| E\left(\int_{R^n} \hat{\Psi}(t,q) \sum_{k,l=1}^{p} C_{k,l} \sum_{j=1}^{n} \frac{1}{2} \left(\partial_j^2 + i\partial_j \hat{b_j}(q) + i\hat{b_j}(q)\partial_j + \hat{b_j}^2(q) \right) \hat{h_k}(q)\theta_l(V) dq \right) \right|,$$

which equal to

$$\left| E\left(\int_{R^n} \hat{\Psi}(t,q) \sum_{k,l=1}^{p} C_{k,l} \left(\sum_{j=1}^{n} \frac{-1}{2} |q|^2 + \sum_{j=1}^{n} \frac{i}{2} \partial_j \hat{b_j}(q) + \sum_{j=1}^{n} \frac{i}{2} \hat{b_j}(q) \left| -qi \right| + \sum_{j=1}^{n} \frac{1}{2} \hat{b_j}^2(q) \right) \hat{h_k}(q)\theta_l(V) dq \right) \right|$$

$$= \left\langle \hat{\Psi}(t,q), \sum_{k,l=1}^{p} C_{k,l} \left(\sum_{j=1}^{n} \frac{-1}{2} |q|^2 + \sum_{j=1}^{n} \frac{i}{2} \partial_j \hat{b_j}(q) + \sum_{j=1}^{n} \frac{i}{2} \hat{b_j}(q) |-qi| + \sum_{j=1}^{n} \frac{1}{2} \hat{b_j}^2(q) \right) \hat{h_k}(q) \theta_l(V) \right\rangle_{L^2(R^n,dq,dV)}$$

$$= \left\langle \hat{\Psi}(t,q), \sum_{k,l=1}^{p} C_{k,l} \left(\frac{-n}{2} |q|^2 + \sum_{j=1}^{n} \frac{i}{2} \partial_j \hat{b_j}(q) + \sum_{j=1}^{n} \frac{i}{2} \hat{b_j}(q) |q| + \sum_{j=1}^{n} \frac{1}{2} \hat{b_j}^2(q) \right) \hat{h_k}(q) \theta_l(V) \right\rangle_{L^2(R^n,dq,dV)}$$

$$= \left\langle \hat{\Psi}(t,q) \left(\frac{-n}{2} |q|^2 + \sum_{j=1}^{n} \frac{i}{2} \partial_j \hat{b_j}(q) + \sum_{j=1}^{n} \frac{i}{2} \hat{b_j}(q) |q| + \sum_{j=1}^{n} \frac{1}{2} \hat{b_j}^2(q) \right), \sum_{k,l=1}^{p} C_{k,l} \hat{h_k}(q) \theta_l(V) \right\rangle_{L^2(R^n,dq,dV)}.$$

The right side after the passage to the Fourier transform gains the form

$$\left\| \sum_{k,l=1}^{p} C_{k,l} \hat{h_k}(q) \theta_l(V) \right\|_{L^2(R^n,dq,dV)}.$$

From this we get

$$\left\| \left(\frac{-n}{2} |q|^2 + \sum_{j=1}^{n} \frac{i}{2} \partial_j \hat{b_j}(q) + \sum_{j=1}^{n} \frac{i}{2} \hat{b_j}(q) |q| + \sum_{j=1}^{n} \frac{1}{2} \hat{b_j}^2(q) \right) \hat{\Psi}(t,q) \right\|_{L^2(R^n,dq,dV)} \leq const. \tag{2.8}$$

In particular

$$\int_{R^n} \left(\frac{-n}{2} |q|^2 + \sum_{j=1}^{n} \frac{i}{2} \partial_j \hat{b_j}(q) + \sum_{j=1}^{n} \frac{i}{2} \hat{b_j}(q) |q| + \sum_{j=1}^{n} \frac{1}{2} \hat{b_j}^2(q) \right)^2 \left| \hat{\Psi}(t,q) \right|^2 dq < +\infty, \tag{2.9}$$

for almost every V and b, i.e. $\Psi(t,x) \in W_1$ for almost every V and b. Besides, $\|\Psi\|^2_{W_1}$ is an integrable function of V.

Further, in just the same way as it was as done while deducing (2.6) one can show that

$$\frac{\partial^2}{\partial t^2} \int E(\Psi(t,x)) h(x) dx = \int E(\Psi(t,x)) \left[\left(\sum_{j=1}^{n} \frac{1}{2} (i\partial_j + b_j(x))^2 + V(x) \right)^2 h(x) \right] dx.$$

$$\frac{\partial^2}{\partial t^2} \int E(\Psi(t,x)) h(x) dx = \int E(\Psi(t,x)) [\left(\sum_{j=1}^{n} \frac{1}{2} (i\partial_j + b_j(x))^2 \right)^2 h(x) + \sum_{j=1}^{n} \frac{1}{2} (i\partial_j + b_j(x))^2 V(x) h(x)$$
$$+ V(x) \sum_{j=1}^{n} \frac{1}{2} (i\partial_j + b_j(x))^2 h(x) + V^2(x) h(x)] dx.$$

Let us multiply the above equation by $\theta(V)$ and integrate over dV, extract the expression containing $\left(\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2\right)^2 h(x)$ and estimate the other terms in this equation.

The term

$$\frac{\partial^2}{\partial t^2}\int E(\Psi(t,x))h(x)\theta(V)dx$$

is estimated in just the same way as in the case of the first derivative, i.e. with the help of the Cauchy integral formula.

The term

$$\int E(\Psi(t,x)h(x)V^2(x))dx$$

admits application of the estimates see equation (2.7). Let us write

$$\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 V(x)h(x)+V(x)\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 h(x)$$

$$=\sum_{j=1}^{n}-\frac{1}{2}\partial_j^2(V(x)h(x))+\sum_{j=1}^{n}\frac{i}{2}\partial_j b_j(x)(V(x)h(x))+\sum_{j=1}^{n}\frac{i}{2}b_j(x)\partial_j(V(x)h(x))+\sum_{j=1}^{n}\frac{1}{2}b_j^2(x)(V(x)h(x))$$

$$+V(x)\sum_{j=1}^{n}-\frac{1}{2}\partial_j^2(h(x))+V(x)\sum_{j=1}^{n}\frac{i}{2}\partial_j b_j(x)(h(x))+V(x)\sum_{j=1}^{n}\frac{i}{2}b_j(x)\partial_j(h(x))+V(x)\sum_{j=1}^{n}\frac{1}{2}b_j^2(x)(h(x))$$

$$=\sum_{j=1}^{n}-\frac{1}{2}\partial_j^2(V(x))h(x)+\sum_{j=1}^{n}-\frac{1}{2}V(x)\partial_j^2(h(x))+2\sum_{j=1}^{n}\left(-\frac{1}{2}\partial_(j)(V(x))\right)\left(-\frac{1}{2}\partial_(j)(h(x))\right)$$

$$+\sum_{j=1}^{n}\frac{i}{2}\partial_j b_j(x)(V(x)h(x))+\sum_{j=1}^{n}\frac{i}{2}b_j(x)\partial_j(V(x))h(x)+\sum_{j=1}^{n}\frac{i}{2}b_j(x)V(x)\partial_j(h(x))+\sum_{j=1}^{n}\frac{1}{2}b_j^2(x)(V(x)h(x))$$

$$+V(x)\sum_{j=1}^{n}-\frac{1}{2}\partial_j^2(h(x))+V(x)\sum_{j=1}^{n}\frac{i}{2}\partial_j b_j(x)(h(x))+V(x)\sum_{j=1}^{n}\frac{i}{2}b_j(x)\partial_j(h(x))+V(x)\sum_{j=1}^{n}\frac{1}{2}b_j^2(x)(h(x))$$

$$=\sum_{j=1}^{n}-\frac{1}{2}\partial_j^2(V(x))h(x)+\sum_{j=1}^{n}-V(x)\partial_j^2(h(x))+2\sum_{j=1}^{n}\left(-\frac{1}{2}\partial_(j)(V(x))\right)\left(-\frac{1}{2}\partial_(j)(h(x))\right)$$

$$+\sum_{j=1}^{n}i\partial_j b_j(x)(V(x)h(x))+\sum_{j=1}^{n}\frac{i}{2}b_j(x)\partial_j(V(x))h(x)+\sum_{j=1}^{n}ib_j(x)V(x)\partial_j(h(x))+\sum_{j=1}^{n}b_j^2(x)(V(x)h(x)).$$

According to the definition of the potential $V(x)$,

$$\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 V(x)=\xi_{j-1,m}\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 V_{j-1,m}\left(x-\left(a_{j-1,m1},a_{j-1,m2},...,a_{j-1,mn}\right)\right)$$

$$+\xi_{j,m}\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 V_{j,m}\left(x-\left(a_{j,m1},a_{j,m2},...,a_{j,mn}\right)\right)$$

for $x\in\Pi_{d=1}^{n}[a_{j,md},a_{j-1,md})$ hence the term

$$\int E(\Psi(t,x)\theta(V)h(x)\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 V(x))dx$$

also admits application of the estimates see equation (2.7). Let us pass in the expression

$$\int E(\Psi(t,x)V(x)\theta(V)\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 h(x))dx$$

$$E(\Psi(t,x)\sum_{j=1}^{n}-\frac{1}{2}\partial_j(V(x))\theta(V)\sum_{j=1}^{n}-\frac{1}{2}\partial_j(h(x)))dx$$

to the Fourier transform over the variable x. Consider, for example, the second one. It will have the following form:-

$$E\left(\int_{R^n}\hat{\Psi}(t,q)\hat{h}(q)\hat{V}(q)\left(\sum_{j=1}^{n}\frac{-1}{2}|q|^2+\frac{1}{2}\sum_{j=1}^{n}i\partial_j\hat{b}_j(q)+\frac{1}{2}\sum_{j=1}^{n}i\hat{b}_j(q)|-iq|+\frac{1}{2}\sum_{j=1}^{n}\hat{b}_j^2(q)\right)dq\theta(V)\right)$$

and its absolute value is less or equal to the expression

$$const\left[E\left(\int_{R^n}|\hat{\Psi}(t,q)|^2\right)\times\left(\sum_{j=1}^{n}\frac{-1}{2}|q|^2+\frac{1}{2}\sum_{j=1}^{n}i\partial_j\hat{b}_j(q)+\frac{1}{2}\sum_{j=1}^{n}i\hat{b}_j(q)|-iq|+\frac{1}{2}\sum_{j=1}^{n}\hat{b}_j^2(q)\right)^2 dq\right]$$

$$\left\|\hat{h}(q)\right\|_{L^2(dx)}\|\theta(V)\|_{L^2(dV)}.$$

Now we get from the above estimate,

$$\int_{R^n}E\left(\Psi(t,x)\left(\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2\right)^2 h(x)\theta(V)\right)dx\le const\,\|h\|_{L^2(R^n,dx)}\|\theta\|_{L^2(R^n,dV)}.$$

We apply also calculations to a random variable of the form $\sum_{k,l=1}^{p}C_{k,l}h_k(x)\theta_l(V)$.

We get the estimate of the form (2.7) where $\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2 h_k(x)$ is replaced by $\left(\sum_{j=1}^{n}\frac{1}{2}\left(i\partial_j+b_j(x)\right)^2\right)^2 h_k(x)$. From this, we get the estimates (2.8) and (2.9) where q is replaced by q^2. Thus $\Psi(t,x)\in W_2$ for almost every V,b. Besides, $\|\Psi\|_{W_2}^2$ is an integrable function of V.

We can continue this arguments. As a result, we get that $\Psi(t,x) \in W_m$ for all $m = 1, 2, ...,$ and for almost every V and $\|\Psi\|^2_{W_m}$ is an integrable function of V. Therefore, $\Psi(t,x)$ is an infinitely differentiable function of the variable x for almost every V. In addition, the function Ψ satisfies, in the classical sense, the following differential equation

$$\frac{\partial \Psi}{\partial t} = \sum_{j=1}^{n} \frac{1}{2} \left(i\partial_j + b_j(x)\right)^2 \Psi(t,x) - V(x)\Psi(t,x)$$

for almost every V.

Let us consider an initial condition which is satisfied by the function Ψ. Since $\Psi(t,x)$ is defined for $t > 0$, we have to find $lim_{t\to 0}\Psi(t,x)$. We record

$$\Psi(t,x) = \int_{R^n} dy \left(\int d\mu^t_x(\omega) \exp\left(-i\int_0^t b(\omega(s))d\omega - \frac{i}{2}\int_0^t divb(\omega(s))ds - \int_0^t V(\omega(s))ds\right)\right)\varphi(y),$$

where the integral converges for almost every V, b, x.

$$\int_{R^n} E\left(\Psi(t,x) - \varphi(x)\right)^2 dx = \int_{R^n} dx E(\int_{R^n} dy(\int d\mu^t_x(\omega) \exp(-i\int_0^t b(\omega(s))d\omega - \frac{i}{2}\int_0^t divb(\omega(s))ds -$$

$$\int_0^t V(\omega(s))ds))\phi(\omega(t)) - \int p(x,y,t)\varphi(x)dy)^2$$

$$= \int_{R^n} dx \int_{R^n}\int_{R^n} dydz \int\int d\mu^t_{x,y}(\omega)d\mu^t_{x,z}(\eta))E(\exp(-i\int_0^t b(\omega(s))d\omega - \frac{i}{2}\int_0^t divb(\omega(s))ds -$$

$$\int_0^t V(\omega(s))ds))\varphi(y) - \varphi(x)(\exp(-i\int_0^t b(\eta(s))d\eta - \frac{i}{2}\int_0^t divb(\eta(s))ds - \int_0^t V(\eta(s))ds)\varphi(z) - \varphi(x)),$$

Now

$$\exp\left(-i\int_0^t b(\gamma(s))d\gamma - \frac{i}{2}\int_0^t divb(\gamma(s))ds - \int_0^t V(\gamma(s))ds\right)$$
$$= 1 - \int_0^1 \left(-i\int_0^t b(\gamma(s))d\gamma - \frac{i}{2}\int_0^t divb(\gamma(s))ds - \int_0^t V(\gamma(s))ds\right)$$
$$.\exp\alpha\left(-i\int_0^t b(\gamma(s))d\gamma - \frac{i}{2}\int_0^t divb(\gamma(s))ds - \int_0^t V(\gamma(s))ds\right)d\alpha \qquad (2.10)$$

We get from (2.10) and from the estimates in (proposition 2.1 in [8]).

$$|E[\int dy \int d\mu^t_{x,y}(\omega)\left[\exp\left(-i\int_0^t b(\omega(s))d\omega - \frac{i}{2}\int_0^t divb(\omega(s))ds - \int_0^t V(\omega(s))ds\right)(\varphi(y) - \varphi(x))\right]$$

$$.[\int dz \int d\mu^t_{x,z}(\omega)\left[\exp\left(-i\int_0^t b(\omega(s))d\omega - \frac{i}{2}\int_0^t divb(\omega(s))ds - \int_0^t V(\omega(s))ds\right)(\varphi(z)-\varphi(x))\right]|$$

$$\leq (\int p(x,y,t)\varphi(y)dy - \varphi(x))(\int p(x,z,t)\varphi(z)dz - \varphi(x)) + const|\int p(x,y,t)\varphi(y)dy - \varphi(x)|t$$

$$+const|(\int p(x,z,t)\varphi(z)dz - \varphi(x))|t + const.t^2$$

since $\varphi(x) \in C_0^\infty$, we have

$$\int_{R^n} E(\Psi(t,x) - \varphi(x))^2 dx \to 0.$$

Thus $\Psi(t,x)$ satisfies the equation $\frac{\partial\Psi}{\partial t} = \sum_{j=1}^n \frac{1}{2}\left(i\partial_j + b_j(x)\right)^2 \Psi(t,x) - V(x)\Psi(t,x)$ with the initial condition $\Psi(t,x) \to \varphi(x)$ in $L^2(R^n, dx, dV)$ as $t \to 0$. Repeating the same estimates, we can show that $\tilde{\Psi}_1(t,x) \to H\varphi(x)$ in $L^2(R^n, dx, dV)$ as $t \to 0$ where $\tilde{\Psi}$ corresponds $H\varphi(x)$, and also $\tilde{\Psi}_m(t,x) \to H^m\varphi(x)$ in $L^2(R^n, dx, dV)$ as $t \to 0$ where $\tilde{\Psi}$ corresponds $H^m\varphi(x)$.

First, we note that since the function $\Psi(t,x)$ satisfies the estimates equation (2.7) and, by lemma (3.1) in [8], may be analytically extended into the mentioned band, then we can repeat literally all the arguments of this section for $x \in R^n \backslash A$ and show that the function $\Psi(t,x)$ is infinitely differentiable if $x \in R^n \backslash A$.

We now consider the case $x \in A$. We assume, that the function $V(x)$ in a neighborhood of $x \in A$ satisfies the following requirements considered in the work of M.D. Gaysinsky (see[3],p.23):

(I) There exists $\epsilon > 0, \delta > 0, k, N$ are some constant such that $0 < V(x) - d(x,A)^{-2-\epsilon} < kd(x,A)^{-N}$, if $0 < d(x,A) < \delta$; where $x \in R^n \backslash A$, $d(x,A)$ is the distance between x and closed set A.

(II) For each $\alpha = (\alpha_1, \alpha_2, , \alpha_n)$ there exists $\delta_\alpha > 0$ such that
$|\frac{\partial^\alpha}{\partial x^\alpha} V(x)| = O(d(x,A)^{-k_\alpha}$, if $0 < d(x,A) < \delta_\alpha$ where k_α are some constants.

We will show now that, in this case, $\Psi(t,x)$ is infinitely differentiable at zero for almost every V, b, and also estimate the derivations of $\Psi(t,x)$ in a neighborhood of $x \in A$ if the support of the function $\varphi(x)$ is disjoint with the closed set A. First, we show that $\Psi(t,x)$ fast decreases as x approach to the set A.

Proposition 2.3. *Let $V \in L^2(R^n \backslash A), \varphi, h \in C_0^\infty$ where A is closed set, $V(x) = +\infty, b_j(x) = +\infty$, if $x \in A$, $0 < V(x) - d(x,A)^{-2-\epsilon} < Kd(x,A)^{-N}$, if $0 < d(x,A) < \delta$; where $x \in R^n \backslash A$, $d(x,A)$ is the distance between x and closed set A, $\epsilon > 0, \delta > 0, k, N$ are some constant,and let for each $\alpha = (\alpha_1, \alpha_2, , \alpha_n)$ there exists $\delta_\alpha > 0$ such that*
$|\frac{\partial^\alpha}{\partial x^\alpha} V(x)| = O(d(x,A)^{-k_\alpha}$, if $0 < d(x,A) < \delta_\alpha$ where k_α are some constants. Let $\varphi \in C_0^\infty$ such that $supp\varphi \cap A = \emptyset$. Then $\Psi(t,x)$ is an infinitely differentable function.

Proof. Following the works of M.D. Gaysinsky [3], we will say that a random variable $\tilde{V}(x)$ has $A-$property if

$$E|\frac{\partial^m}{\partial x^m}\tilde{V}(x)|^r \leq O(d(x,A)^{-k_{m,r}},$$

for $d(x,A) < \delta_{m,r}$ where $\delta_{m,r}, k_{m,r} > 0$ are constant, $m = 1, 2, ...$

$$E|\frac{\partial^m}{\partial x^m}\tilde{V}(x)|^r \leq \exp(C_{m,r}x^2),$$

where $C_{m,r} > 0$ is some constant, $m = 0, 1, 2, ...$.

It is evident that the derivations of a function $(\tilde{V}(x))$, having the $A-$property, also have the $A-$property; the product of functions, having the $A-$property, also has the $A-$property. We prove, by induction, that for any random function $\tilde{V}$ with $A-$property of the function $\tilde{V}\Psi(t,x)$ belongs to the Sobolev space $W_m, m = 0, 1, 2, ...$, for almost every V, b. Consider a sequence of smooth function $\lambda_\nu(x)$ such that

(a) $\lambda_\nu(x) = 0$ if $d(x,A) < \frac{1}{\nu}$ or $d(x,A) > \nu$

(b) $\lambda_\nu(x) = 1$ if $\frac{2}{\nu} < d(x,A) < \nu - 1$

(c) $\max_{|\alpha|\leq 2}\left|\frac{\partial^\alpha}{\partial x^\alpha}\lambda_\nu(x)\right| \leq M\nu^s$, where M, s are constant.

Let $M = 0$. We record

$$E|\Psi(t,x)\tilde{V}(x)|^2 \leq (E(\Psi(t,x))^4)^{\frac{1}{2}}(E(\tilde{V}(x))^4)^{\frac{1}{2}} \leq \tilde{\beta_0}, \tag{2.11}$$

where $\tilde{\beta_0}$ is some constant.

Now, it follows from (2.11) and corollary(2.2)in [8] that

$$E\left(\int_{R^n}|\Psi(t,x)\tilde{V}(x)|^2\right)dx \leq \beta_0,$$

where β_0 is a constant. We can write for any $h(x), \theta(V)$

$$E\left(\int \tilde{V}(x)\Psi(t,x)\left(\sum_{j=1}^n \frac{1}{2}(i\partial_j + b_j)^2\right)h(x)\theta(V)\right)dx$$

$$= \lim_{\nu\to\infty}\left(\int \tilde{V}(x)\Psi(t,x)\left(\sum_{j=1}^n \frac{1}{2}(i\partial_j + b_j)^2\right)\lambda_\nu h(x)\theta(V)\right)dx$$

$$= \lim_{\nu\to\infty}(\int \tilde{V}(x)\Psi(t,x)\left[\left(\sum_{j=1}^n \frac{1}{2}(i\partial_j + b_j)^2\right)(\tilde{V}\lambda_\nu h(x)) + V\tilde{V}\lambda_\nu h(x)\right] - V\tilde{V}\Psi(t,x)\lambda_\nu h(x)$$

$$-\Psi(t,x)\lambda_\nu h(x)\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j)^2\right)\tilde{V}-2\Psi(t,x)\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j(\tilde{V})\right)\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j(\lambda_\nu h)\right))\theta(V)dx.$$

Since the support of the functions $\tilde{h}(x)=\tilde{V}(x)\lambda_\nu h(x)$ is disjoint with the neighborhood of the point $x\in A$, we can repeat literally for $\tilde{h}(x)\theta(V)$ all the arguments which we have stated in the case when $V(x)$ has no singular points. We have then

$$E\left(\int\Psi(t,x)H\tilde{h}(x)\theta(V)\right)dx=E\left(\int\Psi^{(1)}(t,x)\tilde{h}(x)\theta(V)\right)dx,$$

where $\Psi^1(t,x)$ corresponds to the function $\varphi^{(1)}(x)=H\varphi(x)$. Thus, we have

$$\lim_{\nu\to\infty}E\left(\int\Psi(t,x)\left[\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j)^2\right)(\tilde{V}\lambda_\nu h(x))+V\tilde{V}\lambda_\nu h(x)\right]\theta(V)\right)dx$$

$$=\lim_{\nu\to\infty}E\left(\int\Psi^{(1)}(t,x)(\tilde{V}\lambda_\nu h(x))\theta(V)\right)dx.$$

In addition,

$$|E(\int V\tilde{V}\Psi(t,x)h(x)\theta(V))dx|\leq const\int E(V(x)^2|\Psi(t,x)|)^{\frac{1}{2}}E(\tilde{V}(x)^2|\Psi(t,x)|)^{\frac{1}{2}}dx;$$

$$E(\tilde{V}(x)^2|\Psi(t,x)|)\leq E(\tilde{V}(x)^4)^{\frac{1}{2}}E(\Psi(t,x))^2)^{\frac{1}{2}}\leq const\exp\left(-\sum_{k=0}^{l-1}(t_{k+1}-t_k)(b_k)^2\right)\exp\left(-\frac{(x-\alpha)^2}{ct}\right),$$

where $c>0$ is constants. The similar estimate is true if we replace $\tilde{V}$ by $\sum_{j=1}^{n}\frac{1}{2}\partial_j\tilde{V}$ or $\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j)^2\right)(\tilde{V})$ (we use the $A-$property).

Since $\lambda_\nu(x)$ is bounded and $\lambda_\nu(x)\neq 1$ whenever $d(x,A)<\frac{2}{\nu}$ or $d(x,A)>\nu-1$, we have

$$E(\int -V\tilde{V}\Psi(t,x)\lambda_\nu h(x)-\Psi(t,x)\lambda_\nu h(x)\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j)^2\right)\tilde{V}-2\Psi(t,x)\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j(\tilde{V})\right)$$

$$\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j(\lambda_\nu h)\right))\theta(V)dx-E(\int V\tilde{V}\Psi(t,x)h(x)-\Psi(t,x)h(x)\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j)^2\right)\tilde{V}$$

$$-2\Psi(t,x)\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j(\tilde{V})\right)\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j h\right))\theta(V)dx$$

$$\leq const\int_{\{d(x,A)<\frac{2}{\nu}\}\cup\{d(x,A)>\nu-1\}}\exp\left(-\sum_{k=0}^{l-1}(t_{k+1}-t_k)(b_k)^2\right)\exp\left(-\frac{(x-\alpha)^2}{ct}\right)dx,$$

where $c > 0$ is constants.

Therefore,

$$E\left(\int \tilde{V}(x)\Psi(t,x)\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j(x))^2\right)h(x)\theta(V)\right)dx = E\left(\int \Psi^{(1)}(t,x)\tilde{V}(x)h(x)\theta(V)\right)dx$$

$$+E(\int[V(x)\tilde{V}(x)\Psi(t,x)h(x)-\Psi(t,x)h(x)\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j(x))^2\right)\tilde{V}(x)-2\Psi(t,x)\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j(\tilde{V})\right)$$
$$\left(\sum_{j=1}^{n}-\frac{1}{2}\partial_j h\right)\theta(V)])dx.$$

Now, we act in just the same way as in the case of the potential without singularities. Namely, we make the fourier transform by the variable x at the left side. We get the following expression:

$$E(\int (\widehat{\tilde{V}(x)\Psi(t,x)})(k)\left(\sum_{j=1}^{n}\frac{-1}{2}|k|^2+\frac{1}{2}\sum_{j=1}^{n}i\partial_j\hat{b_j}(k)+\frac{1}{2}\sum_{j=1}^{n}i\hat{b_j}(k)|-ki|+\frac{1}{2}\sum_{j=1}^{n}i\hat{b_j^2}(k)\right)\tilde{h}(k)\theta(V)dk. \tag{2.12}$$

Since $V(x)\tilde{V}(x)$ and $\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j(x))^2\right)$ have the $A-$property, we have

$$\left|E\left(V(x)\tilde{V}(x)\Psi(t,x)h(x)\theta(V)\right)dx\right| \le const\,\|h(x)\theta(V)\|_{L^2((R^n,,dx,dV)};\tag{2.13}$$

$$\left|E\left(\int \Psi(t,x)\left(\sum_{j=1}^{n}\frac{1}{2}(i\partial_j+b_j(x))^2\right)\tilde{V}(x)h(x)\theta(V)\right)dx\right| \le const\,\|h(x)\theta(V)\|_{L^2((R^n,,dx,dV)}.\tag{2.14}$$

In the last term we also make the Fourier transform by x. We get the following expression:

$$2iE\left(\int\left(\widehat{\Psi(t,x)\left(\sum_{j=1}^{n}\frac{-1}{2}\partial_j((\tilde{V}))\right)}\right)(k)\tilde{h}(k)(\frac{-nk}{2})\theta(V)\right)dk. \tag{2.15}$$

Now, it follows from (2.12)-(2.15) that

$$|E(\int (\widehat{\tilde{V}(x)\Psi(t,x)})(k)\frac{\sum_{j=1}^{n}\frac{-|k|^2}{2}+\frac{i}{2}\sum_{j=1}^{n}\partial_j\hat{b_j}(k)+\frac{i}{2}\sum_{j=1}^{n}\hat{b_j}(k)|-ki|+\frac{i}{2}\sum_{j=1}^{n}\hat{b_j^2}(k)}{1+|\frac{-nk}{2}|}\tilde{h}(k)\theta(V)dk|$$

$$\leq const\|h(x)\theta(V)\|_{L^2(R^n,dx,dV)}. \quad (2.16)$$

Further the estimate (2.16) is literally transferred onto functions of the form $\sum C_{k,l}h_k(x)\theta_l(V)$. Therefore, $\tilde{V}(x)\Psi(t,x) \in W_1$ for almost every V, b

$$\|\tilde{V}(x)\Psi(t,x)\|^2_{W_1} \leq \beta_1(V),$$

where $E(\beta_1(V)) < +\infty$. Continuting these arguments of induction, we get $\tilde{V}(x)\Psi(t,x) \in W_m$ for almost every V, b

$$\|\tilde{V}(x)\Psi(t,x)\|^2_{W_m} \leq \beta_m(V), \quad (2.17)$$

where $E(\beta_m(V)) < +\infty$.

Thus, we have proved that the function $\Psi(t,x)$ for almost every V, b is infinitely differentiable for all x, in particular, at the point $x \in A$. Besides, for any function with the A-property, the estimates of the form (2.17) take place.

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