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# Total Partial Domination in Graphs under Some Binary Operations

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**Abstract.** Let $G = (V(G), E(G))$ be a simple graph without isolated vertices and let $\alpha \in (0, 1]$. A set $S \subseteq V(G)$ is an *$\alpha$-partial dominating set* in $G$ if $|N[S]| \geq \alpha\, |V(G)|$. The smallest cardinality of an $\alpha$-partial dominating set in $G$ is called the *$\alpha$-partial domination number* of $G$, denoted by $\partial_\alpha(G)$. An $\alpha$-partial dominating set $S \subseteq V(G)$ is a *total $\alpha$-partial dominating set* in $G$ if every vertex in $S$ is adjacent to some vertex in $S$. The *total $\alpha$-partial domination number* of $G$, denoted by $\partial_{T\alpha}(G)$, is the smallest cardinality of a total $\alpha$-partial dominating set in $G$. In this paper, we characterize the total partial dominating sets in the join, corona, lexicographic product and Cartesian product of graphs and determine the exact values or sharp bounds of the corresponding total partial domination number of these graphs.

**2010 Mathematics Subject Classifications**: 05C69, 05C76

**Key Words and Phrases**: Partial domination, Total partial domination, Join, Corona, Lexicographic product, Cartesian product

## 1. Introduction

Let $G = (V(G), E(G))$ be a simple graph and $v \in V(G)$. The *open neighborhood* of $v$ in $G$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. For $X \subseteq V(G)$, the *open neighborhood* of $X$ in $G$ is the set $N_G(X) = N(X) = \bigcup_{v\in X} N_G(v)$ and its *closed neighborhood* is the set $N_G[X] = N[X] = N(X) \cup X$. A set $D \subseteq V(G)$ is a *dominating set* in $G$ if for every $v \in V(G)\backslash D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N[D]= V(G)$. The minimum cardinality of a dominating set in $G$, denoted by $\gamma(G)$, is the *domination number* of

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$G$. Any dominating set in $G$ of cardinality $\gamma(G)$ is referred to as a $\gamma$-*set* in $G$.

Let $G$ be a graph without isolated vertices. A set $T \subseteq V(G)$ is a *total dominating set* in $G$ if $N(T) = V(G)$. The *total domination number* $\gamma_t(G)$ of $G$ is the minimum cardinality of a total dominating set in $G$.

Dominating sets are important in a wide range of applications where some level of service or resource must be provided to each member of a network. However, considerations of scarcity of resources, practicality, or profitability may lead to a necessity for less than complete coverage of the nodes in a network. This gives rise to the notion of partial domination in graphs [1].

For any simple graph $G$ and an $\alpha \in (0,1]$, a set $S \subseteq V(G)$ is an $\alpha$-*partial dominating set* in $G$ if $|N[S]| \geq \alpha\,|V(G)|$. Case et al. [1] and Das [2] independently worked on $\alpha$-partial domination in graphs in 2017. Case et al. focused on $\alpha = \frac{1}{2}$ while Das dealt with general values of $\alpha \in (0,1]$. The $\alpha$-*partial domination number* $\partial_\alpha(G)$ is the minimum cardinality of an $\alpha$-partial dominating set in $G$. $\partial_\alpha(G)$ is denoted by $\mathrm{pd}_\alpha(G)$ in Das [2] and by $\gamma_{\frac{1}{2}}(G)$ in Case et al. [1] when $\alpha = \frac{1}{2}$. An $\alpha$-partial dominating set $S$ in $G$ with $|S| = \partial_\alpha(G)$ is referred to as an $\partial_\alpha$-*set* in $G$.

Case et al. [1] investigated the partial domination number of some special graphs and presented some bounds of the said parameter. Das [2] also studied different bounds on the partial domination number of a graph with respect to several parameters like its order, maximum degree, and domination number. Macapodi, Isla and Canoy [3] characterized the partial dominating sets in the join, corona, lexicographic product and Cartesian product of graphs and determined the exact values or sharp bounds of the corresponding partial domination number of these graphs. They also introduced and examined the concepts of total partial domination and $(\alpha,k)$-partial domination, where $\alpha \in (0,1]$ and $k \in (-\infty,0]$. Let $G$ be a simple graph. A nonempty set $S \subseteq V(G)$ is an $(\alpha,k)$-*partial dominating set* in $G$ if $|N[S]| \geq \alpha\,|V(G)| + k$.

Let $G$ be a graph without isolated vertices. An $\alpha$-partial dominating set $S \subseteq V(G)$ is a *total* $\alpha$-*partial dominating set* in $G$ if every vertex in $S$ is adjacent to some vertex in $S$. In this case, we also say that $G$ is *totally* $\alpha$-*partial dominated by the vertices in* $S$. The *total* $\alpha$-*partial domination number* of $G$, denoted by $\partial_{T\alpha}(G)$, is the minimum cardinality of a total $\alpha$-partial dominating set in $G$. A total $\alpha$-partial dominating set $S$ with $|S| = \partial_{T\alpha}(G)$ is referred to as a $\partial_{T\alpha}$-*set* in $G$.

Let $G$ be a connected graph. Let $\alpha \in (0,1]$ and $k \in (-\infty,0]$. A nonempty set $S \subseteq V(G)$ is a *total* $(\alpha,k)$-*partial dominating set* in $G$ if $|N[S]| \geq \alpha\,|V(G)| + k$ and every element in $S$ is adjacent to an element in $S$. The *total* $(\alpha,k)$-*partial domination number* of $G$, denoted by $\partial_{T\alpha,k}(G)$, is given by $\partial_{T\alpha,k}(G) = \min\{|S| : S \text{ is a total } (\alpha,k)\text{-partial dominating set in } G\}$. Any partial dominating set in $G$ with cardinality $\partial_{T\alpha,k}(G)$ is referred to as a $\partial_{T\alpha,k}$-*set* in $G$.

The *join* $G+H$ of two graphs $G$ and $H$ is the graph with vertex set

$$V(G+H) = V(G) \cup V(H)$$

and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *corona* of two graphs $G$ and $H$, denoted by $G \circ H$, is the graph obtained by taking one copy of $G$ of order $n$ and $n$ copies of $H$, and then joining the $i$-th vertex of $G$ to every vertex in the $i$-th copy of $H$. For every $v \in V(G)$, we denote by $H^v$ the copy of $H$ whose vertices are joined or attached to the vertex $v$. For each $v \in V(G)$, the subgraph $\langle v\rangle + H^v$ of $G \circ H$ will be denoted by $v + H^v$. The *lexicographic product* of two graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The *Cartesian product* of two graphs $G$ and $H$, denoted by $G\Box H$, is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ and edge set $E(G\Box H)$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G\Box H)$ if and only if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

## 2. Preliminary Results

**Remark 1.** *Let $m, n$ and $p$ be positive integers and let $\alpha \in (0, 1]$. Let $G$ be a complete graph $K_m$, a fan graph $F_m$, a star graph $K_{1,n}$ or a wheel graph $W_p$. Then $\partial_{T\alpha}(G) = 2$ for $m \geq 2$, $n \geq 1$ and $p \geq 3$.*

**Remark 2.** *For any $\alpha \in (0, 1]$ and a complete bipartite graph $K_{m,n}$, with $m, n \geq 2$, $\partial_{T\alpha}(K_{m,n}) = 2$.*

**Remark 3.** *Let $G$ be a graph without isolated vertices. If $\gamma_t(G) = 2$, then $\partial_{T\alpha}(G) = 2$ for all $\alpha \in (0, 1]$.*

**Remark 4.** *Let $G$ be a nontrivial graph. Then $\partial_\alpha(G) \leq \partial_{T\alpha}(G) \leq \gamma_t(G)$.*

**Theorem 1.** *Let $n$ be a positive integer and $\alpha = \frac{1}{2}$. Then*

$$\partial_{T\alpha}(P_n) = \partial_{T\alpha}(C_n) = \begin{cases} 2, & 2 \leq n \leq 7 \\ 2r, & n = 8r \\ 2r+1, & n = 8r+s,\ \ s = 1, 2 \\ 2r+2, & n = 8r+s,\ \ s = 3, 4, 5, 6, 7 \end{cases}$$

*where $r$ and $s$ are integers such that $n = 8r + s$, $1 \leq s \leq 7$.*

*Proof.* Let $P_n = [v_1, v_2, ..., v_n]$. If $2 \leq n \leq 7$, then clearly, $\partial_{T\alpha}(P_n) = 2$. Let $n \geq 8$ and consider the following cases:
Case 1: $n = 8r$

Group the vertices of $P_n$ into $r$ disjoint subsets.

$$\begin{aligned} S_1 &= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \\ S_2 &= \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\} \\ &\vdots \\ S_{r-1} &= \{v_{8r-15}, v_{8r-14}, v_{8r-13}, v_{8r-12}, v_{8r-11}, v_{8r-10}, v_{8r-9}, v_{8r-8}\} \end{aligned}$$

$$S_r = \{v_{8r-7}, v_{8r-6}, v_{8r-5}, v_{8r-4}, v_{8r-3}, v_{8r-2}, v_{8r-1}, v_{8r}\}$$

For every induced subgraph $\langle v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_{i+7}\rangle$ of $P_n$, where $i = 1, 9, ..., 8r-7$, the vertices $v_{i+1}$ and $v_{i+2}$ form a total $\alpha$-partial dominating set of $P_n$. Thus, the set $T = \{v_2, v_3, v_{10}, v_{11}, \ldots, v_{8r-6}, v_{8r-5}\}$ is a total $\alpha$-partial dominating set of $P_n$. Since $|T| = 2r$, $\partial_{T\alpha}(P_n) \leq 2r$. Note that every pair of adjacent vertices in $P_n$ can dominate at most 4 vertices. Thus, every total $\alpha$-partial dominating set in $P_n$ contains at least $\lceil \frac{n}{4} \rceil$ vertices. Hence, $\partial_{T\alpha}(P_n) \geq \lceil \frac{n}{4} \rceil = 2r$ since $n = 8r$. Thus, $\partial_{T\alpha}(P_n) = 2r$.

Case 2: $n = 8r + s,\ s = 1, 2$

In Case 1, the first $8r$ vertices of $P_n$ are totally $\alpha$-partial dominated by $2r$ vertices in $T$. Since $n = 8r + 1$ or $8r + 2$, vertices $v_{8r+1}$ and $v_{8r+2}$ of $P_n$ are not totally $\alpha$-partial dominated by $T$. Hence, $\partial_{T\alpha}(P_n) > 2r$. Now, add $v_{8r-4}$ to $T$ so that $T \cup \{v_{8r-4}\}$ is a $\partial_{T\alpha}$-set in $P_n$. Thus, $\partial_{T\alpha}(P_n) = 2r + 1$.

Case 3: $n = 8r + s,\ s = 3, 4, 5, 6, 7$

Consider the total $\alpha$-partial dominating set $T$ in Case 1. Since $\{v_{8r+s} | s = 3, 4, 5, 6, 7\}$ are not totally $\alpha$-partial dominated by vertices in $T \cup \{v_{8r-4}\}$, $\partial_{T\alpha}(P_n) > 2r + 1$. Now, add vertices $v_{8r+2}$ and $v_{8r+3}$ to $T$ so that $T \cup \{v_{8r+2}, v_{8r+3}\}$ is a $\partial_{T\alpha}$-set in $P_n$. Thus, $\partial_{T\alpha}(P_n) = 2r + 2$.

Finally, it can be verified that the total $\alpha$-partial domination number of $P_n$ still holds for $G = C_n$ for $n \geq 3$. □

We now present a realization problem.

**Theorem 2.** *Let $a$ and $b$ be positive integers such that $a = b$ or $b = 2a$ and let $\alpha = \frac{1}{2}$. Then there exists a connected graph $G$ such that $\partial_\alpha(G) = a$ and $\partial_{T\alpha}(G) = b$.*

*Proof.* Consider the following cases:

Case 1: $a = b$

Subcase (i). $a$ is even.

Let $G = G_1$ be the graph shown in Figure 1. It is clear that the set $A = \{x_i : i = 1, 2, ..., a\}$ is both an $\partial_\alpha$-set and a $\partial_{T\alpha}$-set in $G_1$. It follows that $\partial_\alpha(G_1) = \partial_{T\alpha}(G_1) = |A| = a = b$.

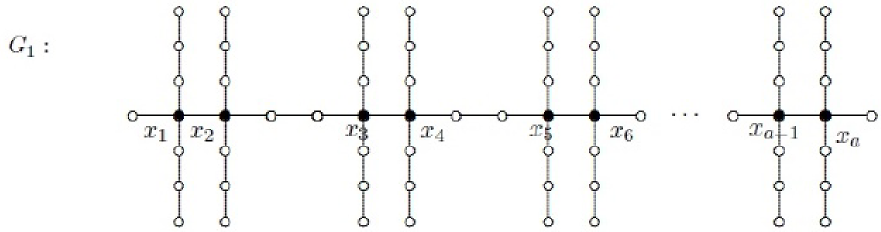


Figure 1: A graph $G_1$ with $\partial_\alpha(G_1) = \partial_{T\alpha}(G_1)$ for $\alpha = \frac{1}{2}$.

Subcase (ii). $a$ is odd.

Let $G = G_2$ be the graph shown in Figure 2. It is clear that the set $A = \{x_i : i = 1, 2, ..., a\}$ is both an $\partial_\alpha$-set and a $\partial_{T\alpha}$-set in $G_2$. It follows that $\partial_\alpha(G_2) = \partial_{T\alpha}(G_2) = |A| = a = b$.

$G_2$ :

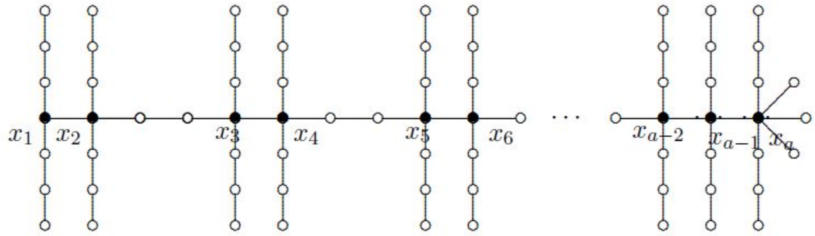


Figure 2: A graph $G_2$ with $\partial_\alpha(G_2) = \partial_{T\alpha}(G_2)$ for $\alpha = \frac{1}{2}$.

Case 2: $b = 2a$

Let $G = G_3$ be the graph shown in Figure 3. Observe that the set $A = \{x_i : i = 1, 2, ..., a\}$ is an $\partial_\alpha$-set and the set $B = A \cup \{z_j : j = 1, 2, ..., a\}$ is a $\partial_{T\alpha}$-set in $G_3$. It follows that $\partial_\alpha(G_3) = |A| = a$ and $\partial_{T\alpha}(G_3) = |B| = 2a = b$.

$G_3$ :

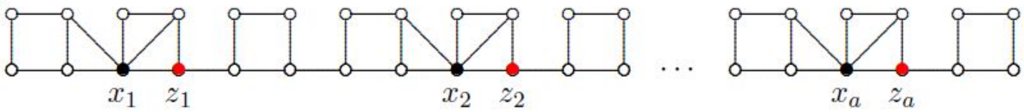


Figure 3: A graph $G$ with $2\partial_\alpha(G) = \partial_{T\alpha}(G)$ for $\alpha = \frac{1}{2}$.

This proves the assertion. $\square$

**Corollary 1.** *Given a positive integer $m$ and $\alpha = \frac{1}{2}$, there exists a connected graph $G$ such that $\partial_{T\alpha}(G) - \partial_\alpha(G) = m$, that is, the difference $\partial_{T\alpha} - \partial_\alpha$ can be made arbitrarily large.*

*Proof.* Let $a = m$ and $b = 2m$. By Theorem 2, there exists a connected graph $G$ with $\partial_{T\alpha}(G) - \partial_\alpha(G) = 2m - m = m$. $\square$

The following characterizations of partial dominating sets in the join, corona, lexicographic product and Cartesian product of graphs are found in Macapodi et al. [3].

**Theorem 3.** *Let $G$ and $H$ be connected graphs of orders $m$ and $n$, respectively, and let $\alpha \in (0, 1]$. Then $C \subseteq V(G + H)$ is an $\alpha$-partial dominating set in $G + H$ if and only if at least one of the following is true:*

*(i)* *$C \subseteq V(G)$ and $C$ is an $(\alpha, (\alpha - 1)n)$-partial dominating set in $G$.*

*(ii)* *$C \subseteq V(H)$ and $C$ is an $(\alpha, (\alpha - 1)m)$-partial dominating set in $H$.*

*(iii)* *$C \cap V(G) \neq \varnothing$ and $C \cap V(H) \neq \varnothing$.*

**Theorem 4.** *Let $G$ be a non-trivial connected graph of order $m$ and $H$ be any graph of order $n$. Let $\alpha \in (0, 1]$ and $C \subseteq V(G \circ H)$. If at least one of the following holds:*

*(i)* $C = \bigcup_{v \in V(G)} S_v$, *where $S_v$ is an $\alpha$-partial dominating set in $H^v$ for each $v \in V(G)$,*

*(ii)* $C \subseteq V(G)$ *where either* $C$ *is a dominating set in* $G$ *and* $|C| \geq \dfrac{\alpha m(n+1)-m}{n}$ *or* $|C| \geq \alpha m$,

*then* $C$ *is an* $\alpha$*-partial dominating set in* $G \circ H$.

**Theorem 5.** *Let* $G$ *and* $H$ *be connected graphs. Let* $\alpha \in (0,1]$ *and* $C = \bigcup_{x\in S} (\{x\} \times T_x) \subseteq V(G[H])$. *If either one of the following holds:*

*(i)* $S$ *is a total* $\alpha$*-partial dominating set in* $G$, *or*

*(ii)* $S$ *is an* $\alpha$*-partial dominating set in* $G$ *and* $T_x$ *is a dominating set in* $H$ *for every* $x \in S \backslash N_G(S)$,

*then* $C$ *is an* $\alpha$*-partial dominating set in* $G[H]$.

**Theorem 6.** *Let* $G$ *and* $H$ *be nontrivial connected graphs and* $\alpha \in (0,1]$. *Then* $C_1 = S_1 \times V(H)$ *and* $C_2 = V(G) \times S_2$ *are* $\alpha$*-partial dominating sets in* $G \Box H$ *if and only if* $S_1$ *and* $S_2$ *are* $\alpha$*-partial dominating sets in* $G$ *and* $H$, *respectively.*

## 3. Main Results

We characterize the total partial dominating sets in the join, corona, lexicographic product and Cartesian product of graphs in this section.

**Remark 5.** *Every total* $(\alpha, (\alpha-1)n)$*-partial dominating set is an* $(\alpha, (\alpha-1)n)$*-partial dominating set.*

**Remark 6.** *Every total* $\alpha$*-partial dominating set is an* $\alpha$*-partial dominating set.*

**Theorem 7.** *Let* $G$ *and* $H$ *be connected graphs of orders* $m$ *and* $n$, *respectively, and let* $\alpha \in (0,1]$. *Then* $C \subseteq V(G+H)$ *is a total* $\alpha$*-partial dominating set in* $G+H$ *if and only if at least one of the following is true:*

*(a)* $C \subseteq V(G)$ *and* $C$ *is a total* $(\alpha, (\alpha-1)n)$*-partial dominating set in* $G$.

*(b)* $C \subseteq V(H)$ *and* $C$ *is a total* $(\alpha, (\alpha-1)m)$*-partial dominating set in* $H$.

*(c)* $C \cap V(G) \neq \varnothing$ *and* $C \cap V(H) \neq \varnothing$.

*Proof.* Suppose $C \subseteq V(G+H)$ is a total $\alpha$-partial dominating set in $G+H$. Then by Remark 6, $C$ is an $\alpha$-partial dominating set in $G+H$. By Theorem 3, at least one of the following is true: $(i)$ $C \subseteq V(G)$ and $C$ is an $(\alpha, (\alpha-1)n)$-partial dominating set in $G$, $(ii)$ $C \subseteq V(H)$ and $C$ is an $(\alpha, (\alpha-1)m)$-partial dominating set in $H$, or $(iii)$ $C \cap V(G) \neq \varnothing$ and $C \cap V(H) \neq \varnothing$. Suppose $(i)$ holds. Since $C$ is a total $\alpha$-partial dominating set in $G+H$ and $C \subseteq V(G)$, it follows that $C$ is a total $\alpha$-partial dominating set in $G$. Hence, $C$ is a total $(\alpha, (\alpha-1)n)$-partial dominating set in $G$, so Condition $(a)$ holds. Similarly, if $(ii)$ holds, then Condition $(b)$ is true. Finally, Condition $(iii)$ is the same as Condition $(c)$.

For the converse, suppose Condition $(a)$ holds. Then $C \subseteq V(G)$ and $C$ is an $(\alpha, (\alpha-1)n)$-partial dominating set in $G$ by Remark 5. Thus, by Theorem 3, $C \subseteq V(G+H)$ is an $\alpha$-partial dominating set in $G+H$. Let $x \in C$. Since $C$ is a total $(\alpha, (\alpha-1)n)$-partial dominating set in $G$, there exists a $y \in C$ such that $xy \in E(\langle C \rangle)$. Thus, $C$ is a total $\alpha$-partial dominating set in $G + H$. Similarly, if Condition $(b)$ holds, it can be shown that if $g \in C \subseteq V(H)$, then there exists an $h \in C$ such that $gh \in E(\langle C \rangle)$. Thus, $C$ is a total $\alpha$-partial dominating set in $G + H$. If Condition $(c)$ holds, then by Theorem 3, $C \subseteq V(G + H)$ is an $\alpha$-partial dominating set in $G + H$. Moreover, $C$ is clearly a total $\alpha$-partial dominating set in $G + H$. □

The next result immediately follows by Remark 5 and Theorem 7.

**Corollary 2.** *Let $G$ and $H$ be connected graphs, $\alpha \in (0, 1]$ and let $C \subseteq V(G+H)$ satisfying one of the following conditions:*

$(i)$ *$C \subseteq V(G)$ is a total $\alpha$-partial dominating set in $G$.*

$(ii)$ *$C \subseteq V(H)$ is a total $\alpha$-partial dominating set in $H$.*

$(iii)$ *$|C \cap V(G)| \geq 1$ and $|C \cap V(H)| \geq 1$.*

*Then $C$ is a total $\alpha$-partial dominating set in $G + H$.*

**Corollary 3.** *Let $G$ and $H$ be connected graphs of orders $m$ and $n$, respectively, and let $\alpha \in (0, 1]$. Then,*

$$\partial_{T\alpha}(G + H) = 2.$$

*Proof.* Pick $x \in V(G), y \in V(H)$. Clearly, $S = \{x, y\}$ is a $\gamma_t$-set, hence a $\partial_{T\alpha}$-set, in $G + H$ and thus, $\partial_{T\alpha}(G + H) = 2$. □

**Theorem 8.** *Let $G$ be a nontrivial connected graph of order $m$ and $H$ be any graph of order $n$. Let $\alpha \in (0, 1]$ and $C \subseteq V(G \circ H)$. If at least one of the following holds:*

$(i)$ *$C = \bigcup_{v \in V(G)} S_v$, where $S_v$ is a total $\alpha$-partial dominating set in $H^v$ for each $v \in V(G)$,*

$(ii)$ *$C \subseteq V(G)$ where either $C$ is a total dominating set in $G$ and $|C| \geq \dfrac{\alpha m(n+1) - m}{n}$ or $|C| \geq \alpha m$ and $C \setminus N_G(C) = \varnothing$,*

*then $C$ is a total $\alpha$-partial dominating set in $G \circ H$.*

*Proof.* Suppose Condition $(i)$ holds. Since $C = \bigcup_{v \in V(G)} S_v$, where $S_v$ is a total $\alpha$-partial dominating set in $H^v$ for each $v \in V(G)$, by Remark 6, $S_v$ is an $\alpha$-partial dominating set in $H^v$ for each $v \in V(G)$. It follows by Theorem 4 that $C$ is an $\alpha$-partial dominating set in $G \circ H$. Moreover, since $S_v$ is a total $\alpha$-partial dominating set in $H^v$ for each $v \in V(G)$, $C$ is a total $\alpha$-partial dominating set in $G \circ H$.

Suppose Condition $(ii)$ holds. Suppose further that $C$ is a total dominating set in $G$ and $|C| \geq \dfrac{\alpha m(n+1)-m}{n}$. Since $C$ is a dominating set in $G$, it follows by Theorem 4 that $C$ is an $\alpha$-partial dominating set in $G \circ H$. Moreover, since $C \subseteq V(G)$ is a total $\alpha$-partial dominating set in $G$, $C$ is a total $\alpha$-partial dominating set in $G \circ H$. Next, if $|C| \geq \alpha m$ and every vertex in $C$ is adjacent to some vertex in $C$, then by Theorem 4, $C$ is an $\alpha$-partial dominating set in $G \circ H$. Since $C \setminus N_G(C) = \varnothing$, $C$ is a total $\alpha$-partial dominating set in $G \circ H$. $\square$

**Remark 7.** *The converse of Theorem* 8 *is not true.*

To see this, consider the graph $P_{12} \circ P_2$ in Figure 4. Let $\alpha = \frac{1}{2}$. The shaded vertices form a $\partial_{T\alpha}$*-set* but neither of Condition $(i)$ nor $(ii)$ holds since $C = \{2,3,6,7,8\}$ is not a total dominating set in $G$ and $|C| < \alpha m$.

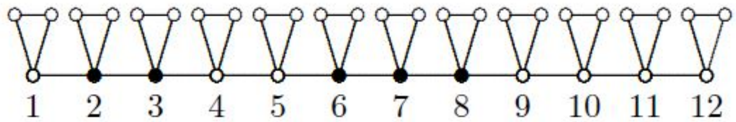


Figure 4: The graph $P_{12} \circ P_2$ with $\partial_{T\alpha}(P_{12} \circ P_2) = 5$ where $\alpha = \frac{1}{2}$.

The next result is an immediate consequence of Theorem 8.

**Corollary 4.** *Let $G$ be a nontrivial connected graph of order $m$ and $H$ be any graph of order $n$ and $\alpha \in (0,1]$ . Then*

$$\partial_{T\alpha}(G \circ H) \leq min\left\{\eta_G^t, \mu_G^t\right\},$$

*where*

$$\eta_G^t = \min\left\{|C| : C \text{ is a total dominating set in } G \text{ with } |C| \geq \frac{\alpha m(n+1)-m}{n}\right\}$$

*and*

$$\mu_G^t = \min\{|C'| : C' \subseteq V(G) \text{ with } \left|C'\right| \geq \alpha m \text{ and } C' \setminus N_G(C') = \varnothing\}.$$

**Remark 8.** *The bound in Corollary* 4 *is sharp. However, the strict inequality can be attained.*

To see this, consider the graphs shown in Figures 5 and 6. Let $\alpha = \frac{2}{5}$. The shaded vertices in Figure 5 form a $\partial_{T\alpha}$-set. Then $\partial_{T\alpha}(P_5 \circ P_5) = 2 = \min\{3,2\} = \min\{\eta_G^t, \mu_G^t\} = \mu_G^t$. Let $\alpha = \frac{3}{5}$. The shaded vertices in Figure 6 form a $\partial_{T\alpha}$-set. Then $\partial_{T\alpha}(P_{12} \circ P_2) = 6 = \min\{6,8\} = \min\{\eta_G^t, \mu_G^t\} = \eta_G^t$ while $\partial_{T\alpha}(P_5 \circ K_1) = 2 < 3 = \min\{3,3\} = \min\{\eta_G^t, \mu_G^t\}$.

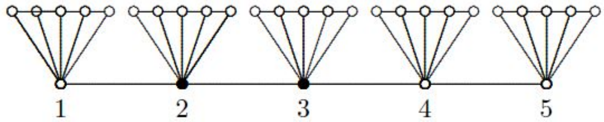


Figure 5: The graph $P_5 \circ P_5$ with $\partial_{T\alpha}(P_5 \circ P_5) = 2 = \min\{\eta_G^t, \mu_G^t\} = \mu_G^t$ where $\alpha = \frac{2}{5}$.

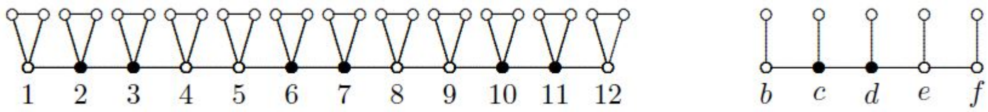


Figure 6: The graphs $P_{12} \circ P_2$ with $\partial_{T\alpha}(P_{12} \circ P_2) = 6 = \min\{\eta_G^t, \mu_G^t\} = \eta_G^t$ and $P_5 \circ K_1$ with $\partial_{T\alpha}(P_5 \circ K_1) = 2 < \min\{\eta_G^t, \mu_G^t\}$ where $\alpha = \frac{3}{5}$.

**Theorem 9.** *Let $G$ be a connected graph and $\alpha \in (0,1]$. If $S$ is an $\alpha$-partial dominating set in $G$, then*

$$\partial_{T\alpha}(G) \leq |S \cap N_G(S)| + 2|S \setminus N_G(S)|.$$

*In particular, $\partial_{T\alpha}(G) \leq 2\partial_\alpha(G)$.*

*Proof.* Let $S$ be an $\alpha$-partial dominating set in $G$. If $S$ is a total $\alpha$-partial dominating set in $G$, then $S \cap N_G(S) = S$ and $S \setminus N_G(S) = \varnothing$. Hence,

$$\partial_{T\alpha}(G) \leq |S| = |S \cap N_G(S)| + 2|S \setminus N_G(S)|.$$

So suppose that $S \setminus N_G(S) \neq \varnothing$. Choose $v_x \in V(G) \cap N_G(x)$ for each $x \in S \setminus N_G(S)$ and let $S_G = \{v_x : x \in S \setminus N_G(S)\}$. Then $|S_G| \leq |S \setminus N_G(S)|$ and $S^* = S \cup S_G$ is a total $\alpha$-partial dominating set in $G$. Thus,

$$\begin{aligned}\partial_{T\alpha}(G) \leq |S^*| &= |S| + |S_G| \\ &\leq |S \cap N_G(S)| + |S \setminus N_G(S)| + |S \setminus N_G(S)| \\ &= |S \cap N_G(S)| + 2|S \setminus N_G(S)|.\end{aligned}$$

In particular, if $S$ is an $\partial_\alpha$-set in $G$, then

$$\partial_{T\alpha}(G) \leq |S \cap N_G(S)| + |S \setminus N_G(S)| + |S \setminus N_G(S)| \leq 2|S| = 2\partial_\alpha(G).$$

□

**Theorem 10.** *Let $G$ and $H$ be connected graphs. Let $\alpha \in (0,1]$ and $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ . If either one of the following holds:*

*(i) $S$ is a total $\alpha$-partial dominating set in $G$, or*

*(ii) $S$ is an $\alpha$-partial dominating set in $G$ and $T_x$ is a total dominating set in $H$ for every $x \in S \setminus N_G(S)$,*

*then $C$ is a total $\alpha$-partial dominating set in $G[H]$.*

*Proof.* Suppose Condition $(i)$ holds. Since $S$ is a total $\alpha$-partial dominating set in $G$, $C$ is an $\alpha$-partial dominating set in $G[H]$ by Theorem 5. Let $(x,a) \in C$. Then $x \in S$. Since $S$ is a total $\alpha$-partial dominating set in $G$, there is a $y \in S$ such that $xy \in E(G)$. Pick $(y,b) \in C$. Then $(x,a)(y,b) \in E(G[H])$. Hence, $C$ is a total $\alpha$-partial dominating set in $G[H]$. Suppose Condition $(ii)$ holds. Since $T_x$, being a total dominating set, is a dominating set in $H$ for every $x \in S \setminus N_G(S)$, it follows by Theorem 5 that $C$ is an $\alpha$-partial dominating set in $G[H]$. Let $x \in S \setminus N_G(S)$. Suppose $(x,a) \in C$. Since $T_x$ is a total dominating set in $H$, there exists a $b \in T_x$ such that $ab \in E(H)$, hence $(x,b) \in C$ and $(x,a)(x,b) \in E(G[H])$. Suppose $x \in S \cap N_G(S)$. Then there exists $y \in S$ such that $xy \in E(G)$. Let $(x,a) \in C$. Pick $(y,b) \in C$. Then $(x,a)(y,b) \in E(G[H])$. Therefore, $C$ is a total $\alpha$-partial dominating set in $G[H]$. □

**Remark 9.** *The converse of Theorem* 10 *is not true.*

To see this, consider the graph $P_5[P_5]$ in Figure 7. The set $C = \{(x,a),(x,b)\}$ form a $\partial_{T\alpha}$-*set*, where $\alpha = \frac{1}{2}$, but neither of Condition $(i)$ nor $(ii)$ holds since $\{x\}$ is not a total $\alpha$-partial dominating set in $G$, and $S = \{x\}$ is an $\alpha$-partial dominating set in $G$ but $T_x = \{a,b\}$ is not a total dominating set in $H$.

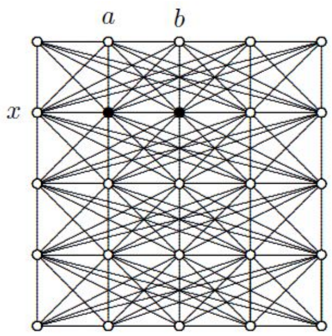


Figure 7: The graph $P_5[P_5]$ with $\partial_{T\alpha}(P_5[P_5]) = 2$ for $\alpha = \frac{1}{2}$.

**Corollary 5.** *Let $G$ and $H$ be nontrivial connected graphs and let $\alpha \in (0,1]$. Then*

$$\partial_{T\alpha}(G[H]) = \partial_{T\alpha}(G).$$

*Proof.* By Theorem 10,

$$\partial_{T\alpha}(G[H]) \leq \min\{\partial_\alpha(G) \cdot \gamma_t(H), \partial_{T\alpha}(G)\}.$$

Clearly, $\partial_{T\alpha}(G[H]) \leq \partial_{T\alpha}(G)$.

Next, let $C = \bigcup_{x\in S} (\{x\} \times T_x)$ be a $\partial_{T\alpha}$-set in $G[H]$. Suppose that there exists $y \in S \cap N_G(S)$ such that $|T_y| \geq 2$. Let $T'_x = T_x$ for all $x \in S \setminus \{y\}$, and let $T'_y = \{a\}$ where $a \in T_y$. Let $C' = \bigcup_{z\in S} (\{z\} \times T'_z)$. Since $y \in S \cap N_G(S)$, $N_{G[H]}[C'] = N_{G[H]}[C]$. Hence,

$|N_{G[H]}[C']| = |N_{G[H]}[C]| \geq \alpha|V(G[H])|$, that is, $C'$ is a total $\alpha$-partial dominating set in $G[H]$. Further, $|C'| < |C|$ since $|T'_y| = 1 < |T_y|$, a contradiction. Thus, $|T_x| = 1$ for all $x \in S \cap N_G(S)$. Now, since every element of $C$ is adjacent to an element of $C$, it follows that $|T_x| \geq 2$ for all $x \in S \setminus N_G(S)$. By Theorem 9,

$$\begin{aligned}\partial_{T\alpha}(G[H]) = |C| &= \sum_{x\in S\cap N_G(S)} |T_x| + \sum_{x\in S\setminus N_G(S)} |T_x| \\ &\geq |S \cap N_G(S)| + 2 \cdot |S \setminus N_G(S)| \\ &\geq \partial_{T\alpha}(G).\end{aligned}$$

Therefore, $\partial_{T\alpha}(G[H]) = \partial_{T\alpha}(G)$. □

**Corollary 6.** *If $\partial_\alpha(G) = 1$ and $\gamma_t(H) = 2$, then $\partial_{T\alpha}(G[H]) = 2$.*

**Theorem 11.** *Let $G$ and $H$ be nontrivial connected graphs and let $\alpha \in (0, 1]$. Then $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are total $\alpha$-partial dominating sets in $G\Box H$ if and only if $S_1$ and $S_2$ are $\alpha$-partial dominating sets in $G$ and $H$, respectively.*

*Proof.* Suppose $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are total $\alpha$-partial dominating sets in $G\Box H$. Then by Remark 6, $C_1$ and $C_2$ are $\alpha$-partial dominating sets in $G\Box H$. By Theorem 6, $S_1$ and $S_2$ are $\alpha$-partial dominating sets in $G\Box H$.

Suppose $S_1$ and $S_2$ are $\alpha$-partial dominating sets in $G$ and $H$, respectively. Then by Theorem 6, $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are $\alpha$-partial dominating sets in $G\Box H$. Let $(x, a) \in C_1 = S_1 \times V(H)$. Since $H$ is connected, there exists a vertex $b \in V(H)$ such that $ab \in E(H)$. Hence, $(x, b) \in C_1$ and $(x, a)(x, b) \in E(G\Box H)$. Similarly, if $(y, c) \in C_2 = V(G) \times S_2$, then since $G$ is connected, there exists $(z, c) \in C_2$ such that $(y, c)(z, c) \in E(G\Box H)$. Thus, $C_1$ and $C_2$ are total $\alpha$-partial dominating sets in $G\Box H$. □

**Corollary 7.** *Let $G$ and $H$ be nontrivial connected graphs. Then $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are $\alpha$-partial dominating sets in $G\Box H$ if and only if $C_1$ and $C_2$ are total $\alpha$-partial dominating sets in $G\Box H$.*

*Proof.* Suppose $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are $\alpha$-partial dominating sets in $G\Box H$. Then by Theorem 6, $S_1$ and $S_2$ are $\alpha$-partial dominating sets in $G$ and $H$, respectively. By Theorem 11, $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are total $\alpha$-partial dominating sets in $G\Box H$.

Conversely, suppose $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are total $\alpha$-partial dominating sets in $G\Box H$. By Remark 6, $C_1 = S_1 \times V(H)$ and $C_2 = V(G) \times S_2$ are $\alpha$- partial dominating sets in $G\Box H$. □

**Corollary 8.** *Let $G$ and $H$ be nontrivial connected graphs of orders $m$ and $n$, respectively, and $\alpha \in (0, 1]$. Then,*

$$\partial_{T\alpha}(G\Box H) \leq \min\left\{m \cdot \partial_\alpha(H), n \cdot \partial_\alpha(G)\right\}.$$

**Remark 10.** *The bound in Corollary 8 is sharp. However, the strict inequality can be attained.*

To see this, consider the graphs shown in Figure 8. Let $\alpha = \frac{1}{2}$. The shaded vertices in each graph form a $\partial_{T\alpha}$-set. Thus,

$$\partial_{T\alpha}(P_4\Box P_6) = 4 = \min\{4,6\} = \min\{4(1),6(1)\} = \min\{4\cdot\partial_\alpha(P_6), 6\cdot\partial_\alpha(P_4)\} = 4\cdot\partial_\alpha(P_6),$$

$$\partial_{T\alpha}(P_6\Box P_3) = 3 = \min\{6,3\} = \min\{6(1),3(1)\} = \min\{6\cdot\partial_\alpha(P_3), 3\cdot\partial_\alpha(P_6)\} = 3\cdot\partial_\alpha(P_6),$$

and

$$\partial_{T\alpha}(P_6\Box P_5) = 4 < 5 = \min\{6(1),5(1)\} = \min\{6\cdot\partial_\alpha(P_5), 5\cdot\partial_\alpha(P_6)\}.$$

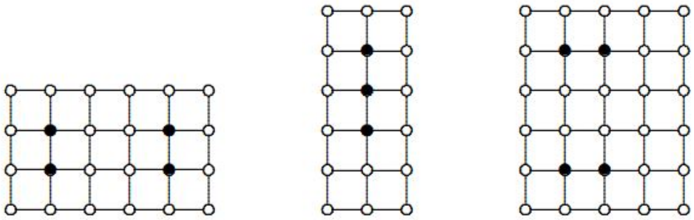

Figure 8: The graphs $P_4\Box P_6$ with $\partial_{T\alpha}(P_4\Box P_6) = 4$, $P_6\Box P_3$ with $\partial_{T\alpha}(P_6\Box P_3) = 3$ and $P_6\Box P_5$ with $\partial_{T\alpha}(P_6\Box P_5) = 4$ where $\alpha = \frac{1}{2}$.

**Corollary 9.** *Let $G$ be a connected graph of order $m$ and $K_n$ be the complete graph of order $n \geq 2$. Then,*

$$\partial_{T\alpha}(G\Box K_n) \leq \min\{m, n\cdot\partial_\alpha(G)\}.$$

**Remark 11.** *The bound in Corollary 9 is sharp. However, the strict inequality can be attained.*

To see this, consider the graphs shown in Figure 9. Let $\alpha = \frac{1}{2}$. The shaded vertices in each graph form a $\partial_{T\alpha}$-set. Thus,

$$\partial_{T\alpha}(P_2\Box K_4) = 2 = \min\{2,4(1)\} = \min\{m, 4\cdot\partial_\alpha(P_2)\} = m,$$

$$\partial_{T\alpha}(P_6\Box K_3) = 3 = \min\{6,3(1)\} = \min\{m, 3\cdot\partial_\alpha(K_3)\} = 3\cdot\partial_\alpha(K_3),$$

and

$$\partial_{T\alpha}(P_6\Box K_4) = 3 < 4 = \min\{6,4(1)\} = \min\{m, 4\cdot\partial_\alpha(P_6)\}.$$

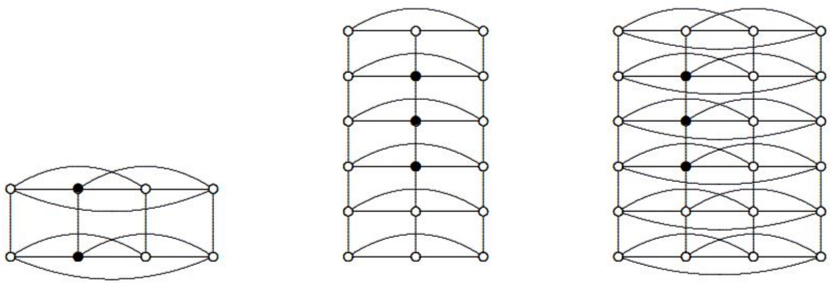

Figure 9: The graphs $P_2\Box K_4$ with $\partial_{T\alpha}(P_2\Box K_4) = 2$ , $P_6\Box C_3$ with $\partial_{T\alpha}(P_6\Box C_3) = 3$ and $P_6\Box K_4$ with $\partial_{T\alpha}(P_6\Box K_4) = 3$ where $\alpha = \frac{1}{2}$ .

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