EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 4, 2019, 1602-1611
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On Embedding Theorems in Grand Grand Nikolskii-Morrey Spaces

Alik M. Najafov[1,*], Azizgul M. Gasimova[2]

[1] *Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
[2] *Sumgait State University, Sumgait, Azerbaijan*

Abstract. In the paper we introduced a grand grand Nikolskii-Morrey spaces. Some differential and differential-difference properties of functions from this spaces are proved by means of the integral representation.

2010 Mathematics Subject Classifications: 46E35, 26B40, 26B40

Key Words and Phrases: Grand grand Nikolskii-Morrey spaces, integral representation, flexible $\lambda-$ horn condition, HÖlder condition

1. Introduction and preliminary notes

It is known that in the middle of the last century, in connection with the study of the regularity properties of differential equations with partial derivatives of a high (integer and non-integer) order, it became necessary with the introduction of Sobolev $W_p^l(G)$ $(l \in N^n)$ [20] and Nikolskii $H_p^l(G)$ $(l \in (0,\infty)^n)$ [15] spaces, etc. These spaces were further developed and generalized by many mathematicians. Considering that the grand grand Nikolskii Morrey $H^l_{p)\varkappa),a,\alpha}(G,\lambda)$ spaces introduced in this paper is wider than all previously considered spaces of this type, it will be interesting to readers.

In this paper we construct a grand grand Nikolskii-Morrey spaces $H^l_{p)\varkappa),a,\alpha}(G,\lambda)$ and we study some differential properties with help of the method of integral representation of functions in view of embedding theory. Let $G \subset R^n$ be a bounded domain, $l \in (0,\infty)^n$, $p \in (1,\infty)$,$a \in [0,1]$,$\varkappa \in (0,\infty)^n$ and $\alpha \geq 0$.

Note that the grand Lebesgue spaces $L_{p)}(G)$ $(|G| < \infty)$ introduced in [5] by T.Iwaniec and C.Sbordone. After a vast amount of research about grand Lebesgue, grand Lebesgue-Morrey, grand-grand Lebesgue-Morrey, grand-grand Sobolev-Morrey spaces (with different norms) has been studied by many mathematicians, (see, e.g. [3, 4, 6–11, 13, 16, 18, 19, 21]) e.t.c.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v12i4.3567

Email addresses: aliknajafov@gmail.com (A. Najafov), ezizgul.qasimova@mail.ru (A. Gasimova)

Definition 1. *By grand grand Nikolskii-Morrey spaces* $H^l_{p),\varkappa),a,\alpha}(G,\lambda)$ *we denote the spaces of all functions* $f \in L_1^{loc}(G)$ $(m_i > l_i - k_i > 0,\ i = 1,2,\ldots,n)$ *with the finite norm*

$$\|f\|_{H^l_{p),\varkappa),a,\alpha}(G,\lambda)} = \|f\|_{p),\varkappa),a,\alpha;G} +$$

$$+\sum_{i=1}^{n} \sup_{0<t<d_0} \frac{\left\|\Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) D_i^{k_i} f\right\|_{p),\varkappa),a,\alpha}}{t^{\lambda_i(l_i-k_i)}}, \tag{1}$$

$$\|f\|_{p),\varkappa),a,\alpha;G} = \|f\|_{L_{p),\varkappa),a,\alpha}(G)} =$$

$$= \sup_{\substack{x\in G,\\ 0<t\le d_0,\\ 0<\varepsilon<s_m}} \left(\frac{1}{t^{|\varkappa|a-\alpha\varepsilon}} \frac{\varepsilon}{|G_{t^\varkappa}(x)|} \int_{G_{t^\varkappa(x)}} |f(y)|^{p-\varepsilon}\,dy\right)^{\frac{1}{p-\varepsilon}}, \tag{2}$$

where, d_0-*diam* $G, m_i \in N, k_i \in N_0, |\varkappa| = \sum_{j=1}^n \varkappa_j,\ s_m = \min\{p-1, \frac{|\varkappa|a}{\alpha}\}$ *and* $x \in R^n$.

$$G_{t^\varkappa}(x) = G \cap I_{t^\varkappa}(x) =$$

$$= G \cap \left\{y : |y_j - x_j| < \frac{1}{2} t^{\varkappa_j}; j = 1,2,\ldots,n\right\}.$$

The Nikolskii-Morrey space $H^l_{p,\lambda}(R^n)$ and Nikolskii-Morrey type space $H^l_{p,\varphi,\beta}(G)$ studied in [1, 17]. Also note that in this paper, in theorem 2.2 it was proved that the Holder "index" is larger than in [1, 12, 14] .

Note that some properties of spaces $L_{p),\varkappa),a,\alpha}(G)$ and $H^l_{p),\varkappa),a,\alpha}(G,\lambda)$.

1) $L_{p),\varkappa),a,\alpha}(G) \to L_{p)}(G),\ H^l_{p),\varkappa),a,\alpha}(G,\lambda) \to H^l_{p)}(G,\lambda)$, i.e.

$$\|f\|_{p),G} \le C\,\|f\|_{p),\varkappa),a,\alpha;G}\,;\ \|f\|_{H^l_{p),}(G,\lambda)} \le C\,\|f\|_{H^l_{p),\varkappa),a,\alpha}(G,\lambda)} \tag{3}$$

where $H^l_p(G,\lambda)$ is grand Nikolskii space with finite norm

$$\|f\|_{H^l_{p)}(G,\lambda)} = \|f\|_{p),G} + \sum_{i=1}^{n} \sup_{0<t<d_0} \frac{\left\|\Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) f\right\|_{p)}}{t^{\lambda_i(l_i-k_i)}},$$

$$\|f\|_{p),G} = \|f\|_{L_p(G)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|G|}\int_G |f(x)|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}}.$$

2) $L_{p),\varkappa),a,\alpha}(G)$ and $H^l_{p),\varkappa),a,\alpha}(G,\lambda)$ are complete.

3) $\|f\|_{p),\varkappa),0,0;G} = \|f\|_{p),G}$ and $\|f\|_{H^l_{p),\varkappa),0,0}(G,\lambda)} = \|f\|_{H^l_{p)}(G,\lambda)}$.

Let $M_i(\cdot,y) \in C_0^\infty(R^n)$ be such that

$$S(M_i) \subset I_1 = \left\{x : |x_j| < \frac{1}{2}, f = 1,2,\ldots,n\right\},$$

Assume $0 < T \leq 1$, $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_j > 0\,(j = 1, 2, \ldots, n)$, and put

$$V = \bigcup_{0<t\leq T} \left\{ y : \left(\frac{y}{t^\lambda}\right) \in S\left(M_i\right) \right\}.$$

Clearly, $V \subset I_{T^\lambda}$ and let U be an open set contained in the domain G; henceforth we always assume that $U + V \subset G$. Put $G_{T^\varkappa}(U) = (U + I_{T^\varkappa}(x)) \bigcap G$. Obviously, if $0 < \varkappa_j \leq \lambda_j\,(j = 1, 2, \ldots, n)$, then $I_{T^\lambda} \subset I_{T^\varkappa}$ and thereby $U + V \subset G_{T^\varkappa}(U) = Q$.

Lemma 1. *Let* $1 < p < q \leq r \leq \infty$;$0 < |\varkappa| \leq \frac{|\lambda|+\alpha\varepsilon}{1+a}$; $0 < t, \eta \leq T \leq d_0$;$0 < \gamma < \gamma_0$; $\nu = (\nu_1, \ldots, \nu_n)\,, \nu_j$ *are integers* $(j = 1, 2, \ldots, n)$; $\Delta_i^{m_i}\left(t^{\lambda_i}\right) f \in L_{p),\varkappa),a,\alpha}(G)$ *and let*

$$\bar{\mu}_i = \lambda_i l_i - |\nu, \lambda| - (|\lambda| - |\varkappa| a - |\varkappa| + \alpha\varepsilon)\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right), \tag{4}$$

$$E_\eta^i(x) = \int_0^\eta t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} \varphi_i(x,t)dt, \tag{5}$$

$$E_{\eta,T}^i(x) = \int_\eta^T t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} \varphi_i(x,t)dt, \tag{6}$$

$$E(x) = \int_{R^n} f(x+y+z)\Omega\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}\right)\Omega^{(\nu)}\left(\frac{z}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}\right) dydz, \tag{7}$$

where

$$|\nu, \lambda| = \sum_{j=1}^n \nu_j \lambda_j,$$

$$\varphi_i(x,t) = \int_{R^n} \int_{-\infty}^{\infty} M_i\left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda}\right) \times$$

$$\times S_i\left(\frac{u}{t^{\lambda_i}}, \frac{\rho_i\left(t^\lambda, x\right)}{2t^{\lambda_i}}, \frac{1}{2}\rho_i'\left(t^{\lambda_i}, x\right)\right) \Delta_i^{m_i}\left(\delta^{\lambda_i} u\right) \times$$

$$\times f\left(x + y + ue_i\right) dudy. \tag{8}$$

Then for any $\bar{x} \in U$ *the following inequalities*

$$\sup_{\bar{x}\in U} \left\|E_\eta^i\right\|_{q-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} \leq$$

$$\leq C_1 \left\|t^{-\lambda_i l_i} \Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) f\right\|_{p)\varkappa),a,\alpha;Q} \varepsilon^{-\frac{1}{p-\varepsilon}} \gamma^{\frac{|\varkappa|(a+1)}{q-\varepsilon}} \eta^{\bar{\mu}_i}\left(\bar{\mu}_i > 0\right), \tag{9}$$

$$\sup_{\bar{x}\in U} \left\|E_{\eta,T}^i\right\|_{q-\varepsilon, U_{\gamma^\varkappa}(\bar{x})} \leq$$

$$\leq C_2 \left\|t^{-\lambda_i l_i} \Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) f\right\|_{p),\varkappa),a,\alpha;Q} \varepsilon^{-\frac{1}{p-\varepsilon}} \gamma^{\frac{|\varkappa|(a+1)}{q-\varepsilon}} \times$$

$$\times\begin{cases} T^{\bar{\mu}_i}, & for\ \ \bar{\mu}_i>0,\\ \ln\frac{T}{\eta}, & for\ \ \bar{\mu}_i=0,\\ \eta^{\bar{\mu}_i}, & for\ \ \bar{\mu}_i<0,\end{cases} \tag{10}$$

$$\sup_{\bar{x}\in U}\|E\|_{q-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}\leq$$

$$\leq C_3\,\|f\|_{p),\varkappa),a,\alpha;Q}\ t^{|\lambda|-(|\lambda|-|\varkappa|-|\varkappa|a)\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)}\varepsilon^{-\frac{1}{p-\varepsilon}}\gamma^{\frac{|\varkappa|(a+1)}{q-\varepsilon}} \tag{11}$$

is hold, where and $U_{\gamma^{\varkappa}}(\bar{x})=\left\{x:|x_j-\bar{x}_j|<\frac{1}{2}\gamma^{\varkappa_j}, j=1,2,\ldots,n\right\}$, C_1 *and* C_2 *are constants independent of* f,γ,η *and* T.

Proof. Applying sequentially the generalized the Minkowskii inequality for any $\bar{x}\in U$

$$\left\|E_{\eta}^{i}\right\|_{q-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}\leq\int_0^{\eta}t^{-1-|\lambda|-|\nu,\lambda|-\lambda_i}\,\|\varphi_i(\cdot,t)\|_{q-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}\,dt, \tag{12}$$

and from the Hölder inequality $(q\leq r)$ we obtain

$$\|\varphi_i(\cdot,t)\|_{q-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}\leq\|\varphi_i(\cdot,t)\|_{r-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}\,\gamma^{|\varkappa|\left(\frac{1}{q-\varepsilon}-\frac{1}{r-\varepsilon}\right)}. \tag{13}$$

Now estimate the norm $\|\varphi_i(\cdot,t)\|_{r-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}$. Let X be a characteristic function of the set $S(M_i)$. Noting that $1<p<r\leq\infty$, $s\leq r$ $\left(\frac{1}{s}=1-\frac{1}{p-\varepsilon}+\frac{1}{r-\varepsilon}\right)$ and

$$\left|M_i\int_{-\infty}^{+\infty}S_i\Delta_i^{m_i}f\,du\right|=\left(\left|\int_{-\infty}^{+\infty}S_i\Delta_i^{m_i}f\,du\right|^{p-\varepsilon}|M_i|^s\right)^{\frac{1}{r-\varepsilon}}\times$$

$$\times\left(\left|\int_{-\infty}^{+\infty}S_i\Delta_i^{m_i}f\,du\right|^{p-\varepsilon}x\right)^{\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}}\left(|M_i|^s\right)^{\frac{1}{s}-\frac{1}{r-\varepsilon}}$$

and apply to $|\varphi_i|$ the Holder inequality $\left(\frac{1}{r-\varepsilon}+\left(\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}\right)+\left(\frac{1}{s}-\frac{1}{r-\varepsilon}\right)=1\right)$,

$$\|\varphi_i(\cdot,t)\|_{r-\varepsilon,U_{\gamma^{\varkappa}}(\bar{x})}\leq$$

$$\leq C^1\sup_{x\in U_{\gamma^{\varkappa}}(\bar{x})}\left(\int_{R^n}\left|\int_{-\infty}^{+\infty}S_i\left(\frac{u}{t^{\lambda_i}},\frac{\rho_i\left(t^{\lambda_i},x\right)}{t^{\lambda_i}},\frac{1}{2}\rho_i'\left(t^{\lambda_i},x\right)\right)\Delta_i^{m_i}\left(t^{\lambda_i}\right)f\left(x+y+ue_i\right)du\right|^{p-\varepsilon}\times\right.$$

$$\left.\times X\left(\frac{y}{t^{\lambda_i}}\right)dy\right)^{\frac{1}{p-\varepsilon}-\frac{1}{r-\varepsilon}}\sup_{y\in v}\times$$

$$\left(\int_{U_{\gamma^{\varkappa}}(\bar{x})}\left|\int_{-\infty}^{+\infty}S_i\left(\frac{u}{t^{\lambda_i}},\frac{\rho_i\left(t^{\lambda_i},x\right)}{t^{\lambda_i}},\frac{1}{2}\rho_i'\left(t^{\lambda_i},x\right)\right)\Delta_i^{m_i}\left(t^{\lambda_i}\right)f\left(x+y+ue_i\right)du\right|^{\frac{1}{p-\varepsilon}}\times\right.$$

$$\times\left(\int_{R^n}\left|M_i^1\left(\frac{y}{t^{\lambda}}\right)\right|^s dy\right)^s, \tag{14}$$

suppose that $|M_i(x,y,z)| \le C^1 \left|M_i^1(x)\right|$.

Obviously, if $|\varkappa| \le \frac{|\lambda|}{1+a}$, $0<t\le 1$, then $Q_{t^\lambda}(x) \subset Q_{t^\varkappa}(x)$. For every $x \in U$ we have

$$\int_{R^n}\left|\int_{-\infty}^{+\infty} S_i\left(\frac{u}{t^{\lambda_i}}, \frac{\rho_i\left(t^{\lambda_i},x\right)}{t^{\lambda_i}}, \frac{1}{2}\rho_i'\left(t^{\lambda_i},x\right)\right)\Delta_i^{m_i}\left(t^{\lambda_i}\right) f(x+y+ue_i du)\right|^{p-\varepsilon} X\left(\frac{y}{t^\lambda}\right)dy \le$$

$$\le \int_{Q_{t^\varkappa}(x)}\left|\int_{-\infty}^{+\infty} S_i\left(\frac{u}{t^{\lambda_i}}, \frac{\rho_i\left(t^{\lambda_i},x\right)}{t^{\lambda_i}}, \frac{1}{2}\rho_i'\left(t^{\lambda_i},x\right)\right)\Delta_i^{m_i}\left(t^{\lambda_i}\right) f(y+ue_i)\right|^{p-\varepsilon} dy \le$$

$$\le \left\|t^{-\lambda_i l_i}\Delta_i^{m_i}\left(t^{\lambda_i}\right)f\right\|_{p-\varepsilon,Q_{t^\varkappa}(x)}^{p-\varepsilon} t^{\lambda_i l_i(p-\varepsilon)} \le$$

$$\le \left\|t^{-\lambda_i}\Delta_i^{m_i}\left(t^{\lambda_i}\right)f\right\|_{p),\varkappa),a,\alpha;Q}^{p-\varepsilon} \varepsilon^{-1} t^{|\varkappa|+|\varkappa|a+\lambda_i l_i(p-\varepsilon)-\alpha\varepsilon}, \tag{15}$$

for $y \in V$

$$\int_{U_{\gamma^\varkappa}(\bar{x})}\left|\int_{-\infty}^{+\infty} S_i\left(\frac{u}{t^{\lambda_i}}, \frac{\rho_i\left(t^{\lambda_i},x\right)}{t^{\lambda_i}}, \frac{1}{2}\rho_i'\left(t^{\lambda_i},x\right)\right)\Delta_i^{m_i}\left(t^{\lambda_i}\right) f(x+y+ue_i du)\right|^{p-\varepsilon} dx \le$$

$$\le \int_{Q_{\gamma^\varkappa}(\bar{x}+y)}\left|\int_{-\infty}^{+\infty} S_i\left(\frac{u}{t^{\lambda_i}}, \frac{\rho_i\left(t^{\lambda_i},x\right)}{t^{\lambda_i}}, \frac{1}{2}\rho_i'\left(t^{\lambda_i},x\right)\right)\Delta_i^{m_i}\left(t^{\lambda_i}\right) f(x+ue_i)du\right|^{p-\varepsilon} dx \le$$

$$\le t^{\lambda_i l_i(p-\varepsilon)}\left\|t^{-\lambda_i l_i}\Delta_i^{m_i}\left(t^{\lambda_i}\right)f\right\|_{p-\varepsilon,Q_{\gamma^\varkappa}(x)}^{p-\varepsilon} \le$$

$$\le \left\|t^{-\lambda_i l_i}\Delta_i^{m_i}\left(t^{\lambda_i}\right)f\right\|_{p),\varkappa),a,\alpha;Q}^{p-\varepsilon} t^{\lambda_i l_i(p-\varepsilon)}\gamma^{|\varkappa|+|\varkappa|a-\alpha\varepsilon}\varepsilon^{-1}. \tag{16}$$

$$\int_{R^n}\left|M_i^1\left(\frac{y}{t^\lambda}\right)\right|^s dy = t^{|\lambda|}\left\|M_1\right\|_s^s \tag{17}$$

From inequalities (13)-(17) for $r=q$ that

$$\left\|E_\eta^i\right\|_{q-\varepsilon,U_{\gamma^\varkappa}(\bar{x})} \le C_1\left\|t^{-\lambda_i l_i}\Delta_i^{m_i}(t^{\lambda_i})f\right\|_{p),\varkappa),a,\alpha,Q}\varepsilon^{-\frac{1}{p-\varepsilon}}\gamma^{\frac{|\varkappa|a+|\varkappa|-\alpha\varepsilon}{q-\varepsilon}}\times$$

$$\times t^{|\lambda|-(|\lambda|-|\varkappa|-|\varkappa|a+\alpha\varepsilon)\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)} \tag{18}$$

Unseating this inequality in (12), for all $\bar{x} \in U$, we see that

$$\left\|E_\eta^i\right\|_{q-\varepsilon,U_{\gamma^\varkappa}(\bar{x})} \le C_2\left\|t^{-\lambda_i l_i}\Delta_i^{m_i}(t^{\lambda_i})f\right\|_{p),\varkappa),a,\alpha;Q}\varepsilon^{-\frac{1}{p-\varepsilon}}\gamma^{\frac{|\varkappa|(1+a)}{q-\varepsilon}}\eta^{\bar{\mu}_i}(\mu_i>0)$$

Similarly, we can prove (10) and (11).

2. Main results

.

We proved two theorems on the properties of the functions from spaces $H^l_{p),\varkappa),a,\alpha}(G,\lambda)$.

Theorem 1. *Let $G \subset R^n$ be an open bounded set satisfy the flexible $\lambda-$ horn condition (see [2]); $1 < p < q \leq \infty$; $|\varkappa| \leq \frac{\lambda+\alpha\varepsilon}{1+a}$; $\nu = (\nu_1, \dots, \nu_n)$, $\nu_j \geq 0$ are integers $(j = 1, \dots, n)$; $\bar{\mu}_i > 0 (i = 1, 2, \dots, n)$ and let $f \in H^l_{p),\varkappa),a,\alpha}(G,\lambda)$.*

Then $D^\nu : H^l_{p),\varkappa),a,\alpha}(G,\lambda) \to L_{q-\varepsilon}(G)$ hold for any $\varepsilon \in (0, s_m)$, and moreover, the following inequality is valid

$$\|D^\nu f\|_{q-\varepsilon,G} \leq C(\varepsilon) \Big(T^{\bar{\mu}_0} \|f\|_{p),\varkappa),a,\alpha;G} +$$

$$+ \sum_{i=1}^n T^{\bar{\mu}_i} \sup_{0<t<d_0} \left\| \frac{\Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) f}{t^{\lambda_i l_i}} \right\|_{p),\varkappa),a,\alpha} \Bigg) \tag{19}$$

In particular, if $\bar{\mu}_{i,0} = \lambda_i l_i - |\nu,\lambda| - (|\lambda| - |\varkappa| - |\varkappa| a + \alpha\varepsilon) \frac{1}{p-\varepsilon} > 0$ $(i = 1, 2, \dots, n)$ if $D^\nu f$ is continuous on G and

$$\sup_{x \in G} |D^\nu f(x)| \leq C(\varepsilon) \Big(T^{\bar{\mu}_{0,0}} \|f\|_{p),\varkappa),a,\alpha;G} +$$

$$+ \sum T^{\bar{\mu}_{i,0}} \sup_{0<t<d_0} \left\| \frac{\Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) f}{t^{\lambda_i l_i}} \right\|_{p),\varkappa),a,\alpha} \Bigg) \tag{20}$$

moreover $0 < T \leq d_0$, $C(\varepsilon) = C\varepsilon^{-\frac{1}{p-\varepsilon}}$ and C is a constant independent of f, T and ε.

Proof. At first note that in the conditions of our theorem there exists a generalized derivatives $D^\nu f$ on G Indeed, from the condition $\bar{\mu}_i > 0\,(i = 1, 2, \dots, n)$ it follows that for $f \in H^l_{p),\varkappa),a,\alpha}(G,\lambda) \to H^l_{p)}(G,\lambda) \to H^l_{p-\varepsilon}(G,\lambda)$ $(p - \varepsilon > 1)$. Then $D^\nu f$ exists on G and belongs to $L_{p-\varepsilon}(G)$ and for almost each point $x \in G$ the integral representation in [2].

$$D^\nu f(x) = f^{(\nu)}_{T^\lambda}(x) + (-1)^{|\nu|} \int_0^T \sum_{i=1}^n \int_{R^n} \int_{-\infty}^{\infty} t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} \times$$

$$\times \Psi_i^{(\nu)} \left(\frac{y}{t^\lambda}, \frac{\rho(t^\lambda, x)}{t^\lambda} \right) S_i \left(\frac{u}{t^{\lambda_i}}, \frac{\rho(t^\lambda, x)}{2t^{\lambda_i}}, \frac{1}{2}\rho_i'(t^{\lambda_i}, x) \right) \times$$

$$\times \Delta_i^{m_i}(\delta^{\lambda_i} u) f(x + y + ue_i) du dy dt, \tag{21}$$

$$f^{(\nu)}_{T^\lambda}(x) = (-1)^{|\nu|} T^{-2|\lambda|-|\nu,\lambda|} \int_{R^n} \int_{R^n} f(x + y + z) \times$$

$$\times \Omega \left(\frac{y}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda} \right) \Omega^{(\nu)} \left(\frac{z}{t^\lambda}, \frac{\rho\left(t^\lambda, x\right)}{t^\lambda} \right) dy dz, \tag{22}$$

$0 < T \leq d_0$ and $\Omega(\cdot, y), \Psi_i(\cdot, y) \in C_0^\infty(R^n), S_i(\cdot, y, z) \in C_0^\infty(R)$.
Recall that the flexible $\lambda-$ horn and $x + V$ is the support of the representation (21) and (22). Applying the Minkowski inequality, from identities (21) and (22) we get

$$\|D^\nu f\|_{q-\varepsilon,G} \leq \left\|f_{T^\lambda}^{(\nu)}\right\|_{q-\varepsilon,G} + \sum_{i=1}^{n} \left\|E_T^i\right\|_{q-\varepsilon,G}. \tag{23}$$

By (11) for $U = G$, $M_i = \Omega$, $t = T$ we get

$$\left\|f_{T^\lambda}^{(\nu)}\right\|_{q-\varepsilon,G} \leq C_1(\varepsilon)\|f\|_{p),\varkappa),a,\alpha;G} \cdot T^{\bar{\mu}_0}, \tag{24}$$

by (11) for $U = G$, $M_i = \Psi_i$, $\eta = T$ we get

$$\left\|E_T^i\right\|_{q-\varepsilon,G} \leq C_2(\varepsilon)\left\|t^{-\lambda_i l_i}\Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) f\right\|_{p),\varkappa),a,\alpha} T^{\bar{\mu}_i}. \tag{25}$$

Substituting (25) and (24) in (23), we get inequality (19).

Now let conditions $\bar{\mu}_{i,0} > 0\,(i = 1, 2, \ldots, n)$. Show that $D^\nu f$ is continuous on G. By (21) and (22), using (23) for $q = \infty$ and $\bar{\mu}_i(q = \infty) = \bar{\mu}_{i,0} > 0$ $(i = 1, 2, \ldots, n)$ we obtain

$$\left\|D^\nu f - f_{T^\lambda}^{(\nu)}\right\|_{\infty,G} \leq C(\varepsilon)\sum_{i=1}^{n} T^{\bar{\mu}_{i,0}} \sup_{0<t<d_0}\left\|\frac{\Delta_i^{m_i}\left(t^{\lambda_i}, G_{t^\lambda}\right) f}{t^{\lambda_i l_i}}\right\|_{p),\varkappa),a,\alpha}.$$

As $T \to 0$, the left side of this inequality tends to zero, since $f_{T^\lambda}^{(\nu)}(x)$ is continuous on G and the convergence in $L_\infty(G)$ coincides with the uniform convergence. Then the limit function $D^\nu f$ is continuous on G. Theorem 2.1 is proved.

Let ξ be an $n-$ dimensional vector.

Theorem 2. *Suppose that the domain G the parameters p, q and vector ν satisfy the condition of theorem 2.1. If $\bar{\mu}_i > 0\,(i = 1, \ldots, n)$ then $D^\nu f$ satisfies the Holder condition with exponent σ on G in the metric of $L_{q-\varepsilon}$; more exactly*

$$\|\Delta(\xi, G)\, D^\nu f\|_{q-\varepsilon,G} \leq C(\varepsilon)\|f\|_{H^l_{p),\varkappa),a,\alpha}(G,\lambda)}|\xi|^\sigma, \tag{26}$$

σ is an arbitrary number satisfying the inequalities:

$$\begin{gathered} 0 \leq \sigma \leq 1, \quad if \quad \frac{\bar{\mu}^0}{\lambda_0} > 1; \\ 0 \leq \sigma < 1, \quad if \quad \frac{\bar{\mu}^0}{\lambda_0} = 1; \\ 0 \leq \sigma \leq \frac{\bar{\mu}^0}{\lambda_0}, \quad if \quad \frac{\bar{\mu}^0}{\lambda_0} < 1, \end{gathered} \tag{27}$$

where $\bar{\mu}^0 = \min(\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_n)$, $\lambda_0 = \max\limits_{j=1,\ldots,n} \lambda_j$.

If $\bar{\mu}_{i,0} > 0 \ \ (i = 1, \ldots, n)$, *then*

$$\sup_{x \in G} |\Delta(\xi, G) D^{\nu} f(x)| \leq C(\varepsilon) \|f\|_{H^{l}_{p),\varkappa),a,\alpha}(G,\lambda)} |\xi|^{\sigma_0}, \tag{28}$$

where σ_0 *satisfy the some conditions as* σ *with* $\bar{\mu}_{i,0}$ *instead of* $\bar{\mu}_i$ *and* $C(\varepsilon) = C\varepsilon^{-\frac{1}{p-\varepsilon}}$ *and* C *is a constant independent of* f *and* ε.

Proof. By Lemma 8.6 of [2] there is a domain $G_{\omega} \subset G(\omega = k\, r_{\lambda}(x),\, k > 0, r_{\lambda}(x) = \rho_{\lambda}(x, \partial G), x \in G)$. Suppose that $|\xi|_{\lambda} < \omega$, then segment joining the points of the segment with the some kernels. Making simple transformations, we obtain

$$|\Delta(\xi, G) D^{\nu} f(x)| \leq C_1 T^{-2|\lambda| - |\nu,\lambda|} \times$$

$$\times \int_{R^n} \int_{R^n} |f(x+y+z)| \left| \Omega^{(\nu)} \left(\frac{y-\xi}{t^{\lambda}}, \frac{\rho(t^{\lambda}, x)}{t^{\lambda}} \right) - \Omega^{(\nu)} \left(\frac{y}{t^{\lambda}}, \frac{\rho(t^{\lambda}, x)}{t^{\lambda}} \right) \right| dydz +$$

$$C_2 \sum_{i=1}^{n} \left\{ \int_{0}^{|\xi|^{\frac{1}{\lambda_0}}} t^{-1-|\lambda|-\lambda_i-|\nu,\lambda|} \int_{R^n} \int_{-\infty}^{+\infty} \left| S_i \left(\frac{u}{t^{\lambda_i}}, \frac{\rho(t^{\lambda_i}, x)}{t^{\lambda_i}}, \frac{1}{2} \rho_i'(t^{\lambda_i}, x) \right) \right| \times \right.$$

$$\times \left\| \Psi_i^{(\nu)} \left(\frac{y}{t^{\lambda}}, \frac{\rho(t^{\lambda}, x)}{t^{\lambda}} \right) \right\| \left| \Delta_i^{m_i} \left(\delta^{\lambda_i} u \right) f(x+y+ue_i) \right| dudydt +$$

$$+ \int_{|\xi|^{\frac{1}{\lambda_0}}}^{T} t^{-|\lambda|-\lambda_i-|\nu,\lambda|} \int_{R^n} \int_{-\infty}^{+\infty} \left| S_i \left(\frac{u}{t^{\lambda_i}}, \frac{\rho(t^{\lambda_i}, x)}{t^{\lambda_i}}, \frac{1}{2} \rho_i'(t^{\lambda_i}, x) \right) \right|$$

$$\left. \times \left\| \Psi_i^{(\nu)} \left(\frac{y}{t^{\lambda}}, \frac{\rho(t^{\lambda}, x)}{t^{\lambda}} \right) \right\| \int_{0}^{1} \left| \Delta_i^{m_i} \left(\delta^{\lambda_i} u \right) f(x+y+ue_i+\omega\xi) \right| dudydtd\omega \right\} =$$

$$= C_1 A(x, \xi) + C_2 \sum_{i=1}^{n} \left(B(x, \xi) + F(x, \xi) \right), \tag{29}$$

where $0 < T < t_0$. We also assume that $|\xi| < T^{\lambda}$, and consequently $|\xi| \leq \min(\omega^{\lambda_0}, T^{\lambda_0})$. If $x \in G \backslash G_{\omega}$ then by definition $\Delta(\xi, G) D^{\nu} f(x) = 0$.
By (29)

$$\|\Delta(\xi, G) D^{\nu} f\|_{q-\varepsilon, G} = \|\Delta(\xi, G) D^{\nu} f\|_{q-\varepsilon, G_{\omega}} \leq C_1 \|A(\cdot, \xi)\|_{q-\varepsilon, G_{\omega}} +$$

$$+ C_2 \sum \left(\|B(\cdot, \xi)\|_{q-\varepsilon, G_{\omega}} + \|F(\cdot, \xi)\|_{q-\varepsilon, G_{\omega}} \right) \tag{30}$$

$$A(x, \xi) \leq \sum_{j=1}^{n} T^{-\lambda_j - 2|\lambda| - |\nu,\lambda|} \int_{0}^{|\xi|} d\gamma \times$$

$$\times \int_{R^n}\int_{R^n} |f(x+y+z+\xi e_\gamma)| \left| D_j \Omega^{(\nu)} \left(\frac{y}{T^\lambda}, \frac{p\left(T^\lambda, x\right)}{T^\lambda} \right) \Omega \left(\frac{z}{T^\lambda}, \frac{p\left(T^\lambda, x\right)}{T^\lambda} \right) \right| dydz.$$

taking into account $\xi e_\gamma + G_\omega \subset G$ and applying the generalized Minkowski inequality and by (11) for $U = G$ we have

$$\|A(\cdot,\xi)\|_{q-\varepsilon,G_\omega} \le C_1(\varepsilon)\,|\xi|\,\|f\|_{p),\varkappa),a;G}\,. \tag{31}$$

By means of inequality (10) for $U = G$, $M_i = \Psi_i, \eta = |\xi|^{\frac{1}{\lambda_0}}$ we obtain

$$\|B(\cdot,\xi)\|_{q-\varepsilon,G_\omega} \le C_2(\varepsilon) \left\| t^{-\lambda_i l_i} \Delta_i^{m_i}(t^{\lambda_i}, G_{t^\lambda}) f \right\|_{p),\varkappa),a,\alpha} |\xi|^{\frac{\bar{\mu}_i}{\lambda_0}}, \tag{32}$$

and by means inequality (10) for $U = G$, $M_i = \Psi_i, \eta = |\xi|^{\frac{1}{\lambda_0}}$ we obtain

$$\|F(\cdot,\xi)\|_{q-\varepsilon,G_\omega} \le C_3(\varepsilon)\,|\xi|^\sigma \left\| t^{-\lambda_i l_i} \Delta_i^{m_i}(t^{\lambda_i}, G_{t^\lambda}) f \right\|_{p),\varkappa),a,\alpha}. \tag{33}$$

From inequalities (30)-(33) we get the required inequality.
Now suppose that $|\xi| > \min(\omega^{\lambda_0}, T^{\lambda_0})$, then

$$\|\Delta(\xi,G)\,D^\nu f\|_{q-\varepsilon,G} \le 2\,\|D^\nu f\|_{q-\varepsilon,G} \le C(\omega,T)\,\|D^\nu f\|_{q-\varepsilon,G}\,|\xi|^\sigma.$$

Estimating $\|D^\nu f\|_{q-\varepsilon,G}$ by means of (19) we obtain the sought inequality in this case as well. The theorem is proved.

References

[1] A Akbulut, A Eroglu, and A Najafov. Some embedding theorems on the Nikolskii-Morrey type spaces. *Advances in Analysis*, 1(1), 2016.

[2] O V Besov, V P Ilyin, and S M Nikolskii. *Integral representations of functions and embeddings theorems.* M.Nauka, 1996.

[3] A Fiorenza, Formica, and A Gogatishvili. On grand and small Lebesgue and Sobolev spaces and some applications to PDE's. *Differential Equations and Applications*, 10(1), 2018.

[4] A Fiorenza and C E Karadzhov. Grand and small Lebesgue spaces and their analogs. *J.Anal. Appl.*, 23(4), 2004.

[5] T Iwaniec and C Sbordone. On the integrability of the Jacobian under minimal hypoteses. *Arch. Ration. mach. Anal.*, 119:129–143, 1992.

[6] V Kokilashvili. The Riemann boundary value problem for analytic functions in the frame of grand $\Re$ spaces. *Bull. Georgian Nat. Acad. Sci.*, 4(1), 2010.

[7] V Kokilashvili and A Meskhi. Trace inequalities for fractional integrals in grand Lebesgue spaces. *Studia Math.*, 210(2), 2012.

[8] V Kokilashvili, A Meskhi, and H.Rafeiro. Estimates for nondivergence elliptic equations with VMO coefficients in generalized grand Morrey spaces. *Comp.Var. Ellip. Equations*, 8(59), 2014.

[9] A Meskhi. Maximal functions, potentials and singular integrals in grand Morrey spaces. *Comp.Var. Ellip. Equations.*, 56(10-11), 2011.

[10] Y Mizuta and T Ohno. Tradingers exponential integrability for Riesz potentials of function in generalized grand Morrey spaces. *J.Math. Anal. Appl.*, 420(1), 2014.

[11] A M Najafov and S T Alekberli. On properties functions from grand grand Sobolev Morrey spaces. *Journal of Baku Engineering University*, 2(1), 2018.

[12] A M Najafov and A T Orujova. On the solution of a class of partial differential equations. *Electron. J. Qual. Theory Differ. Equ.*, (44), 2017.

[13] A M Najafov and N R Rustamova. Some differential properties of anisotropic grand Sobolev-Morrey spaces. *Trans.of A. Razmadze Math. Ins.*, 172(1), 2018.

[14] A M Najafov, N R Rustamova, and S T Alekberli. On solvability of a quasi-elliptic partial differential equations. *Journal of Elliptic and Parabolic equations*, 5(1), 2019.

[15] S M Nikolskii. Properties of certain classes of functions of several variables on differentiable manifolds.

[16] H Rafeiro. A note on boundednes of operators in grand grand Morrey spaces. *Advances in Harmonic analysis and operator theory*, 229:349–356, 2013.

[17] I Ross. A Morrey -Nikolskii inequality. *Proc. Amer. Math.Soc.*, 78:97–102, 1980.

[18] S G Samko and S M Umarkhadzhiev. On Iwaniec-Sbordone spaces on sets which may have infinite measure. *Azerb. Journal of Math.*, 1(1), 2011.

[19] C Sbordone. Grand Sobolev spaces and their applications to variational problems. *Le Mathematiche.*, 1(2), 1996.

[20] S L Sobolev. On a theorem of functional analysis. *Math. Sbor.*, 4(46), 1938.

[21] S M Umarkhadzhiev. The boundedness of the Riesz potential operator from generalized grand Lebesque spaces to generalized grand Morrey spaces. *Advances and Applications Basel.*, 242:363–373, 2014.