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On Weak Projectivity in Arithmetic

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Abstract. In the 19th century, non-Euclidean geometries were discovered and studied. In the 20th century, non-Diophantine arithmetics were discovered and studied. Construction of non-Diophantine arithmetics is based on very general mathematical structures, which are called abstract prearithmetics, as well as on the projectivity relation between abstract prearithmetics. In a similar way, as set theory gives a foundation for mathematics, the theory of abstract prearithmetics provides foundations for the theory of the Diophantine and non-Diophantine arithmetics. In this paper, we study relations between operations in abstract prearithmetics exploring how properties of operations in one prearithmetic impact properties of operations in another prearithmetic. In addition, we explore how to build new prearithmetics from existing ones.

Key Words and Phrases: arithmetic, prearithmetic, vector expansion, matrix expansion, addition, multiplication, projectivity, category

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1. Introduction

One of the most basic objects in mathematics is the arithmetic $\boldsymbol{N}$ of all natural numbers. People in general and mathematicians in particular think that the laws of this arithmetic are universal and unique. The formula $2 \times 2 = 4$ is regarded a perpetual unconditional truth. However, for a long time the best thinkers had reservations with respect to universality of $\boldsymbol{N}$ considering numerous situations when the rules of this arithmetic, which is called the Diophantine arithmetic, are not true (cf., for example, [6, 11, 13, 21, 22, 28, 29, 41, 42, 67, 73]).

Here we present only three of such examples although there are much more.

(i) One raindrop added to another raindrop does not make two raindrops but only one [73]. Mathematically, it is described by the equality $1 + 1 = 1$.

(ii) If one puts a lion and a rabbit in a cage, one will not find two animals in the cage later on (cf. [42, 67]). In terms of numbers, it will mean $1 + 1 = 1$.

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(iii) When a cup of milk is added to a cup of popcorn, then only one cup of mixture will result because the cup of popcorn will very nearly absorb a whole cup of milk without spillage [21]. So, in this case, we also have $1 + 1 = 1$.

In addition, recently the expression $1 + 1 = 3$ has become a very popular metaphor for synergy in a variety of areas: in business and industry (cf., for example, ([2, 33, 34, 38, 45, 58])), in economics and finance (cf., for example, [9], in psychology and sociology (cf., for example,[4, 12, 25, 26, 40, 55, 72]), library studies (cf., for example, [57]), biochemistry and bioinformatics (cf., for example, [46]), computer science (cf., for example, [24, 30, 50]) , physics (cf., for example, [49]), medicine (cf., for example, [15, 64, 71]) and pedagogy (cf., for example, [62]).

All these examples indicated existence of other non-Diophantine arithmetics, in which it would be possible to explain all these cases in a rigorous mathematical way. Some researchers predicted this (cf., for example, [29, 41, 66]. These predictions became true when the first class of non-Diophantine arithmetics was discovered and explored in 1975 although the first publication appeared in 1977 [10]. Later other classes of non-Diophantine arithmetics were constructed [7, 11]. Recently non-Diophantine arithmetics found explicit applications in physics [17, 18, 20] and psychology [19] although implicit utilization non-Diophantine arithmetics in physics and psychology existed for quite a while (cf., for example, [52–54, 56]).

Following the classical understanding of arithmetic, here non-Diophantine arithmetics are treated as arithmetics of natural numbers although it is also possible, for example, to consider non-Diophantine arithmetics of real or integer numbers.

Construction of non-Diophantine arithmetics is based on more general mathematical structures, which are called abstract prearithmetics, as well as on the projectivity relation between abstract prearithmetics [7, 10, 11]. In a similar way, as set theory forms a foundation for mathematics, the theory of abstract prearithmetics provides foundations for the theory of the Diophantine arithmetic and non-Diophantine arithmetics.

The term *arithmetic* means not only a mathematical structure but also a branch of mathematics aimed at the study of number systems with operations and relations. This allows treating abstract prearithmetics as the basic structures in the field of arithmetic. In addition, the theory of abstract prearithmetics includes theories of various conventional mathematical structures, such as rings, semirings, fields, ordered rings, ordered fields, lattices and Boolean algebras, as its subtheories. This allows using constructions from the theory of abstract prearithmetics for its subtheories of conventional mathematical structures. For instance, it is possible to study projectivity relations for rings or Boolean algebras. Abstract prearithmetics also provide a unified algebraic context for some traditional mathematical constructions, such as logarithmic scales, modular arithmetics and computer arithmetics, which are used in many applications in mathematics, science and technology.

In essence, an abstract prearithmetic is a universal algebra (algebraic system) with two binary operations and a partial order. Operations are called addition and multiplication but in a general case, there are no restrictions on these operations. Some of abstract prearithmetics are numerical, that is, their elements are numbers, e.g., natural numbers or real numbers. A numerical prearithmetic that satisfies additional conditions, in particular, containing all natural numbers and no other elements is called an arithmetic of natural numbers. A numerical prearithmetic that satisfies additional conditions, in particular, contains all integer numbers and no other elements is called an arithmetic of integer numbers. Everybody knows the conventional Diophantine arithmetic $\boldsymbol{N}$ of natural numbers. However, there are also many non-Diophantine arithmetics of natural numbers introduced and studied in [5–7, 10, 11].

An important relation between prearithmetics or arithmetics is projectivity as it is demonstrated in [5, 7, 11]. It has three types: weak projectivity, projectivity *per se* and exact projectivity. These relations allow deducing properties of one arithmetic or prearithmetic from properties of another arithmetic or prearithmetic. Besides, they are used for building new prearithmetics and arithmetics. Projectivity between two prearithmetics (arithmetics) means that both operations – addition and multiplication - of one them are expressed using the corresponding operation in the second one by means of the same function (parameter of the projectivity).

The goal of this paper is the further development of the theory of abstract prearithmetics by considering different forms of weak projectivity, in which projectivity connects separate operations, e.g., addition or multiplication. That is why it is called partial weak projectivity. It allows building new prearithmetics or arithmetics from the existing ones by changing only one operation or in a different way changing both operations – addition and multiplication – using specific parameters for each of them. In turn, we also obtain more flexible tools for finding relations between properties of prearithmetics or arithmetics connected by partial weak projectivity relations.

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2. Abstract prearithmetics

An *abstract prearithmetic* is a set (often a set of numbers) A with a partial order $\leq$ and two binary operations $+$ (addition) and $\circ$ (multiplication), which are defined for all its elements. It is denoted by $\boldsymbol{A} = (A; +, \circ, \leq)$. The set A is called the *set of elements or set of numbers* or the *carrier* of the prearithmetic $\boldsymbol{A}$. As always, if $x \leq y$ and $x \neq y$, then we denote this relation by $x < y$. Operation $+$ is called *addition* and operation $\circ$ is called *multiplication* in the abstract prearithmetic $\boldsymbol{A}$.

Note that an abstract prearithmetic can have more than two operations and more than one order relation.

Example 1. *Naturally, the conventional Diophantine arithmetic* $\boldsymbol{N}$ *of all natural numbers, the conventional arithmetic* $\boldsymbol{W}$ *of all whole numbers, the conventional arithmetic* $\boldsymbol{Z}$ *of all integer numbers, the conventional arithmetic* $\boldsymbol{Q}$ *of all rational numbers, the conventional arithmetic* $\boldsymbol{R}$ *of all real numbers and the conventional arithmetic* $\boldsymbol{C}$ *of all complex numbers are abstract prearithmetics.*

Example 2. *Another example of abstract prearithmetics is modular arithmetic, which is sometimes known as residue arithmetic or clock arithmetic [48]. It is studied in mathematics and used in physics and computing. In modular arithmetic, operations of addition and multiplication are defined but in contrast to the conventional arithmetic, its numbers form a cycle upon reaching a certain value, which called the modulus. A rigorous approach to the theory of modular arithmetic was worked out by Carl Friedrich Gauss.*

Example 3. *Many algebraic structures studied in algebra are abstract prearithmetics with a trivial order, i.e., any ring, lattice, Boolean algebra, linear algebra, field,* Ω*-group,* Ω*-ring,* Ω*-algebra [1, 48], topological ring, topological field, normed ring, normed algebra, normed field, and in essence, any universal algebra with two operations is an abstract prearithmetic with a trivial order. The same structures with nontrivial order are also abstract prearithmetics.*

Examples are given by ordered rings, ordered linear algebras and ordered fields. Besides, it is possible to treat universal algebras with one operation as abstract prearithmetics with a trivial order and trivial multiplication.

All these examples show that conventional mathematical structures are abstract prearithmetics. However, there are many unusual abstract prearithmetics.

Example 4. *Let us consider the set* N *of all natural numbers with the standard order* $\leq$*, addition* $+$*, multiplication* $\cdot$ *and introduce the following operations:*

$$a \oplus b = a \cdot b$$

$$a \otimes b = a^b$$

Then the system $\boldsymbol{A} = (N; \oplus, \otimes, \leq)$ *is an abstract prearithmetic with addition* $\oplus$ *and multiplication* $\otimes$*.*

Example 5. *Let us consider the set* R^{++} *of all positive real numbers is with the standard order* $\leq$*, addition* $+$*, multiplication* $\cdot$*, division* $\div$ *and introduce the following operations:*

$$a \boxplus b = a + b$$

$$a \circledast b = a \div b$$

Then the system $\boldsymbol{B} = (R^{++}; \boxplus, \circledast, \leq)$ *is an abstract prearithmetic with addition* $\boxplus$ *and multiplication* $\circledast$*.*

Example 6. *Semirings in general and idempotent semirings, in particular, are abstract prearithmetics with a trivial order ([47] Golan, 1999). Many researchers utilized idempotent semirings and matrices over such semirings for solving various applied problems in computer science and discrete mathematics (cf., for example, [14, 16, 32, 39, 63, 74, 75]). Idempotent semi-rings also have many other applications, in particular, as the basic structure of idempotent analysis [44, 59, 60] and of its special case tropical analysis [51, 70].*

Example 7. *Let $\boldsymbol{R}_{max}$ be the set $A = \boldsymbol{R} \cup \{-\infty\}$ with the operations $\oplus = max$ and $\odot = +$, which is the usual addition in $\boldsymbol{R}$ and defining $\boldsymbol{0} = -\infty$ and $\boldsymbol{1} = 0$. By construction, $\boldsymbol{R}_{max}$ is a commutative idempotent semi-ring and thus, a prearithmetic. It is very useful in idempotent analysis [44, 59, 60].*

Example 8. *Let $\boldsymbol{R}_{min}$ be the set $A = \boldsymbol{R} \cup \{+\infty\}$ with the operations $\oplus = min$ and $\odot = +$, which is the usual addition in $\boldsymbol{R}$ and defining $\boldsymbol{0} = +\infty$ and $\boldsymbol{1} = 0$. By construction, $\boldsymbol{R}_{min}$ is a commutative idempotent semi-ring and thus, a prearithmetic. It is also very useful in idempotent analysis [44, 59, 60].*

Example 9. *Tropical semirings [65] and subtropical algebras with max or min as multiplication (Shiozawa, 1998) are prearithmetics.*

There is a possibility to assemble different algebraic constructions similar to modules and vector spaces using abstract prearithmetics instead of rings or fields. For instance, taking an abstract prearithmetic $\boldsymbol{A} = (A; +, \circ, \leq)$ and a natural number n, it is possible to build the abstract prearithmetic of n-dimensional A-vectors $V^n\boldsymbol{A} = (V^nA; +, \circ, \leq)$, elements of which are vectors in $\boldsymbol{A}$. Namely, elements of n-dimensional A-vector prearithmetic $V^n\boldsymbol{A} = (V^nA; +, \circ, \leq)$, i.e., A-vectors, have the form $(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n$ are elements from the abstract prearithmetic $\boldsymbol{A}$. The prearithmetic $V^n\boldsymbol{A}$ is called a *vector expansion* of the abstract prearithmetic $\boldsymbol{A}$.

In a similar way, taking an abstract prearithmetic $\boldsymbol{A} = (A; +, \circ, \leq)$ and a pair of natural numbers n and m, it is also possible to build the abstract prearithmetic of $n \times m$-dimensional A-matrices $M^{n\times m}\boldsymbol{A} = (M^{n\times m}A; +, \circ, \leq)$, elements of which are matrices in $\boldsymbol{A}$. Namely, elements of $n \times m$-dimensional A-matrix prearithmetic $M^{n\times m}\boldsymbol{A} = (M^{n\times m}A; +, \circ, \leq)$, i.e., A-matrices, have the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13}\cdots & a_{1n} & \\ a_{21} & a_{22} & a_{23}\cdots & a_{2n} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3}\cdots & a_{mn} & \end{pmatrix}$$

where all $a_{ij} (i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n)$ are elements from the abstract prearithmetic $\boldsymbol{A}$. The prearithmetic $M^{n\times m}\boldsymbol{A}$ is called a *matrix expansion* of the abstract prearithmetic $\boldsymbol{A}$.

Addition and multiplication in these prearithmetics are defined coordinate-wise. For instance, taking the arithmetic $\boldsymbol{Z}$ of integer numbers and two two-dimensional Z-vectors (2, 3) and (4, 5) from the prearithmetic $V^2\boldsymbol{Z}$ of Z-vectors, we define their sum as $(2, 3) + (4, 5) = (2 + 4, 3 + 5) = (6, 8)$ and their product as $(2, 3) \circ (4, 5) = (2 \cdot 4, 3 \cdot 5) = (8, 15)$.

Order in $V^n\boldsymbol{A}$ is defined by the following condition:

If ($a_1, a_2, \ldots, a_n$) and $(b_1, b_2, \ldots, b_n)$ are vectors from V^nA, then

($a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$ if and only if $a_j \leq b_j$ for all $j = 1, 2, 3, \ldots, n$

For matrices, addition, multiplication and order are defined in a similar way.

Note that the defined multiplication is *scalar multiplication* of vectors and matrices, which is different from vector and matrix multiplication.

Prearithmetics $V^n\boldsymbol{A}$ and $M^{n\times m}\boldsymbol{A}$ preserve many properties of the abstract prearithmetic $\boldsymbol{A}$. For instance, we have the following results.

Proposition 2.1. If addition is commutative in an abstract prearithmetic $\boldsymbol{A}$, then addition is commutative in the vector prearithmetic $V^n\boldsymbol{A}$ and in the matrix prearithmetic $M^{n\times m}\boldsymbol{A}$.

Indeed, if the identity $a + b = b + a$ is true in the abstract prearithmetic $\boldsymbol{A}$, then in the vector prearithmetic $V^n\boldsymbol{A}$, we have

$(a_1, a_2, \ldots, a_n)+(b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n) = (b_1+a_1, b_2+a_2, \ldots, b_n+ a_n) = (b_1, b_2, \ldots, b_n) + (a_1, a_2, \ldots, a_n)$

It means that addition is commutative in the vector prearithmetic $V^n\boldsymbol{A}$.

Commutativity of addition in the matrix prearithmetic $M^{n\times m}\boldsymbol{A}$ is proved in a similar way.

The same is true for multiplication.

Proposition 2.2. If multiplication is commutative in an abstract prearithmetic $\boldsymbol{A}$, then multiplication is commutative in the vector prearithmetic $V^n\boldsymbol{A}$ and in the matrix prearithmetic $M^{n\times m}\boldsymbol{A}$.

Proof is similar to the proof of Proposition 2.1.

Proposition 2.3. If addition is associative in an abstract prearithmetic $\boldsymbol{A}$, then addition is associative in the vector prearithmetic $V^n\boldsymbol{A}$ and in the matrix prearithmetic $M^{n\times m}\boldsymbol{A}$.

Proof is similar to the proof of Proposition 2.1.

The same is true for multiplication.

Proposition 2.4. If multiplication is associative in an abstract prearithmetic $\boldsymbol{A}$, then multiplication is associative in the vector prearithmetic $V^n\boldsymbol{A}$ and in the matrix prearithmetic $M^{n\times m}\boldsymbol{A}$.

Proof is similar to the proof of Proposition 2.1.

In the Diophantine arithmetic $\boldsymbol{N}$, multiplication is distributive with respect to addition, i.e., the following identities hold

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

$$(y + z) \cdot x = y \cdot x + z \cdot x$$

However, in abstract prearithmetics, multiplication is not always commutative and we need to discern three kinds of distributivity. Namely, *distributivity from the left*

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

and *distributivity from the right*

$$(y+z)\cdot x = y\cdot x + z\cdot x$$

Besides, multiplication is *distributive* with respect to addition when both identities hold.

Proposition 2.5. If multiplication is distributive (distributive from the left or distributive from the right) with respect to addition in an abstract prearithmetic $\boldsymbol{A}$, then multiplication is distributive (distributive from the left or distributive from the right) with respect to addition in the vector prearithmetic $V^n\boldsymbol{A}$ and in the matrix prearithmetic $M^{n\times m}\boldsymbol{A}$.

Proof is similar to the proof of Proposition 2.1.

Remark 1. *Having an abstract prearithmetic $\boldsymbol{A} = (A; +, \circ, \leq)$, it is possible to build not only abstract prearithmetics of A-vectors and A-matrices but also abstract prearithmetics of multidimensional matrices or arrays in $\boldsymbol{A}$ of arbitrary dimensions, i.e., multidimensional A-matrices or A-arrays, and form their prearithmetics exploring what properties they inherit from the initial abstract prearithmetic $\boldsymbol{A}$.*

Another way to build new prearithmetics from the existing ones utilizes projectivity relations, different kinds of which are studied in the next section.

3. Weak projectivity in abstract prearithmetics

Let us take two abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ and $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$.

Definition 1. *a) Addition $+_1$ in the abstract prearithmetic $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \cdot \leq_1)$ is called weakly projective with respect to addition $+_2$ in the abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ if there are three mappings $g_1 : A_1 \to A_2, g_2 : A_1 \to A_2$ and $h : A_2 \to A_1$ and the following equality is valid for all elements a and b from $\boldsymbol{A}_1$:*

$$a +_1 b = h(g_1(a) +_2 g_2(b))$$

b) The mappings g_1 and g_2 are called the projectors and the mapping h is called the coprojector for the pair $(+_1, +_2)$.

c) In this case, we say that addition in $\boldsymbol{A}_2$ is weakly projected onto addition in $\boldsymbol{A}_1$, while addition in $\boldsymbol{A}_1$ is a weak projection of addition in $\boldsymbol{A}_2$.

We also say that there is a *weak projectivity* between addition in the prearithmetic $\boldsymbol{A}_1$ and addition in the prearithmetic $\boldsymbol{A}_2$ and there is a *weak inverse projectivity* between addition in the prearithmetic $\boldsymbol{A}_2$ and addition in the prearithmetic $\boldsymbol{A}_1$.

This means that there is *partial weak projectivity* between the prearithmetics $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. This type of partial weak projectivity is called*additive weak projectivity.*

Note that studied in [5, 7, 11] weak projectivity connects both operations in prearithmetics while partial weak projectivity connects only one operation in prearithmetics.

Example 10. *It is possible to treat numerical average as a projection of the conventional addition of numbers. Indeed, taking* $g_1(a) = g_2(a) = g(a) = a$ *and* $h(x) = (\frac{1}{2})x$, *the average of numbers* a *and* b *is*

$$\frac{1}{2}(a+b) = h(g(a)+g(b)) = a \oplus b$$

It is possible to extend these projections to averages of any quantity of numbers, i.e.,

$$(1/n)(a_1 + \cdots + a_n) = h(g(a_1) + \cdots + g(a_n)) = a_1 \oplus \cdots \oplus a_n$$

A special case of this projection, i.e., when g *is a bijection,*$h(x) = g^{-1}(1/2x)$ *in the binary case and* $h(x) = g^{-1}((1/n)x)$, *in a general case, was introduced and studied by Kolmogorov, Nagumo and de Finetti [23, 43, 61]. It is also used in the book [35].*

Example 11. *Weighted sum of numbers is a projection of the conventional addition of numbers. Indeed, with the functions* $g_1(a) = w_1a, g_2(a) = w_2a$ *as projectors and* $h(x) = x$ *as the coprojector, the weighted sum of numbers* a *and* b *is presented as*

$$a \oplus b = w_1a + w_2b$$

Example 12. *Weighted normalized sum of numbers is a projection of the conventional addition of numbers. Indeed, with the functions* $g_1(a) = w_1a, g_2(a) = w_1a$ *as projectors and* $h(x) = x/(w_1 + w_2)$ *as the coprojector, the weighted sum of numbers* a *and* b *is presented as*

$$a \oplus b = (w_1a + w_2b)/(w_1 + w_2)$$

It is necessary to remark that weak projectivity is intrinsically related to such mathematical constructions as fiber bundles [37] and bidirectional named sets [8] as well as to information processes of coding and decoding [68].

Let us consider two abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ and $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$.

Proposition 3.1. If the operation $+_1$ is commutative and weakly projective with respect to the commutative operation $+_2$ with the projectors g_1 and g_2, then the operation $+_1$ is commutative and weakly projective with respect to the operation $+_2$ with the projectors g_2 and g_1.

Indeed, for any elements a and b from A_1 , we have

$$a +_1 b = h(g_1(a) +_2 g_2(b)) = h(g_2(b) +_2 g_1(a)) = b +_1 a$$

This implies

$$b +_1 a = h(g_2(b) +_2 g_1(a))$$

i.e., the operation $+_1$ is weakly projective with respect to the operation $+_2$ with the projectors g_2 and g_1.

Proposition 3.1 allows to show when addition in one abstract prearithmetic is not weakly projective with respect to addition in another abstract prearithmetic.

Example 13. *Let us consider the conventional Diophantine arithmetic $\boldsymbol{N}$ of all natural numbers and the abstract prearithmetic $\boldsymbol{A} = (N; \oplus, \otimes, \leq)$ where N is the set of all natural numbers,$\leq$ is the natural order on the set of all natural numbers, multiplication $\otimes$ is the same as in $\boldsymbol{N}$, while addition is defined by the following formula*

$$a \oplus b = a$$

Addition $\oplus$ in the abstract prearithmetic $\boldsymbol{A}$ is not weakly projective with respect to addition in $\boldsymbol{N}$ because otherwise by Proposition 3.1, it would be commutative and it is not commutative.

At the same time, as we will see later, multiplication $\otimes$ in the abstract prearithmetic $\boldsymbol{A}$ is weakly projective with respect to multiplication in $\boldsymbol{N}$.

If we investigate properties of weak projectivity in the class of abstract prearithmetics, we find that it is a transitive relation. Let us consider three abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1), \boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ and $\boldsymbol{A}_3 = (A_3; +_3, \circ_3, \leq_3)$.

Proposition 3.2. If the operation $+_1$ is weakly projective with respect to the operation $+_2$ and the operation $+_2$ is weakly projective with respect to the operation $+_3$, then the operation $+_1$ is weakly projective with respect to the operation $+_3$.

Proof. Let us assume that the operation $+_1$ in an abstract prearithmetic $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ is weakly projective with respect to the operation $+_2$ an abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ with the projectors $g_{11} : A_1 \to A_2, g_{12} : A_1 \to A_2$, and the coprojector $h : A_2 \to A_1$ for the pair $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ and the operation $+_2$ is weakly projective with respect to the operation $+_3$ in an abstract prearithmetic $\boldsymbol{A}_3 = (A_3; +_3, \circ_3, \leq_3)$ with the projectors $g_{21} : A_2 \to A_3, g_{22} : A_2 \to A_3$, and the coprojector $l : A_3 \to A_2$ for the pair $(\boldsymbol{A}_2, \boldsymbol{A}_3)$. Then we can define mappings $q_i = g_{1i}g_{2i} : A_1 \to A_3 (i = 1, 2)$ and $p = hl : A_3 \to A_1$. Let us consider relations between operations $+_1$ and $+_3$.

$$\begin{aligned} a +_1 b \quad &= h(g_{11}(a) +_2 g_{12}(b)) = h(l(g_{21}(g_{11}(a)) +_3 g_{22}(g_{12}(b)))) \\ &= hl(g_{21}g_{11}(a) +_3 g_{22}g_{12}(b)) = p(q_1(a) +_3 q_2(b)) \end{aligned}$$

Consequently,

$$a +_1 b = p(q_1(a) +_3 q_2(b))$$

for any elements a and b from A_1. It means the operation $+_1$ is weakly projective with respect to the operation $+_3$.

Proposition is proved.

Proposition 3.2 allows proving the following result.

Theorem 1. *Abstract prearithmetics with weak projectivity relations for addition form the category* ***AAWP*** *where objects are abstract prearithmetics and morphisms are weak projectivity relations between additions.*

Indeed, the identity function defines weak projectivity relations for addition of an abstract prearithmetic with itself and by Proposition 3.2, the sequential composition of weak projectivity relations is a weak projectivity relation.

In this category, the identity morphism of an abstract prearithmetic $\boldsymbol{A}$ is the weak projectivity in which both projectors and the coprojector are identity mappings of this prearithmetic.

An important special case of weak projectivity is obtained when both projections coincide.

Definition 2. *a) Addition $+_1$ in the abstract prearithmetic $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ is called weakly monoprojective with respect to addition $+_2$ in the abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ if $+_1$ is weakly projective with respect to addition $+_2$ and $g_1 = g_2$, i.e., there is only one projector.*

b) In this case, we say that addition in $\boldsymbol{A}_2$ is weakly monoprojected onto addition in $\boldsymbol{A}_1$ while addition in $\boldsymbol{A}_1$ is a weak monoprojection of addition in $\boldsymbol{A}_2$.

We also say that there is a *weak monoprojectivity* between addition in the prearithmetic $\boldsymbol{A}_1$ and addition in the prearithmetic $\boldsymbol{A}_2$ and there is an *inverse weak monoprojectivity* between addition in the prearithmetic $\boldsymbol{A}_2$ and addition in the prearithmetic $\boldsymbol{A}_1$. This relation is also called*additive weak monoprojectivity* between prearithmetics $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$.

Example 14. *Weak projectivity in non-Diophantine arithmetics is an example of weak monoprojectivity between addition in one prearithmetic and addition in another prearithmetic [5, 11].*

Let us consider two abstract prearithmetics$\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ and $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$.

Proposition 3.3. If the operation $+_1$ is weakly monoprojective with respect to the operation $+_2$ with the projector g and the operation $+_2$ is commutative in the prearithmetic $\boldsymbol{A}_2$, then the operation $+_1$ is commutative in the prearithmetic $\boldsymbol{A}_1$.

Indeed, for any elements a and b from A_1, we have

$$a +_1 b = h(g(a) +_2 g(b)) = h(g(b) +_2 g(a)) = b +_1 a$$

This shows that inverse weak projectivity preserves commutativity of addition.

To preserve associativity of addition in inverse weak projectivity, we need stronger conditions.

Proposition 3.4. If the operation $+_1$ is weakly monoprojective with respect to the operation $+_2$ with the coprojector h and the projector g, which is a homomorphism with respect to addition $+_1$, i.e., $g(a +_1 b) = g(a) +_2 g(b)$ for arbitrary elements a and b from $\boldsymbol{A}_1$, and the operation $+_2$ is associative in the prearithmetic $\boldsymbol{A}_2$, then the operation $+_1$ is associative in the prearithmetic $\boldsymbol{A}_1$.

Proof. Assuming that g is a homomorphism with respect to addition and the operation $+_2$ is associative in the prearithmetic $\boldsymbol{A}_2$, let us take arbitrary elements a and b from $\boldsymbol{A}_1$. Then by definition, we have

$$g(a +_1 b) = g(h(g(a) +_2 g(b))) = g(a) +_2 g(b) \tag{1}$$

Equality (1) implies the following equalities for arbitrary elements a , b and c from $\boldsymbol{A}_1$

$$(a +_1 b) +_1 c = h(g(h(g(a) +_2 g(b))) +_2 g(c)) = h((g(a) +_2 g(b))) +_2 g(c))$$

$$a +_1 (b +_1 c) = h(g(a) +_2 g(h(g(b) +_2 g(c)))) = h(g(a) +_2 (g(b) +_2 g(c))))$$

As the operation $+_2$ is associative in the prearithmetic $\boldsymbol{A}_2$, we have

$$(a +_1 b) +_1 c = h((g(a) +_2 g(b)) +_2 g(c)) =$$
$$h(g(a) +_2 (g(b)) +_2 g(c))) = a +_1 (b +_1 c)$$

Proposition is proved.

Proposition 3.4 implies the following result.

Let us consider three abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1), \boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ and $\boldsymbol{A}_3 = (A_3; +_3, \circ_3, \leq_3)$.

Proposition 3.5. If the operation $+_1$ is weakly monoprojective with respect to the operation $+_2$ and the operation $+_2$ is weakly monoprojective with respect to the operation $+_3$, then the operation $+_1$ is weakly monoprojective with respect to the operation $+_3$.

Proof is similar to the proof of Proposition 3.2.

Proposition 3.5 allows proving the following result.

Theorem 2. *Abstract prearithmetics with weak monoprojectivity relations for addition form the category* ***AAWMP*** *where objects are abstract prearithmetics and morphisms are weak monoprojectivity relations between additions.*

Proof is similar to the proof of Theorem 1.

Monoprojectivity allows turning a universal algebra with one binary operation into an abstract prearithmetic.

Let us consider a universal algebra $\boldsymbol{A} = (A; \bullet)$ and with one binary operation $\bullet$, i.e., a groupoid, and an abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$.

Proposition 3.6. For any two mappings $g : A \to A_2$ and $h : A_2 \to A$, it is possible to extend the algebra $\boldsymbol{A}$ to an abstract prearithmetic $\boldsymbol{A}_1 = (A; +, \bullet, \leq)$, in which addition $+$ is weakly monoprojective with respect to addition $+_2$.

Indeed, it is possible to take the trivial partial order on A and define addition $+$ in a by the following formula

$$a + b = h(g(a) +_2 g(b))$$

The abstract prearithmetic $\boldsymbol{A}_1 = (A; +, \bullet, \leq)$ is called the *extension* of $\boldsymbol{A}$ by the pair (g, h).

The utilized construction implies the following result.

Let us consider three mappings $g : A \to A_2, h : A_2 \to A$ and $f : A_2 \to A$.

Proposition 3.7. If mappings f and h coincide on the image $g(A)$ of A , then the extensions of $\boldsymbol{A}$ by the pairs (g, h) and (g, f) coincide.

Proof follows directly from definitions.

Abstract prearithmetics have two operations. This gives three more concepts of weak projectivity and three more concepts of weak monoprojectivity.

Definition 3. *a) Multiplication $\circ_1$ in the abstract prearithmetic $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ is called weakly projective with respect to multiplication $\circ_2$ in the abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ if there are three mappings $g_1 : A_1 \to A_2, g_2 : A_1 \to A_2$ and $h : A_2 \to A_1$ and the following equality is valid for all elements a and b from $\boldsymbol{A}_1$:*

$$a \circ_1 b = h(g_1(a) \circ_2 g_2(b))$$

b) The mappings g_1 and g_2 are called the projectors and the mapping h is called the coprojector for the pair $(\circ_1, \circ_2)$.

c) In this case, we say that multiplication in $\boldsymbol{A}_2$ is weakly projected onto addition in $\boldsymbol{A}_1$ while multiplication in $\boldsymbol{A}_1$ is a weak projection of addition in $\boldsymbol{A}_2$.

We also say that there is a *weak projectivity* between multiplication in the prearithmetic $\boldsymbol{A}_1$ and multiplication in the prearithmetic $\boldsymbol{A}_2$ and there is a *weak inverse projectivity* between multiplication in the prearithmetic $\boldsymbol{A}_2$ and multiplication in the prearithmetic $\boldsymbol{A}_1$.

This means that there is *partial weak projectivity* between the prearithmetics $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. This type of partial weak projectivity is called *multiplicative weak projectivity.*

Let us consider two abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ and
$\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$.

Proposition 3.8. If multiplication $\circ_1$ is commutative and weakly projective with respect to the commutative multiplication $\circ_2$ with the projectors g_1 and g_2, then multiplication $\circ_1$ is commutative and weakly projective with respect to multiplication $\circ_2$ with the projectors g_2 and g_1.

Proof is similar to the proof of Proposition 3.1.

Proposition 3.8 allows to show when multiplication in one abstract prearithmetic is not weakly projective with respect to multiplication in another abstract prearithmetic.

Example 15. *Let us consider the conventional Diophantine arithmetic $\boldsymbol{N}$ of all natural numbers and the abstract prearithmetic $\boldsymbol{A} = (N; \oplus, \otimes, \leq)$ where N is the set of all natural numbers, $\leq$ is the natural order on the set of all natural numbers, addition $\oplus$ is the same as in $\boldsymbol{N}$, while multiplication $\otimes$ is defined by the following formula*

$$a \otimes b = b$$

Multiplication $\otimes$ in the abstract prearithmetic $\boldsymbol{A}$ is not weakly projective with respect to multiplication in $\boldsymbol{N}$ because otherwise by Proposition 3.8, it would be commutative and it is not commutative. At the same time, addition $\oplus$ in the abstract prearithmetic $\boldsymbol{A}$ is weakly projective with respect to addition in $\boldsymbol{N}$.

It is also possible that both operations in two abstract prearithmetics are not weakly projective with respect to one another.

Example 16. *Let us consider the conventional Diophantine arithmetic* $\boldsymbol{N}$ *of all natural numbers and the abstract prearithmetic* $\boldsymbol{A} = (N; \oplus, \otimes, \leq)$ *where* N *is the set of all natural numbers and* $\leq$ *is the natural order on the set of all natural numbers, while addition* $\oplus$ *and multiplication* $\otimes$ *are defined by the following formulas*

$$a \oplus b = a$$

$$a \otimes b = b$$

Multiplication $\otimes$ in the abstract prearithmetic $\boldsymbol{A}$ is not weakly projective with respect to multiplication in $\boldsymbol{N}$ because otherwise by Proposition 3.8, it would be commutative and it is not commutative. Addition $\oplus$ in the abstract prearithmetic $\boldsymbol{A}$ is not weakly projective with respect to addition in $\boldsymbol{N}$ because otherwise by Proposition 3.1, it would be commutative and it is not commutative.

Similarly to monoprojectivity of addition, we define weak monoprojectivity of multiplication.

Definition 4. *a) Multiplication* $\circ_1$ *in the abstract prearithmetic* $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ *is called weakly monoprojective with respect to multiplication* $\circ_2$ *in the abstract prearithmetic* $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ *if* $\circ_1$ *is weakly projective with respect to* $\circ_2$ *and* $g_1 = g_2$ *, i.e., there is only one projector.*

b) In this case, we say that multiplication in $\boldsymbol{A}_2$ *is weakly monoprojected onto multiplication in* $\boldsymbol{A}_1$ *while multiplication in* $\boldsymbol{A}_1$ *is a weak monoprojection of multiplication in* $\boldsymbol{A}_2$*.*

We also say that there is a *weak monoprojectivity* between multiplication in the prearithmetic $\boldsymbol{A}_1$ and multiplication in the prearithmetic $\boldsymbol{A}_2$ and there is an *inverse weak monoprojectivity* between multiplication in the prearithmetic $\boldsymbol{A}_2$ and multiplication in the prearithmetic $\boldsymbol{A}_1$. This relation is also called *multiplicative weak monoprojectivity* between prearithmetics $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$.

Example 17. *Weak projectivity in non-Diophantine arithmetics is an example of weak monoprojectivity between multiplication in one prearithmetic and multiplication in another prearithmetic [5, 11].*

Note that in a general case, addition and multiplication in an abstract prearithmetic are simply names of two operations without any additional properties. That is why it is possible to directly convert addition to multiplication or multiplication to addition by renaming. As a result, it is possible to deduce properties of weak projectivity or weak monoprojectivity between multiplication and multiplication from the properties of weak projectivity or weak monoprojectivity between addition and addition. For instance, we have the following results.

Let us consider three abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1), \boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ and $\boldsymbol{A}_3 = (A_3; +_3, \circ_3, \leq_3)$.

Proposition 3.9. If the operation $\circ_1$ is weakly projective (monoprojective) with respect to the operation $\circ_2$ and the operation $\circ_2$ is weakly projective (monoprojective) with respect to the operation $\circ_3$, then the operation $\circ_1$ is weakly projective (monoprojective) with respect to the operation $\circ_3$.

Proof is similar to the proof of Proposition 3.2.

Proposition 3.9 allows proving the following result.

Theorem 3. *Abstract prearithmetics with weak projectivity relations for multiplication form the category* ***AMWP*** *(category* ***AMWMP****) where objects are abstract prearithmetics and morphisms are weak projectivity (monoprojectivity) relations between multiplications.*

Proof is similar to the proof of Theorem 1.

Taking addition and multiplication, we obtain two new concepts.

Definition 5. *Addition $+_1$ in the abstract prearithmetic $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ is called weakly projective with respect to multiplication $\circ_2$ in the abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ if there are three mappings $g_1 : A_1 \to A_2, g_2 : A_1 \to A_2$ and $h : A_2 \to A_1$ and the following equality is valid for all elements a and b from $\boldsymbol{A}_1$:*

$$a +_1 b = h(g_1(a) \circ_2 g_2(b))$$

b) The mappings g_1 and g_2 are called the projectors and the mapping h is called the coprojector for the pair $(+_1, \circ_2)$.

c) In this case, we say that multiplication in $\boldsymbol{A}_2$ is weakly projected onto addition in $\boldsymbol{A}_1$ while addition in $\boldsymbol{A}_1$ is a weak projection of multiplication in $\boldsymbol{A}_2$.

We also say that there is a *weak projectivity* between addition in the prearithmetic $\boldsymbol{A}_1$ and multiplication in the prearithmetic $\boldsymbol{A}_2$ and there is a *weak inverse projectivity* between multiplication in the prearithmetic $\boldsymbol{A}_2$ and addition in the prearithmetic $\boldsymbol{A}_1$.

In a similar way, we define weak monoprojectivity.

Definition 6. *a) Addition $+_1$ in the abstract prearithmetic $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ is called weakly monoprojective with respect to multiplication $\circ_2$ in the abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$if $+_1$ is weakly projective with respect to $\circ_2$ and $g_1 = g_2$, i.e., there is only one projector.*

b) In this case, we say that multiplication in $\boldsymbol{A}_2$ is weakly monoprojected onto addition in $\boldsymbol{A}_1$ while addition in $\boldsymbol{A}_1$ is a weak monoprojection of multiplication in $\boldsymbol{A}_2$.

In this case, we also say that there is a *weak monoprojectivity* between addition in the prearithmetic $\boldsymbol{A}_1$ and multiplication in the prearithmetic $\boldsymbol{A}_2$ and there is an *inverse weak monoprojectivity* between multiplication in the prearithmetic $\boldsymbol{A}_2$ and addition in the prearithmetic $\boldsymbol{A}_1$.

Renaming of operations in abstract prearithmetics allows obtaining properties of weak projectivity or weak monoprojectivity between addition and multiplication from the properties of weak projectivity or weak monoprojectivity between addition and addition.

Taking multiplication and addition, we obtain the following concepts.

Definition 7. *Multiplication* $\circ_1$ *in the abstract prearithmetic* $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ *is called weakly projective with respect to addition* $+_2$ *in the abstract prearithmetic* $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ *if there are three mappings* $g_1 : A_1 \to A_2, g_2 : A_1 \to A_2$ *and* $h : A_2 \to A_1$ *and the following equality is valid for all elements* a *and* b *from* $\boldsymbol{A}_1$*:*

$$a \circ_1 b = h(g_1(a) +_2 g_2(b))$$

b) The mappings g_1 *and* g_2 *are called the projectors and the mapping* h *is called the coprojector for the pair* $(\circ_1, +_2)$*.*

c) In this case, we say that addition in $\boldsymbol{A}_2$ *is weakly projected onto multiplication in* $\boldsymbol{A}_1$ *while multiplication in* $\boldsymbol{A}_1$ *is a weak projection of addition in* $\boldsymbol{A}_2$*.*

We also say that there is a *weak projectivity* between multiplication in the prearithmetic $\boldsymbol{A}_1$ and addition in the prearithmetic $\boldsymbol{A}_2$ and there is a *weak inverse projectivity* between addition in the prearithmetic $\boldsymbol{A}_2$ and multiplication in the prearithmetic $\boldsymbol{A}_1$.

In a similar way, we define weak monoprojectivity.

Definition 8. *a) Multiplication* $\circ_1$ *in the abstract prearithmetic* $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ *is called weakly monoprojective with respect to addition* $+_2$ *in the abstract prearithmetic* $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ *if* $\circ_1$ *is weakly projective with respect to* $+_2$ *and* $g_1 = g_2$ *, i.e., there is only one projector.*

b) In this case, we say that addition in $\boldsymbol{A}_2$ *is weakly monoprojected onto multiplication in* $\boldsymbol{A}_1$ *while multiplication in* $\boldsymbol{A}_1$ *is a weak monoprojection of addition in* $\boldsymbol{A}_2$*.*

We also say that there is a *weak monoprojectivity* between multiplication in the prearithmetic $\boldsymbol{A}_1$ and addition in the prearithmetic $\boldsymbol{A}_2$ and there is an *inverse weak monoprojectivity* between addition in the prearithmetic $\boldsymbol{A}_2$ and multiplication in the prearithmetic $\boldsymbol{A}_1$.

Example 18. *Implicitly people started using weak monoprojectivity between multiplication and addition with the invention (discovery) of logarithms in the early* 17^{th} *century. Indeed, the logarithmic weak monoprojectivity is defined by the projector* $g_1(a) = g_2(a) = \log a$ *and coprojector* $h(x) = \exp x$ *of the arithmetic* $\boldsymbol{R}$ *of real numbers into itself. Namely, we have*

$$a \otimes b = 2^{\log_2 a + \log_2 b} = 2^{\log -2 a \cdot b} = a \cdot b$$

or

$$a \otimes b = 10^{\log_{10} a + \log_{10} b} = 10^{\log_{10} a \cdot b} = a \cdot b$$

Renaming of operations in abstract prearithmetics allows obtaining properties of weak projectivity or weak monoprojectivity between multiplication and addition from the properties of weak projectivity or weak monoprojectivity between addition and addition.

While weak projectivity connects two operations, biprojectivity connects all operations from two abstract prearithmetics.

Let us take two abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ and $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ and consider six mappings $g_1 : A_1 \to A_2, g_2 : A_1 \to A_2, g_3 : A_1 \to A_2, g_4 : A_1 \to A_2, h_1 : A_2 \to A_1$ and $h_2 : A_2 \to A_1$.

Definition 9. *An abstract prearithmetic $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ is called weakly biprojective with respect to an abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ if addition $+_1$ in $\boldsymbol{A}_1$ is weakly projective with respect to addition $+_2$ in $\boldsymbol{A}_2$ with the projectors g_1 and g_2 and the coprojector h_1 while multiplication $\circ_1$ in $\boldsymbol{A}_1$ is weakly projective with respect to multiplication $\circ_2$ in $\boldsymbol{A}_2$ with the projectors g_3 and g_4 and the coprojector h_2 .*

We also say that there is a *weak biprojectivity* between the prearithmetic $\boldsymbol{A}_1$ and the prearithmetic $\boldsymbol{A}_2$.

Example 19. *Weak projectivity of abstract prearithmetics studied in [5, 7, 11] is a special case of weak biprojectivity when all projectors coincide, i.e., $g_1 = g_2 = g_3 = g_4$, and both coprojectors coincide, i.e., $h_1 = h_2$.*

Example 20. *Let us consider the conventional Diophantine arithmetic $\boldsymbol{N}$ of all natural numbers and the abstract prearithmetic $\boldsymbol{A} = (N; \oplus, \otimes, \leq)$ where N is the set of all natural numbers and $\leq$ is the natural order on the set of all natural numbers. To define multiplication $\otimes$ and addition $\oplus$, we take the following functions*

$$g_1(n) = g_2(n) = n + 5$$

$$g_3(n) = g_4(n) = 3n$$

$$h_1(n) = h_2(n) = 1_N$$

Then for arbitrary natural numbers m and n, we have

$$m \oplus n = (m + 5) + (n + 5) = (m + n) + 10$$

$$m \otimes n = (3m) \otimes (3n) = 9mn$$

We see that the abstract prearithmetic $\boldsymbol{A}$ is weakly biprojective with respect to the arithmetic $\boldsymbol{N}$ although projectors for addition and multiplication are different. This shows that in a general case, weak biprojectivity of abstract prearithmetics does not coincide with weak projectivity of abstract prearithmetics studied in [5, 7, 11].

Let us consider some properties of weak biprojectivity taking three abstract prearithmetics $\boldsymbol{A}_1 = (A_1; +_1, \circ_1, \leq_1)$, $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ and $\boldsymbol{A}_3 = (A_3; +_3, \circ_3, \leq_3)$.

Proposition 3.10. If the abstract prearithmetic $\boldsymbol{A}_1$ is weakly biprojective with respect to the abstract prearithmetics $\boldsymbol{A}_2$ and the abstract prearithmetic $\boldsymbol{A}_2$ is weakly biprojective with respect to the abstract prearithmetic $\boldsymbol{A}_3$, then the abstract prearithmetic $\boldsymbol{A}_1$ is weakly biprojective with respect to the abstract prearithmetic $\boldsymbol{A}_3$.

Proof is similar to the proof of Proposition 3.2.

As weak projectivity studied in [5, 7, 11] is a particular case of weak biprojectivity, we have the following result.

Corollary 1. *If the abstract prearithmetic $\boldsymbol{A}_1$ is weakly projective with respect to the abstract prearithmetic $\boldsymbol{A}_2$ and the abstract prearithmetic $\boldsymbol{A}_2$ is weakly projective with respect to the abstract prearithmetic $\boldsymbol{A}_3$, then the abstract prearithmetic $\boldsymbol{A}_1$ is weakly projective with respect to the abstract prearithmetic $\boldsymbol{A}_3$.*

Proposition 3.10 allows proving the following result.

Theorem 4. *Abstract prearithmetics with weak biprojectivity relations form the category* ***AWBP*** *where objects are abstract prearithmetics and morphisms are weak biprojectivity relations between abstract prearithmetics.*

Proof is similar to the proof of Theorem 1.

Corollary 2. *Abstract prearithmetics with weak projectivity relations form the category* ***AWP*** *where objects are abstract prearithmetics and morphisms are weak biprojectivity relations between abstract prearithmetics.*

Corollary 3. *The category* ***AWP*** *is a wide subcategory of the category* ***AWBP***.

Corollary 4. *The categories* ***AAWP*** *and* ***AMWP*** *are wide subcategories of the category* ***AWBP***.

Biprojectivity allows turning an arbitrary set into an abstract prearithmetic.

Let us consider a set A and an abstract prearithmetic $\boldsymbol{A}_2 = (A_2; +_2, \circ_2, \leq_2)$.

Proposition 3.11. For any six mappings $g_1 : A \to A_2, g_2 : A \to A_2, g_3 : A \to A_2, g_4 : A \to A_2, h_1 : A_2 \to A$, and $h_2 : A_2 \to A$, it is possible to define an abstract prearithmetic $\boldsymbol{A} = (A; +, \circ, \leq)$, in which is weakly biprojective with respect to the abstract prearithmetic $\boldsymbol{A}_2$ with the projectors g_1 and g_2 and the coprojector h_1 for addition and the projectors g_3 and g_4 and the coprojector h_2 for multiplication.

Indeed, it is possible to take the trivial partial order on A and define addition $+$ in A by the formula

$$a + b = h_1(g_1(a) +_2 g_2(b))$$

multiplication $\circ$ in a by the formula

$$a \circ b = h_2(g_3(a) \circ_2 g_4(b))$$

The obtained abstract prearithmetic $\boldsymbol{A} = (a; +, \circ, \leq)$ is called the *biprojective extension* of A by the function vector $(g_1, g_2, g_3, g_4, h_1, h_2)$.

The utilized construction implies the following result.

Let us consider eight mappings $g_1 : A \to A_2, g_2 :\to A_2, g_3 :\to A_2, g_4 :\to A_2, h_1 : A_2 \to A$, and $h_2 : A_2 \to A, f_1 : A_2 \to A$, and $f_2 : A_2 \to A$.

Proposition 3.12. If $g_1 = g_2, g_3 = g_4$, mappings f_1 and h_1 coincide on the abstract prearithmetic $PA(g_1(A))$ generated by image $g_1(A)$ of A in A_2 and mappings f_2 and h_2 coincide on the abstract prearithmetic $PA(g_3(A))$ generated by image $g_3(A)$ of A in A_2, then the biprojective extensions of A by the function vectors $(g_1, g_2, g_3, g_4, h_1, h_2)$ and $(g_1, g_2, g_3, g_4, f_1, f_2)$ coincide.

Proof follows directly from definitions.

4. Conclusion

We have explained that abstract prearithmetics encompass a wide range of various mathematical systems, which are used in traditional and novel mathematical domains and applications. We also demonstrated how projectivity relations between abstract prearithmetics allow one to deduce properties of one abstract prearithmetic from properties of another one. Techniques for building new abstract prearithmetics from given ones were elaborated and studied.

It is necessary to remark that traditionally the main relation between algebraic systems is homomorphism with its special types such as monomorphism, epimorphism and isomorphism. The basic property of homomorphisms is that they preserve operations. Systems of algebraic systems such as groups, vector spaces or rings with their homomorphisms form categories.

In the theory of non-Diophantine arithmetics, another basic relation between algebraic systems is introduced. It is called projectivity and has three basic types: weak projectivity, projectivity *per se* and exact projectivity. In this work, we show that there also partial and total weak projectivity while partial weak projectivity has three types: additive weak projectivity, multiplicative weak projectivity and weak biprojectivity. The key property of projectivity relations is that they transfer operations from one prearithmetic to another. Similar to homomorphisms, systems of prearithmetics with their projectivity relations of a fixed type form categories as it is demonstrated in this paper.

The obtained results open potential directions for future research. For instance, it would be interesting to study properties of categories of abstract prearithmetics with different types of partial weak projectivity or monoprojectivity relations as morphisms. In particular, we can explore relations between these categories and traditionally studied categories, such as categories of sets or categories of groups.

In this paper, we study abstract prearithmetics and partial weak projectivity between them. That is why another appealing direction for future research is exploration of partial weak projectivity and monoprojectivity relations between operations in numerical prearithmetics and arithmetics, which form an important class of prearithmetics containing non-Diophantine arithmetics.

One more attractive direction of research in this area is introduction and study of stronger relations of partial projectivity, biprojectivity and monoprojectivity between operations in prearithmetics. These relations can disclose closer ties between operations in related prearithmetics.

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