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The Mathematical Modelling of Refuse Build-up on the University Campus: A Case Study of KNUST Campus

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Abstract. In this paper, $xyzw$ model is introduced which characterizes the solid waste generated by the four departments on the KNUST. Thus, the refuse on the street $x(t)$, in the gutters $y(t)$, in the dustbins $z(t)$ and dumpsite $w(t)$. From the qualitative analysis of $xyzw$ model, it revealed that the refuse in these departments piles up as the time increases indefinitely. Based on the data, it revealed that refuse keeps on piling up. Thus, the trucks are not able to adequately carry refuse from departments: street, gutters and dustbins to the dumpsite as expected by the university authority. This comes as a result of overflows from the dustbins at some vantage points in the university. In addition, the waste in gutters and on street are collected and deposit it in the dustbins (with varying volumes) everyday resulting in overflows of waste on street and in gutters.

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1. Introduction

Solid waste is an issue of great concern to the world especially developing countries since it has devastating effects on the environment. This can be attributed to the rapid population growth rate which eventually translates into generation of copious amounts of solid waste, and also the lack of technical and financial resources of authorities in Africa to address the related challenges. Waste usually comes from domestic homes, commercial and industrial activities. Gilpin (1996) observed that waste as all unwanted and economically unusable by-products or residuals at any given place and time. Waste is described from different perceptives by different researchers across the globe. Palmer (1998) also viewed waste as unwanted materials arising entirely from human activities which are discarded into the environment. Interestly, there are number of ways of classifying waste. Usually, waste is identified by its physical state: solid, liquid and gaseous by its primary use and its origin (White et al., 1995).

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In developing countries like Ghana, huge quantities of waste are generated daily. Improper handling of waste has catastrophic effects on the individuals, as well as, the environs where these wastes are kept. Diseases such as cholera, typhoid, dysentery and malaria are all acquired by people who live in refuse-piled environments. These often result in low productivity and more importantly, loss of human resource needed in the country. Some strategics have been adopted to manage waste worldwide. Waste management aims at minimizing the impacts of waste on the health of citizens and the environment and it includes the characterization and measurement, collection and transportation, separation and resource recovery, processing and disposal (Suh*ü*beler, 1996).

However, there have been studies which look at the disposal of waste across the globe. Boyer et al. (2013) applied optimization technique, biobjective mixed integer programming model, for identifying the optimal location-routing to transport industrial waste from the source to the dumpsite. Badgie et al. (2012) made use of material flow analysis method for the piling up of solid wastes at Kuala Terengganu.

Recently, the solid waste disposal is a major issue on the campuses of the universities in Ghana. In Kwame Nkrumah University of Science and Technology (KNUST), students are often seen littering the lecture theaters, streets, hall of residence, etc. These waste are usually in the form of rappers of toffees, polythene bags which create unpleasant sanitary conditions for the users of the university facilities. Moreover, withering of dry leaves of the trees on the street and in gutters on the campus brings a huge burden on the sanitary workers (zoomlion) of the university. Every morning, the sanitary workers sweep the streets, lecture theaters and disilt gutters, and place them in the mounted refuse bins before finally, conveyed to the dumpsite by a truck.

Revealing the literature extensively, little works have been made with the use of mathematics, mainly most of these works involve optimization techniques to predict the flow of the municipal waste in Ghana and beyond. In this paper, we model the refuse build-up on the KNUST campus using a system of ordinary differential equations ODEs.

2. Main Result

Definition 1. *Let V be a linear space over $\mathbf{R}$. A norm on V is a real-valued function*

$$\|\cdot\| : V \to [0, \infty)$$

such that for any $u, v \in V$ and $\alpha \in \mathbf{R}$ the following conditions are met:

$$\begin{aligned} \|u\| &\geq 0, \; and \; \|u\| = 0, \; iff \; u = 0 \\ \|\alpha u\| &= |\alpha| \|u\|, \;\; \forall\, u \in V \; and \; \alpha \in \mathbf{R} \\ \|u \pm v\| &\leq \|u\| + \|v\|, \;\; \forall\, u, v \in V \end{aligned}$$

The norm of a vector u can be generated by the inner product $(,)$

$$\|u\| = \sqrt{(u, u)}.$$

See [7].

In this paper, the flow of waste at each collection point and between collection points are regarded as a continuous process. In practice, a mathematical model which incorporates few (dependent) variables and conveys the information of waste build-up on the KNUST campus is paramount. A mathematical model is formulated to describe the refuse build-up on the KNUST campus. The $xyzw$ model involves the flow of waste on the street $x(t)$, in gutters $y(t)$, in dustbins $z(t)$ and dumpsite $w(t)$. The rate at which an individual deposits waste on the streets on campus is denoted by α, the rate at which waste from streets gets into the gutters by β and κ is the rate at which the waste on the street is washed by the rain into the gutters. The flow of waste from the gutters to the mounted dustbins is denoted by ψ. The rates at which the street and gutters receive waste, as a result of withering of leaves of the trees, are α and ν, respectively. Setting γ be the rate at which individuals deposit waste in the mounted dustbins on the campus and ϕ is the rate at which refuse is transported to the dumpsite.

2.1. The Assumptions of $xyzw$ Model

(i) The $xyzw$ model incorporates only four refuse collection points on the KNUST campus; the refuse on the street $x(t)$, the refuse in gutters $y(t)$, the refuse in dustbins $z(t)$ and the refuse sent to dumpsite $w(t)$ as a continuous process.

(ii) The overflow of waste from the dustbins to the street is negligible on the grounds that, the quantum of municipal waste on the street is small compared to the quantum of the entire waste on the KNUST campus.

(iii) The students on campus indiscriminately throw waste in the dustbins, on the streets, or in the gutters. In addition, the rain washes the refuse on the streets into the gutters. Due to the withering of leaves of the trees, both street and gutter receive waste by the activities of the trees.

(iv) The laborers or sanitary workers (Zoomlion) sweep the street daily and put the waste in (mounted) dustbins on the KNUST campus.

(v) There is no interactions of waste pile-ups among the three compartments.

Based on the figure 1, we obtain the following system of ODEs as:

$$
\begin{aligned}
\frac{dx}{dt} &= \alpha x - \kappa x - \beta x \\
\frac{dy}{dt} &= \beta x + \nu y - \psi y \\
\frac{dz}{dt} &= \kappa x + \psi y + \gamma z - \phi w \\
\frac{dw}{dt} &= \phi w.
\end{aligned} \tag{1}
$$

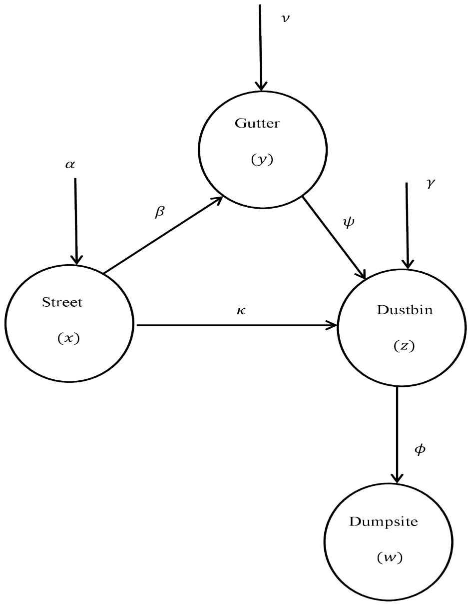

Figure 1: Shows the dynamics of refuse pile-up on KNUST campus.

Rewriting the equation (1) in a matrix form

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \\ \frac{dw}{dt} \end{pmatrix} = \begin{bmatrix} (\alpha-\kappa-\beta) & 0 & 0 & 0 \\ \beta & (\nu-\psi) & 0 & 0 \\ \kappa & \psi & \gamma & -\phi \\ 0 & 0 & 0 & \phi \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix}. \tag{2}$$

Rewriting equation (2) is in form:

$$\frac{dX}{dt} = AX,$$

where,

$$\frac{dX}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \\ \frac{dw}{dt} \end{pmatrix},$$

$$A = \begin{bmatrix} (\alpha-\kappa-\beta) & 0 & 0 & 0 \\ \beta & (\nu-\psi) & 0 & 0 \\ \kappa & \psi & \gamma & -\phi \\ 0 & 0 & 0 & \phi \end{bmatrix}$$

is the community matrix and

$$X = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix}.$$

In order to find the eigenvalues of the community matrix at $(x^*, y^*, z^*, w^*) = (0,0,0,0)$, we make use of the formula:

$$\begin{aligned} det(A-\lambda I) &= 0. \\ \Rightarrow det(A-\lambda I) &= \begin{vmatrix} (\alpha-\kappa-\beta)-\lambda & 0 & 0 & 0 \\ \beta & (\nu-\psi)-\lambda & 0 & 0 \\ \kappa & \psi & \gamma-\lambda & -\phi \\ 0 & 0 & 0 & \phi-\lambda \end{vmatrix} = 0. \end{aligned}$$

The eigenvalues of the fixed point $(x^*, y^*, z^*) = (0,0,0,0)$ is:

$$\begin{aligned} \Rightarrow \lambda_1 &= (\alpha-\kappa-\beta). \\ \lambda_2 &= \nu-\psi. \\ \lambda_3 &= \gamma \\ \lambda_4 &= \phi. \end{aligned}$$

2.2. Analytic Solution of $xyzw$ model

In this section, we search for analytic solution of a system of ODEs. Using the formula:

$$det(A-\lambda I)V = 0 \tag{3}$$

Substituting $\lambda_1 = \alpha-\kappa-\beta$ into equation (3) yields

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta & (\nu-\psi)-(\alpha-\kappa-\beta) & 0 & 0 \\ \kappa & \psi & \gamma-(\alpha-\kappa-\beta) & \phi \\ 0 & 0 & 0 & \phi-(\alpha-\kappa-\beta) \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Rewriting the above equation in the augmented matrix, we obtain

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \beta & (\nu-\psi)-(\alpha-\kappa-\beta) & 0 & 0 & 0 \\ \kappa & \psi & \gamma-(\alpha-\kappa-\beta) & 0 & 0 \\ 0 & 0 & 0 & \phi-(\alpha-\kappa-\beta) & 0 \end{array}\right]$$

Interchanging row one and row three, then row three and four yield

$$\left[\begin{array}{cccc|c} \kappa & \psi & \gamma-(\alpha-\kappa-\beta) & 0 & 0 \\ \beta & (\nu-\psi)-(\alpha-\kappa-\beta) & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi-(\alpha-\kappa-\beta) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Using the row operation $\frac{1}{\kappa}R_1 \longrightarrow R_1$, we obtain

$$\left[\begin{array}{cccc|c} 1 & \frac{\psi}{\kappa} & \frac{\gamma-(\alpha-\kappa-\beta)}{\kappa} & 0 & 0 \\ \beta & (\nu-\psi)-(\alpha-\kappa-\beta) & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi-(\alpha-\kappa-\beta) & 0 \\ 0 & 0 & 0 & 0 & \end{array}\right].$$

Using the row operations: $R_1 - \frac{1}{\beta}R_2 \longrightarrow R_2$ followed by $\frac{\kappa\beta}{\beta\psi+\kappa\left((\alpha-\kappa-\beta)-(\nu-\psi)\right)}R_2 \longrightarrow R_2$ yield

$$\left[\begin{array}{cccc|c} 1 & \frac{\psi}{\kappa} & \frac{\gamma-(\alpha-\kappa-\beta)}{\kappa} & 0 & 0 \\ 0 & 1 & \frac{\beta\left(\gamma-(\alpha-\kappa-\beta)\right)}{\beta\psi+\kappa\left((\alpha-\kappa-\beta)-(\nu-\psi)\right)} & 0 & 0 \\ 0 & 0 & 0 & \phi-(\alpha-\kappa-\beta) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Rewriting the above augmented matrix in equation form, we obtain

$$\left[\begin{array}{cccc} 1 & \frac{\psi}{\kappa} & \frac{\gamma-(\alpha-\kappa-\beta)}{\kappa} & 0 \\ 0 & 1 & \frac{\beta\left(\gamma-(\alpha-\kappa-\beta)\right)}{\beta\psi+\kappa\left((\alpha-\kappa-\beta)-(\nu-\psi)\right)} & 0 \\ 0 & 0 & 0 & \phi-(\alpha-\kappa-\beta) \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus,

$$k_1 + \frac{\psi}{\kappa}k_2 + \frac{\gamma-(\alpha-\kappa-\beta)}{\kappa}k_3 = 0$$

$$k_2 + \frac{\beta\Big(\gamma - (\alpha - \kappa - \beta)\Big)}{\beta\psi + \kappa\Big((\alpha - \kappa - \beta) - (\nu - \psi)\Big)} k_3 = 0$$
$$\phi - (\alpha - \kappa - \beta)k_4 = 0$$

We can see that k_3 is arbitrary. Thus choosing $k_3 = 1$ yields

$$\begin{aligned} k_4 &= 0 \\ k_2 &= \frac{-\beta\Big(\gamma - (\alpha - \kappa - \beta)\Big)}{\beta\psi + \kappa\Big((\alpha - \kappa - \beta) - (\nu - \psi)\Big)} \\ k_1 &= \frac{\psi\beta\Big(\gamma - (\alpha - \kappa - \beta)\Big) + \Big((\alpha - \kappa - \beta) - \psi\Big)\Big(\beta\psi + \kappa\Big((\alpha - \kappa - \beta) - (\nu - \psi)\Big)\Big)}{\kappa\Big(\beta\psi + \kappa\Big((\alpha - \kappa - \beta) - (\nu - \psi)\Big)\Big)} \end{aligned}$$

We observed that the eigenvector

$$v_1 = \begin{pmatrix} \frac{\psi\beta\Big(\gamma-(\alpha-\kappa-\beta)\Big)+\Big((\alpha-\kappa-\beta)-\psi\Big)\Big(\beta\psi+\kappa\Big((\alpha-\kappa-\beta)-(\nu-\psi)\Big)\Big)}{\kappa\Big(\beta\psi+\kappa\Big((\alpha-\kappa-\beta)-(\nu-\psi)\Big)\Big)} \\ \frac{-\beta\Big(\gamma-(\alpha-\kappa-\beta)\Big)}{\beta\psi+\kappa\Big((\alpha-\kappa-\beta)-(\nu-\psi)\Big)} \\ 1 \\ 0 \end{pmatrix}.$$

which corresponds to $\lambda_1 = (\alpha - \kappa - \beta)$.

Similarly, substituting $\lambda_2 = \nu - \psi$ into equation (3) and writing the resulting equation in augmented matrix yields

$$\left[\begin{array}{cccc|c} (\alpha - \kappa - \beta) - (\nu - \psi) & 0 & 0 & 0 & 0 \\ \beta & (\nu - \psi) - (\nu - \psi) & 0 & 0 & 0 \\ \kappa & \psi & \gamma - (\nu - \psi) & 0 & 0 \\ 0 & 0 & 0 & \phi - (\nu - \psi) & 0 \end{array}\right].$$

Using the following sequential row operations $R_1 \longleftrightarrow R_3$, $\frac{1}{\kappa}R_1 \longrightarrow R_1$, $R_1 - \frac{1}{\beta}R_2 \longrightarrow R_2$, $R_1 - \frac{1}{(\alpha-\kappa-\beta)-(\nu-\psi)}R_3 \longrightarrow R_3$, $\frac{\kappa}{\psi}R_2 \longrightarrow R_2$ and $R_2 - \frac{\kappa}{\psi}R_3 \longrightarrow R_3$, we obtain

$$\left[\begin{array}{cccc|c} 1 & \frac{\psi}{\kappa} & \frac{\gamma-(\alpha-\kappa-\beta)}{\kappa} & -\frac{\phi}{\kappa} & 0 \\ 0 & 1 & \frac{\gamma-(\nu-\psi)}{\psi} & -\frac{\phi}{\psi} & 0 \\ 0 & 0 & 0 & \phi - (\nu - \psi) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The above augmented matris yields

$$\begin{bmatrix} 1 & \frac{\psi}{\kappa} & \frac{\gamma-(\nu-\psi)}{\kappa} & -\frac{\phi}{\kappa} \\ 0 & 1 & \frac{\gamma-(\nu-\psi)}{\psi} & -\frac{\phi}{\psi} \\ 0 & 0 & 0 & \phi-(\nu-\psi) \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where, $v_2 = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}$ is the eigenvector which corresponds to $\lambda_2 = \nu - \psi$. We see that:

$$\begin{aligned} m_1 \quad + \quad & \frac{\psi}{\kappa} m_2 + \frac{\gamma - (\nu - \psi)}{\kappa} m_3 - \frac{\phi}{\kappa} m_4 = 0 \\ & m_2 + \frac{\gamma - (\nu - \psi)}{\psi} m_3 - \frac{\phi}{\psi} m_4 = 0 \\ & \phi - (\nu - \psi) m_4 \end{aligned}$$

The vector coordinate k_3 arbitrary and choosing $k_3 = 1$, and back solving the above equation, we obtain

$$\begin{aligned} m_4 &= 0 \\ m_2 &= -\Big(\frac{\gamma - (\nu - \psi)}{\psi}\Big) \\ m_1 &= 0 \end{aligned}$$

Hence,

$$v_2 = \begin{pmatrix} 0 \\ -\Big(\frac{\gamma-(\nu-\psi)}{\psi}\Big) \\ 1 \\ 0 \end{pmatrix}.$$

which corresponds to $\lambda_2 = (\nu - \psi)$.

Similarly, substituting $\lambda_3 = \gamma$ into equation (3) yields

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Similarly, substituting $\lambda_4 = \phi$ into equation (3) yields

$$\left[\begin{array}{cccc|c} (\alpha - \kappa - \beta) - \phi & 0 & 0 & 0 & 0 \\ \beta & (\nu - \psi) - \phi & 0 & 0 & 0 \\ \kappa & \phi & \gamma - \phi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using the following sequential row operations $R_1 \longleftrightarrow R_3,\ \frac{1}{\kappa}R_1 \longrightarrow R_1,\ R_1 - \frac{1}{\beta}R_2 \longrightarrow R_2,\ R_1 - \frac{1}{(\alpha-\kappa-\beta)-\phi}R_3 \longrightarrow R_3,$

$$\left[\begin{array}{cccc|c} 1 & \frac{\psi}{\kappa} & \frac{\gamma-\phi}{\kappa} & \frac{-\phi}{\kappa} & 0 \\ 0 & \left(\frac{\psi}{\kappa} - \frac{(\nu-\psi)-\phi}{\beta}\right) & \frac{\gamma-\phi}{\kappa} & \frac{-\phi}{\kappa} & 0 \\ 0 & \frac{\psi}{\kappa} & \frac{\gamma-\phi}{\kappa} & \frac{-\phi}{\kappa} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{\beta\kappa}{\psi\beta+(\phi-(\gamma-\psi))}R_2 \longrightarrow R_2,$

$$\left[\begin{array}{cccc|c} 1 & \frac{\psi}{\kappa} & \frac{\gamma-\phi}{\kappa} & \frac{-\phi}{\kappa} & 0 \\ 0 & 1 & \frac{\beta(\gamma-\phi)}{\psi\beta+\kappa(\phi-(V-\psi))} & \frac{-\beta\phi}{\psi\beta+\kappa(\phi-(V-\psi))} & 0 \\ 0 & 0 & \frac{\beta(\gamma-\phi)}{\psi\beta+\kappa(\phi-(V-\psi))} - \frac{\gamma-\phi}{\kappa} & \frac{-\beta\phi}{\psi\beta+\kappa(\phi-(V-\psi))} + \frac{\phi}{\psi} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{\kappa(\psi\beta+\kappa(\phi-(v-\psi)))}{(\gamma-\phi)(\beta\kappa-\psi\beta+\kappa(\phi-(v-\psi)))}R_3 \longrightarrow R_3,$

$$\left[\begin{array}{cccc|c} 1 & \frac{\psi}{\kappa} & \frac{\gamma-\phi}{\kappa} & \frac{-\phi}{\kappa} & 0 \\ 0 & 1 & \frac{\beta(\gamma-\phi)}{\psi\beta+\kappa(\phi-(V-\psi))} & \frac{-\beta\phi}{\psi\beta+\kappa(\phi-(V-\psi))} & 0 \\ 0 & 0 & 1 & \frac{\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{\psi((\gamma-\phi)(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$d_1 + \frac{\psi}{\kappa}d_2 + \frac{\gamma-\phi}{\kappa}d_3 - \frac{\phi}{\kappa}d_4 = 0$$

$$d_2 + \frac{\beta(\gamma-\phi)}{\psi\beta+\kappa(\phi-(V-\psi))}d_3 - \frac{\beta\phi}{\psi\beta+\kappa(\phi-(V-\psi))}d_4 = 0$$

$$d_3 + \frac{\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{\psi((\gamma-\phi)(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))}d_4 = 0$$

choose $d_4 = 1$

$$d_3 = -\frac{\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{\psi((\gamma-\phi)(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))}$$

$$d_2 = \frac{\beta\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{[\psi\beta+\kappa(\phi-(V-\psi))][\psi(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))]}$$

$$d_1 = \frac{\phi}{\kappa} - \frac{\beta[\beta\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))]}{[\psi\beta+\kappa(\phi-(V-\psi))][\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi))]} + \frac{[-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi)))]}{\psi[\beta\kappa-\phi\beta+\kappa(\phi-(V-\psi))]}$$

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} \frac{\phi}{\kappa} - \frac{\beta[\beta\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))]}{[\psi\beta+\kappa(\phi-(V-\psi))][\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi))]} + \frac{[-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi)))]}{\psi[\beta\kappa-\phi\beta+\kappa(\phi-(V-\psi))]} \\ \frac{\beta\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{[\psi\beta+\kappa(\phi-(V-\psi))][\psi(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))]} \\ -\frac{\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{\psi((\gamma-\phi)(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))} \\ 1 \end{pmatrix}$$

we obtain as the corresponding eigenvector. Hence, the general solution of xyz model is:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = c_1 \begin{pmatrix} \frac{\psi\beta\Big(\gamma-(\alpha-\kappa-\beta)\Big)+\Big((\alpha-\kappa-\beta)-\psi\Big)\Big(\beta\psi+\kappa\Big((\alpha-\kappa-\beta)-(\nu-\psi)\Big)\Big)}{\kappa\Big(\beta\psi+\kappa\Big((\alpha-\kappa-\beta)-(\nu-\psi)\Big)\Big)} \\ \frac{-\beta\Big(\gamma-(\alpha-\kappa-\beta)\Big)}{\beta\psi+\kappa\Big((\alpha-\kappa-\beta)-(\nu-\psi)\Big)} \\ 1 \\ 0 \end{pmatrix} e^{(\alpha-\kappa-\beta)t} + c_2 \begin{pmatrix} 0 \\ -\Big(\frac{\gamma-(\nu-\psi)}{\psi}\Big) \\ 1 \\ 0 \end{pmatrix} e^{(\nu-\psi)t}$$

$$+ c_4 \begin{pmatrix} \frac{\phi}{\kappa} - \frac{\beta[\beta\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))]}{[\psi\beta+\kappa(\phi-(V-\psi))][\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi))]} + \frac{[-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi)))]}{\psi[\beta\kappa-\phi\beta+\kappa(\phi-(V-\psi))]} \\ \frac{\beta\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{[\psi\beta+\kappa(\phi-(V-\psi))][\psi(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))]} \\ -\frac{\kappa(-\beta\phi\psi+\phi(\psi\beta+\kappa(\phi-(V-\psi))))}{\psi((\gamma-\phi)(\beta\kappa-\psi\beta+\kappa(\phi-(V-\psi)))} \\ 1 \end{pmatrix} e^{\phi t}$$

where, c_1, c_2, c_3 and c_4 are constants.

2.3. Numerical Analysis of $xyzw$ model

The characteristic polynomial is obtained as:

$$\lambda^4 - (\alpha-\kappa-\beta+\gamma+\nu-\psi+\phi)\lambda^3$$
$$+ \Big(\alpha\gamma+\alpha\nu-\alpha\psi+\alpha\phi-\kappa\gamma-\kappa\nu+\kappa\psi-\kappa\phi-\beta\gamma-\beta\nu+\beta\psi$$

$$
\begin{aligned}
- & \quad \beta\phi + \nu\gamma + \phi\nu - \psi\gamma - \gamma\phi\Big)\lambda^2 \\
- & \quad \Big(- \alpha\nu\gamma + \alpha\psi\gamma - \kappa\nu\gamma - \kappa\psi\gamma - \beta\nu\gamma + \beta\psi\gamma + \alpha\phi\nu - \alpha\psi\phi - \kappa\phi\nu + \kappa\phi\psi - \beta\phi\nu \\
+ & \quad \beta\psi\phi + \alpha\gamma\phi - \kappa\gamma\phi - \beta\gamma\nu - \gamma\psi\phi\Big)\lambda = 0. \qquad (3)
\end{aligned}
$$

The following parameter values in table 1 were estimated based on the data collected on the KNUST campus from June, 2017 to April, 2018.

Table 1: Shows the estimation of model parameters.

Parameter	Description	Typical Value
κ	rate at which waste is transferred into dustbins	0.38
β	rate at which waste from streets enter the gutters	0.26
ψ	waste collected from gutters to dustbin	0.15
α	rate at which individuals deposit waste on the streets	0.08
ν	waste in gutters on campus	0.05
γ	rate at which individuals deposit waste directly into dustbin	0.69
ϕ	rate at which waste is transferred to the dump site	0.92

Substituting the parameter values in table 1 into equation 3 yields

$$\begin{aligned} &\lambda^4 - 0.95\lambda^3 - 0.3718\lambda^2 + 0.3288\lambda - 0.035549 = 0 \\ \Rightarrow \quad &\lambda_1 = -0.56,\ \lambda_2 = -0.1\ \ \lambda_3 = 0.69\ and\ \lambda_4 = 0.92. \end{aligned}$$

The corresponding eigenvectors are obtained from the following equation:

$$\begin{bmatrix} (\alpha-\kappa-\beta)-\lambda & 0 & 0 & 0 \\ \beta & (\nu-\psi)-\lambda & 0 & 0 \\ \kappa & \psi & \gamma-\lambda & -\phi \\ 0 & 0 & 0 & \phi-\lambda \end{bmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4}$$

where,

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix}$$

is the eigenvector which corresponds to eigenvalue λ.

Substituting $\lambda_1 = -0.56$ into the equation (4) yields

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.26 & 0.46 & 0 & 0 \\ 0.38 & 0.15 & 1.25 & -0.92 \\ 0 & 0 & 0 & 0.92 \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the eigenvector which corresponds to λ_1 is:

$$v_1 = \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \\ 0 \end{pmatrix}.$$

Similarly, substituting λ_2, λ_3 and $\lambda_4$into equation (4) yield

$$v_2 = \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \\ 0 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} and\ v_4 = \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix},$$

respectively.

The solution of the equation (1) is obtained as:

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = c_1 \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \\ 0 \end{pmatrix} e^{-0.56t} + c_2 \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \\ 0 \end{pmatrix} e^{-0.1t} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{0.69t}$$

$$+ \quad c_4 \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix} e^{0.92t}.$$

Setting the initial conditions $x(0) = 1,\ y(0) = 2,\ z(0) = 1,$ and, $w(0) = 10$, we obtain the particular solution

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = 24 \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \\ 0 \end{pmatrix} e^{-0.56t} - 11.148 \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \\ 0 \end{pmatrix} e^{-0.1t} - 52.317 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{0.69t}$$
$$+ \quad 10 \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix} e^{0.92t}. \tag{5}$$

2.4. Long Term Behaviour of the Refuse Build-up on the KNUST Campus

In this section, the long behaviour of the refuse build-up on the KNUST campus as the time increases is obtained. Taking the limit of both sides of equation 5 as the time t approaches infinity yields

$$\lim_{t\to\infty} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = \lim_{t\to\infty} \left\{ 24 \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \\ 0 \end{pmatrix} e^{-0.56t} - 11.148 \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \\ 0 \end{pmatrix} e^{-0.1t} \right.$$
$$\left. - \quad 52.317 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{0.69t} + 10 \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix} e^{0.92t} \right\}$$

$$\lim_{t\to\infty} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = \lim_{t\to\infty} 24 \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \end{pmatrix} e^{-0.56t} - \lim_{t\to\infty} 11.148 \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \\ 0 \end{pmatrix} e^{-0.1t}$$
$$- \quad \lim_{t\to\infty} 52.316 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{0.69t} + \lim_{t\to\infty} 10 \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix} e^{0.92t}$$

$$\lim_{t\to\infty} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = \infty.$$

This implies that the solution of equation (1) does not approach any limiting value. Thus, the the solution of the system of ODEs is unstable. Hence, the refuse on the KNUST campus keeps on piling up as time t increases indefinitely.

Also, we determine the behaviour of the solution of system of ODEs by finding the norm of the both sides of equation (5) which yields

$$\begin{aligned}
\left\| \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} \right\| &= \left\| 24 \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \\ 0 \end{pmatrix} e^{-0.56t} - 11.148 \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \end{pmatrix} e^{-0.1t} \right. \\
&- \left. 52.316 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{0.69t} + 10 \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix} e^{0.92t} \right\| \\
\Rightarrow \left\| \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} \right\| &\leq \left\| 24 \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \\ 0 \end{pmatrix} e^{-0.56t} \right\| + \left\| 11.148 \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \\ 0 \end{pmatrix} e^{-0.1t} \right\| \\
&+ \left\| 52.316 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{0.69t} \right\| + \left\| +10 \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix} e^{0.92t} \right\| \\
\Rightarrow \left\| \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} \right\| &= \left\| 24 \begin{pmatrix} 0.15 \\ -1.86 \\ 1 \\ 0 \end{pmatrix} \right\| \left\| e^{-0.56t} \right\| + \left\| \begin{pmatrix} 0 \\ \frac{-79}{15} \\ 1 \\ 0 \end{pmatrix} \right\| \left\| e^{-0.1t} \right\| \\
&+ \left\| \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\| \left\| e^{0.69t} \right\| + \left\| 10 \begin{pmatrix} 6.3636 \\ -1.121 \\ 4 \\ 1 \end{pmatrix} \right\| \left\| e^{0.92t} \right\| \\
\Rightarrow \left\| \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} \right\| &\leq \infty.
\end{aligned}$$

This inequality implies that there is no boundedness constant, which in turn shows that, the solution in equation (5) is unstable as the time t approaches increases. This method of analysis of the solution of system of ODEs confirms the asymptotic analysis.

3. Conclusion

Based on the analysis of data from the KNUST campus the refuse keeps on piling up. This reveals that the trucks are not able to carry adequate refuse from three departments: street, gutters and dustbins to the dumpsite as expected by the university authority. This come as result of overflows from the dustbins at some vantage points in the university. In practice, the waste in gutters and on street are collected and deposit it in these dustbins (with varying volumes) everyday, but the trucks are not able to convey all the quantum of waste in these dustbins to the dumpsite thereby resulting in refuse pile up on campus of the university. Also, we observed that $xywz$ model exhibits analytic solution which is characterized by eigenvalues and their corresponding eigenvectors.

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