EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 12, No. 4, 2019, 1689-1700
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Sensitivity of fuzzy nonautonomous dynamical systems

Yaoyao Lan

Department of Mathematics, Chongqing University of Arts and Sciences, Yongchuan, China

Abstract. This paper is devoted to a study of relations between two forms of sensitivity of nonautonomous dynamical system and its induced fuzzy systems. More specially, we study strong sensitivity and mean sensitivity in an original nonautonomous system and its connections with the same ones in its induced systems, including set-valued system and fuzzified system.

2010 Mathematics Subject Classifications: 03E72, 37B55

Key Words and Phrases: Nonautonomous, dynamical systems, fuzzy, strong sensitivity, mean sensitivity

1. Introduction

Let $f_n : X \to X$ be a sequence of continuous maps acting on a compact metric space (X, d). A nonautonomous discrete dynamical systems is a pair $(X, \{f_n\}_{n=1}^{\infty})$ defined by:

$$x_{n+1} = f_n(x_n), \quad n \geq 1, \tag{1}$$

Note that the autonomous dynamical system is a special case of system (1) when $f_n = f$ for all $n \geq 1$.

For other notions and notations mentioned in this section, we refer to Section 2.

The dynamics of autonomous dynamical system have been extensively studied and many elegant results have been obtained [1, 2, and the references therein]. Nonautonomous systems, also called sequences of dynamical systems, present situations that the dynamics vary with time. These systems can be very complicated and naturally appear as a suitable model to describe real processes. The rich dynamics of non-autonomous discrete systems attract the interest of several researchers, obtaining results on chaotic properties [3]-[7].

Sensitivity is essential for the concept of chaos. A study of stronger forms of sensitivity has been initiated by Moothathu [8]. Along this line, several elegant results have been obtained [9, 10]. A series of research focus on mean sensitivity [11, 12]. Until very recently, sensitivity of nonautonomous dynamical system has been discussed [13]. Motivated by the idea in [9], we discuss different kinds of sensitivities in nonautonomous dynamical systems in this paper.

DOI: https://doi.org/10.29020/nybg.ejpam.v12i4.3573

Email addresses: `yylanmath@163.com` (Y. Lan)

On the other hand, it is well known that every given discrete dynamical system uniquely induces its fuzzified counterpart, i.e., a discrete system on the space of fuzzy sets. It is natural to investigate the relation between dynamical properties of the original and fuzzified systems. Actually, there are quite a few elegant results have been obtained [14]-[21].

In this paper, we initiate a preliminary study of relations between several forms of sensitivity of the original and its fuzzified nonautonomous dynamical systems. Below, basic notions are introduced in Section 2. Main results are presented in Section 3, where the relations between two forms of sensitivity of the original and fuzzified systems have been discussed, respectively.

2. Basic concepts and notations

2.1. Metric space of fuzzy sets

Let (X, d) denote a compact metric space and let $\mathbb{K}(X)$ be the class of all non-empty and compact subsets of X. Define the ε-neighborhood of a nonempty subset A in X to be the set

$$\mathbb{U}_d(A, \varepsilon) = \{x \mid d(x, A) < \varepsilon\},$$

where $d(x, A) = \inf_{a\in A} \|x - a\|$.

The Hausdorff separation $\rho(A, B)$ of $A, B \in \mathbb{K}(X)$ is defined by

$$\rho(A, B) = \inf\{\varepsilon > 0 |\ A \subseteq \mathbb{U}(B, \varepsilon)\},$$

The Hausdorff metric d_H on $\mathbb{K}(X)$ is defined by letting

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

For a compact metric space X, the topology generated by d_H coincides with the finite topology. It is known that the set of all finite subsets of X, denote by $\mathbb{L}(X)$, is dense in $\mathbb{K}(X)$.

Define $\mathbb{F}(X)$ as the class of all upper semicontinuous fuzzy sets $u : X \to [0, 1]$ such that $[u]_\alpha \in \mathbb{K}(X)$, where *$\alpha$-cuts* and the *support* of u are defined by

$$[u]_\alpha = \{x \in X | u(x) \geq \alpha\}, \alpha \in [0, 1],$$

and

$$supp(u) = \overline{\{x \in X | u(x) > 0\}},$$

respectively.

Moreover, for each $x \in X$, we denote $\hat{x}$ the characteristic function of x, it is clear that for for all $x \in X$, $\hat{x} \in \mathbb{F}(X)$ and $[\hat{x}]_\alpha = \{x\}$ for $\alpha \in (0, 1]$. Denote $\emptyset_X$ the *empty fuzzy set* ($\emptyset_X(x) = 0$ for all $x \in X$).

A *levelwise metric* d_∞ on $\mathbb{F}(X)$ is defined by

$$d_\infty(u, v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha),$$

for all $u, v \in \mathbb{F}(X)$. It is well known that if (X, d) is complete, then $(\mathbb{F}(X), d_\infty)$ is also complete but is not compact and is not separable.

2.2. Zadeh's and set-valued extension

The set-valued extension of a discrete dynamical system (X, f) is a map $\bar{f} : \mathbb{K}(X) \to \mathbb{K}(X)$ defined by $\bar{f}(A) = f(A)$ for any $A \in \mathbb{K}(X)$. It is shown that $\bar{f}$ is continuous in Hausdorff metric if and only if f is continuous [14].

The Zadeh's extension of (X, f) is a map $\hat{f} : \mathbb{F}(X) \to \mathbb{F}(X)$ defined by

$$[\hat{f}(u)](x) = \sup_{y \in f^{-1}(x)} \{u(y)\}$$

for any $u \in \mathbb{F}(X)$ and $x \in X$. It is known that for compact X, $\hat{f} : \mathbb{F}(X) \to \mathbb{F}(X)$ is continuous if and only if $f : X \to X$ is continuous [15].

Lemma 1 ([16],[17]). *Let X be a metric space. If $f : X \to X$ is continuous, then $[\hat{f}(u)]_\alpha = f([u]_\alpha)$.*

A fuzzy set u is *piecewise constant* if there exists a strictly decreasing sequence of closed subsets $\{C_1, C_2, \cdots, C_k\}$ of X and a strictly increasing sequence of real numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_k\} \subseteq (0, 1]$ such that $[u]_\alpha = C_{i+1}$, where $\alpha \in (\alpha_i, \alpha_{i+1}]$.

Lemma 2 ([18]). *For any $v \in \mathbb{F}(X)$ and $\varepsilon > 0$ there exists a piecewise constant $u \in \mathbb{F}(X)$ such that $d_\infty(u, v) < \varepsilon$, i.e., the set of piecewise constant fuzzy sets is dense in $\mathbb{F}(X)$.*

Denote by $\mathbb{SF}(X)$ the set of piecewise constant fuzzy sets.

2.3. Nonautonomous discrete dynamical systems

For a compact metric space X, let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous maps, where $f_n : X \to X$. An *orbit* $\{x_n\}_{n=1}^\infty$ of a point $x_1 \in X$ is defined as follows:

$$x_{n+1} = f_n(x_n), \quad n = 1, 2, \cdots$$

The set-valued extension of $(X, \{f_n\}_{n=1}^\infty)$ is denoted by $(\mathbb{K}(X), \{\bar{f}_n\}_{n=1}^\infty)$.

Denote $F_n : X \to X$ and $\bar{F}_n : \mathbb{K}(X) \to \mathbb{K}(X)$ by

$$F_n(x) = f_n \circ f_{n-1} \cdots \circ f_2 \circ f_1(x),$$

and

$$\bar{F}_n(x) = \bar{f}_n \circ \bar{f}_{n-1} \cdots \circ \bar{f}_2 \circ \bar{f}_1(x),$$

respectively.

3. Main Results

In this section, we investigate the relations between several forms of sensitivity of nonautonomous dynamical system and its induced fuzzy systems.

Let (X, d) be a compact metric space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous maps on X.

For $(X, \{\hat{f}_n\}_{n=1}^{\infty})$, its *Zadeh's extension* (or *fuzzification*) is a sequence of continuous maps $\hat{f}_n : \mathbb{F}(X) \to \mathbb{F}(X)$ defined by

$$[\hat{f}_n(u)](x) = sup_{y \in f_n^{-1}(x)}\{u(y)\},$$

for any $u \in \mathbb{F}(X)$ and $x \in X$.

An *orbit* $\{u_n\}_{n=1}^{\infty}$ of a point $u_1 \in \mathbb{F}(X)$ is defined as follows:

$$u_{n+1} = \hat{f}_n(u_n), \quad n = 1, 2, \cdots .$$

Define $\hat{F}_n : \mathbb{F}(X) \to \mathbb{F}(X)$ by

$$\hat{F}_n(u) = \hat{f}_n \circ \hat{f}_{n-1} \cdots \circ \hat{f}_2 \circ \hat{f}_1(u),$$

for any $u \in \mathbb{F}(X)$.

Definition 1. *We say that* $\{f_n\}_{n=1}^{\infty}$ *is*

strong sensitive *if there is a constant* $\delta > 0$ *such that for every point* x *and every neighborhood* A *of* x*, there is a* $y \in A$ *and an integer* n_0 *such that* $d(F_k(x), F_k(y)) > \delta$ *for every* $n \geq n_0$.

mean sensitive *if there is a constant* $\delta > 0$ *such that for every point* $x \in X$ *and every neighborhood* A *of* x*, there is a* $y \in A$ *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(F_i(x), F_i(y)) > \delta.$$

We call (x, y) *a mean sensitive pair.*

Definition 2. *We say that* $\{\hat{f}_n\}_{n=0}^{\infty}$ *is* **strong sensitive** *if there is a constant* $\delta > 0$ *such that for every fuzzy set* $u \in \mathbb{F}(X)$ *and every neighborhood* U *about* u*, there is a* $v \in U$ *and an integer* n_0 *such that* $d_\infty(\hat{F}_k(u), \hat{F}_k(v)) \geq \delta$ *for every* $n \geq n_0$.

mean sensitive *if there is a constant* $\delta > 0$ *such that for every fuzzy set* $u \in \mathbb{F}(X)$ *and every neighborhood* U *of* u*, there is a* $v \in U$ *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_\infty(\hat{F}_i(u), \hat{F}_i(v)) > \delta.$$

Proposition 1. *Let* $u \in \mathbb{F}(X)$ *and* $\hat{F}_n : \mathbb{F}(X) \to \mathbb{F}(X)$*. Then* $[\hat{F}_n(u)]_\alpha = F_n([u]_\alpha)$ *for* $\alpha \in [0, 1]$.

Proof. Take $u = \omega_1$. Since $[\hat{f}(\omega)]_\alpha = f([\omega]_\alpha)$ and $\hat{f}_n(\omega_n) = \omega_{n+1}$ for $n = 1, 2, \cdots$, then

$$\begin{aligned}
[\hat{F}_n(u)]_\alpha &= [\hat{F}_n(\omega_1)]_\alpha \\
&= [\hat{f}_n \circ \hat{f}_{n-1} \circ \cdots \circ \hat{f}_1(\omega_1)]_\alpha) \\
&= [\hat{f}_n \circ \hat{f}_{n-1} \circ \cdots \circ \hat{f}_2(\omega_2)]_\alpha) \\
&= [\hat{f}_n \circ \hat{f}_{n-1} \circ \cdots \circ \hat{f}_3(\omega_3)]_\alpha) \\
&= \cdots \\
&= [\hat{f}_n(\omega_n)]_\alpha \\
&= f_n([\omega_n]_\alpha) \\
&= f_n([\hat{f}_{n-1}(\omega_{n-1})]_\alpha) \\
&= f_n \circ f_{n-1}([\omega_{n-1}]_\alpha) \\
&= f_n \circ f_{n-1} \circ \cdots \circ f_1([\omega_1]_\alpha) \\
&= F_n([\omega_1]_\alpha) \\
&= F_n([u]_\alpha).
\end{aligned}$$

Theorem 1. *If $\{\hat{f}_n\}_{n=1}^\infty$ is strongly sensitive, then $\{f_n\}_{n=1}^\infty$ is strongly sensitive.*

Proof. Let $x \in X$. Take $u = \hat{x} \in \mathbb{F}(X)$. Since $\{\hat{f}_n\}_{n=1}^\infty$ is strongly sensitive, there exist $\delta > 0$ and an integer n_0 such that

$$\begin{aligned}
d_\infty(\hat{F}_n(u), \hat{F}_n(\nu)) &= d_\infty(\hat{F}_n(\hat{x}), \hat{F}_n(\nu)) \\
&= \sup_{\alpha\in[0,1]} d_H([\hat{F}_n(\hat{x})]_\alpha, [\hat{F}_n(\nu)]_\alpha) \\
&= \sup_{\alpha\in[0,1]} d_H(F_n([\hat{x}]_\alpha), F_n([\nu]_\alpha)) \\
&= \sup_{\alpha\in[0,1]} d_H(\bar{F}_n(\{x\}), \bar{F}_n([\nu]_\alpha)) \\
&= \sup_{\alpha\in[0,1]} \{ \sup_{y\in[\nu]_\alpha} d(F_n(x), F_n(y))\} \\
&= \sup_{y\in[\nu]_0} d(F_n(x), F_n(y)) > \delta.
\end{aligned}$$

for all $n \geq n_0$. Thus it follows from the continuity of $\{f_n\}_{n=1}^\infty$ and the compactness of $[\nu]_0$ that there exists $y^* \in [\nu]_0$ such that

$$d_\infty(\hat{F}_n(\hat{x}), \hat{F}_n(\nu)) = d(F_n(x), F_n(y^*)) > \delta.$$

On the other hand, since $\nu \in \mathbb{U}_{d_\infty}(\hat{x}, \varepsilon)$, we have $[\nu]_0 \subset \mathbb{U}_{d_H}(\{x\}, \varepsilon)$ and then $y^* \in \mathbb{U}_d(x, \varepsilon)$. Consequently, $\{f_n\}_{n=1}^\infty$ is strongly sensitive in X.

Claim 1 If $\{\bar{f}_n\}_{n=1}^\infty$ is strongly sensitive in $\mathbb{L}(X)$, then it is strongly sensitive in $\mathbb{K}(X)$.

Proof. Let $B \in \mathbb{L}(X)$. Since $\mathbb{L}(X)$ is dense in $\mathbb{K}(X)$, for any $\varepsilon > 0$, there exists $A \in \mathbb{L}(X)$ such that $A \in \mathbb{U}_{d_H}(B, \varepsilon)$. Due to the strong sensitivity of $\{\bar{f}_n\}_{n=1}^\infty$ in $\mathbb{L}(X)$, there exist a constant $\delta > 0$ and an integer n_0 such that $d_H(\bar{F}_k(A), \bar{F}_k(B)) \geq \delta$ for all $n \geq n_0$. This completes the proof.

Apply the similar technique to $(\mathbb{F}(X), \{\hat{f}_n\}_{n=1}^{\infty})$, the following result is obtained:

Claim 2 If $\{\hat{f}_n\}_{n=1}^{\infty}$ is strongly sensitive in $\mathbb{SF}(X)$, then it is strongly sensitive in $\mathbb{F}(X)$.

Proposition 2. *The following conditions are equivalent:*
(1) $\{\hat{f}_n\}_{n=1}^{\infty}$ *is strongly sensitive.*
(2) $\{\bar{f}_n\}_{n=1}^{\infty}$ *is strongly sensitive.*

Proof. (1) $\Rightarrow$ (2) Since $\mathbb{L}(X)$ is dense in $\mathbb{K}(X)$, by Claim 1, it is sufficient to show that $\{\bar{f}_n\}_{n=1}^{\infty}\,|_{\mathbb{L}(X)}$ is strongly sensitive.

Let $\{\hat{f}_n\}_{n=1}^{\infty}$ be strongly sensitive with sensitive constant δ and $A = \{x_1, x_2, \cdots, x_k\} \in \mathbb{L}(X)$. Take $u_i = \hat{x}_i$ for $1 \leq i \leq k$, then $u_i \in \mathbb{F}(X)$. Since $\{\hat{f}_n\}_{n=1}^{\infty}$ is strongly sensitive, for each u_i, there exist $v_i \in \mathbb{U}_{d_\infty}(u_i, \varepsilon)$ and an integer n_i such that $d_\infty(\hat{F}_r(u_i), \hat{F}_r(v_i)) > 2\delta$ for all $r > n_i$, where $i = 1, 2, \cdots, k$. Set $N = \max\{n_i : 1 \leq i \leq k\}$.

Now we show that $d_H(\bar{F}_n(A), \bar{F}_n(B)) > \delta$ for all $B \in \mathbb{U}_{d_H}(A, \varepsilon)$ and $n > N$.

Let $n > N$. Then for any u_i, there exists $v_i \in \mathbb{U}_{d_\infty}(u_i, \varepsilon)$ such that $d_\infty(\hat{F}_n(u_i), \hat{F}_n(v_i)) > 2\delta$. Set $C = \{w_i\}_{i=1}^{k}$. Without loss of generality, let

$$w_i = \begin{cases} v_i, & \text{if } d_\infty(\hat{F}_n(u_1), \hat{F}_n(u_i)) \leq \delta, \\ u_i, & \text{if } d_\infty(\hat{F}_n(u_1), \hat{F}_n(u_i)) > \delta. \end{cases}$$

More specifically,
if $w_i = u_i$, then $d_\infty(\hat{F}_n(u_1), \hat{F}_n(w_i)) = d_\infty(\hat{F}_n(u_1), \hat{F}_n(u_i)) > \delta$;
if $w_i = v_i$, then

$$\begin{aligned} 2\delta \quad & < d_\infty(\hat{F}_n(u_i), \hat{F}_n(v_i)) = d_\infty(\hat{F}_n(u_i), \hat{F}_n(w_i)) \\ & < d_\infty(\hat{F}_n(u_i), \hat{F}_n(u_1)) + d_\infty(\hat{F}_n(u_1), \hat{F}_n(w_i)) \\ & \leq \delta + d_\infty(\hat{F}_n(u_1), \hat{F}_n(w_i)). \end{aligned}$$

Thus $d_\infty(\hat{F}_n(u_1), \hat{F}_n(w_i)) > \delta$ and then

$$\begin{aligned} d_\infty(\hat{F}_n(u_1), \hat{F}_n(w_i)) \quad & = d_\infty(\hat{F}_n(\hat{x}_1), \hat{F}_n(w_i)) \\ & = \sup_{\alpha \in [0,1]} d_H([\hat{F}_n(\hat{x}_1)]_\alpha, [\hat{F}_n(w_i)]_\alpha) \\ & = \sup_{\alpha \in [0,1]} d_H(F_n([\hat{x}_1]_\alpha), F_n([w_i]_\alpha)) \\ & = \sup_{\alpha \in [0,1]} d_H(\{F_n(x_1)\}, F_n([w_i]_\alpha)) > \delta. \end{aligned}$$

Therefore, there exists $y_i \in [w_i]_\alpha$ such that $d(F_n(x_1), F_n(y_i)) > \delta$ for each i. Take $B = \{y_i\}_{i=1}^{k}$. Then $d_H(\bar{F}_n(A), \bar{F}_n(B)) > \delta$ holds for all $B \in \mathbb{U}_{d_H}(A, \varepsilon)$ and $n > N$.

(2) $\Rightarrow$ (1) Assume $\{\bar{f}_n\}_{n=1}^{\infty}$ is strongly sensitive with sensitive constant δ. To show that $\{\hat{f}_n\}_{n=1}^{\infty}$ is strongly sensitive in $\mathbb{F}(X)$, it is sufficient to prove that $\{\hat{f}_n\}_{n=1}^{\infty}\,|_{\mathbb{SF}(X)}$

is strongly sensitive, as $\mathbb{SF}(X)$ is dense in $\mathbb{F}(X)$. Let $u \in \mathbb{SF}(X)$, then there exist a sequence of nested closed subsets $\{A_1, A_2, \cdots, A_k\}$ of X and a sequence of real numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ such that

$$[u]_\alpha = A_{i+1}, \text{ where } \alpha \in (\alpha_i, \alpha_{i+1}], 1 \leq i \leq k.$$

Since $\{\bar{f}_n\}_{n=1}^{\infty}$ is strongly sensitive, for A_k and any $B \in \mathbb{K}(X)$ with $B \in \mathbb{U}_{d_H}(A_k, \frac{\varepsilon}{2})$, there exists an integer n_0 such that for all $n > n_0$,

$$d_H(\bar{F}_n(A_k), \bar{F}_n(B)) > \delta. \tag{2}$$

Set $X_1 = X$ and $C_1 = u^{-1}(\alpha_k) \bigcap X_1$. In general, define $\{X_i\}_{i=1}^{k}$ and $\{C_i\}_{i=1}^{k}$ by the following

$$X_i = X_{i-1} \setminus \mathbb{U}_{d_H}(C_{i-1}, \frac{\varepsilon}{4}), \quad C_i = u^{-1}(\alpha_{k-i+1}) \bigcap X_i.$$

Let $D_i = \overline{\bigcup_{j=1}^{i} \mathbb{U}_{d_H}(C_i, \frac{\varepsilon}{4})}$, then we obtain an incresing sequence $D_1 \subset D_2 \subset \cdots \subset D_k$ of closed sets in $\mathbb{K}(X)$. Consequently, we have a piecewise constant fuzzy set $\omega \in \mathbb{SF}(X)$ satisfying

$$[\omega]_\alpha = D_{i+1}, \quad \text{where} \quad \alpha \in (\alpha_i, \alpha_{i+1}].$$

It follows from the construction and Lemma 2.2 that

$$d_\infty(u, \omega) < \frac{\varepsilon}{4}. \tag{3}$$

Thus we have for each $i = 1, 2, \cdots, k$,

$$d_H(B, D_i) \leq d_H(B, A_k) + d_H(A_k, D_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}. \tag{4}$$

Take $\nu \in \mathbb{SF}(X)$ such that

$$[\nu]_\alpha = \begin{cases} B, & \text{if } \alpha \in (\alpha_{k-1}, \alpha_k] \\ B \bigcup D_i, & \text{if } \alpha \in (\alpha_i, \alpha_{i+1}], \quad i = 1, 2, \cdots, k-2 \end{cases}$$

Then from (4), we have

$$d_\infty(\nu, \omega) < \frac{3\varepsilon}{4}. \tag{5}$$

Hence it follows from (3) and (5) that

$$d_\infty(u, \nu) < d_\infty(u, \omega) + d_\infty(\omega, \nu) < \frac{\varepsilon}{4} + \frac{3\varepsilon}{4} = \varepsilon.$$

On the other hand, by (2) and Lemma 2.1, the following

$$\begin{aligned} d_\infty(\hat{F}_n(u), \hat{F}_n(\nu)) &= \sup_{\alpha \in [0,1]} d_H([\hat{F}_n(u)]_\alpha, [\hat{F}_n(\nu)]_\alpha) \\ &= \sup_{\alpha \in [0,1]} d_H(F_n([u]_\alpha), F_n([\nu]_\alpha)) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\alpha\in[0,1]} d_H(\bar{F}_n([u]_\alpha), \bar{F}_n([\nu]_\alpha)) \\
&\geq d_H(\bar{F}_n([u]_{\alpha_k}), \bar{F}_n([\nu]_{\alpha_k})) \\
&= d_H(\bar{F}_n(A_k), \bar{F}_n(B)) > \delta
\end{aligned}
$$

holds, the strong sensitivity of $\{\hat{f}_n\}_{n=1}^{\infty}$ follows.

Theorem 2. *If $(\mathbb{F}(X), \{\hat{f}_n\}_{n=1}^{\infty})$ is mean sensitive, then $(X, \{f_n\}_{n=1}^{\infty})$ is also mean sensitive.*

Proof. Let $(\mathbb{F}(X), \{\hat{f}_n\}_{n=1}^{\infty})$ be mean sensitive with sensitive constant δ, then for every $u \in \mathbb{F}(X)$ and every $\varepsilon > 0$ there exists $v_1 \in \mathbb{U}_{d_\infty}(u, \varepsilon)$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_\infty(\hat{F}_i u, \hat{F}_i v_1) > \delta.$$

Taking $u = \hat{x} \in \mathbb{F}(X)$ we have that

$$
\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_\infty(\hat{F}_i \hat{x}, \hat{F}_i v_1) \quad &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{\alpha\in[0,1]} d_H([\hat{F}_i(\hat{x})]_\alpha, [\hat{F}_i(v_1)]_\alpha) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{\alpha\in[0,1]} d_H(F_i([x]_\alpha), F_i([v_1]_\alpha)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{\alpha\in[0,1]} d_H(\bar{F}_i(\{x\}), \bar{F}_i([v_1]_\alpha)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{y\in[v_1]_0} d(F_i(x), F_i(y)) > \delta.
\end{aligned}
$$

Thus it follows from the continuity of $\{f_n\}_{n=1}^{\infty}$ and the compactness of $[v_1]_0$ that there exist $y_1 \in [v_1]_0$ and an integer n_1 such that

$$\sum_{i=0}^{n_1-1} d(F_i(x), F_i(y_1)) > n_1\delta.$$

If (x, y_1) forms a mean sensitive pair, then the proof is done. If not, then there exists an integer k_1 with $k_1 > n_1$ such that $\sum_{i=0}^{n-1} d(F_i(x), F_i(y_1)) \leq n_1\delta$ for all $n \geq k_1$. Thus we can find a neighborhood U_1 of y_1 with $U_1 \subset U_d(x, \varepsilon)$ such that $\sum_{i=0}^{n_1-1} d(F_i(x), F_i(z)) > n_1\delta$ for all $z \in U_1$. Furthermore, there exists $\varepsilon_1 > 0$ such that

$$\overline{U_d(y_1, \varepsilon_1)} \subset U_1.$$

Using the mean sensitivity of $\{\hat{f}_n\}_{n=1}^{\infty}$ again, we have $v_2 \in U_{d_\infty}(\hat{y_1}, \varepsilon_1)$ such that $(\hat{y_1}, v_2)$ is a mean sensitive pair, that is,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_\infty(\hat{F}_i \hat{y_1}, \hat{F}_i v_2) \quad = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{\alpha\in[0,1]} d_H([\hat{F}_i(\hat{y_1})]_\alpha, [\hat{F}_i(v_2)]_\alpha)$$

$$= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{\alpha\in[0,1]} d_H(\bar{F}_i(y_1), \bar{F}_i([v_2]_\alpha))$$
$$= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sup_{y\in[v_2]_0} d(F_i(y_1), F_i(y)) > \delta.$$

Therefore, there exist $y_2 \in [v_2]_0$ and an integer $n_2 > k_1 > n_1$ such that

$$\sum_{i=0}^{n_2-1} d(F_i(y_1), F_i(y_2)) > n_2\delta,$$

and then

$$\sum_{i=0}^{n_2-1} d(F_i(x), F_i(y_2)) > \sum_{i=0}^{n_2-1} d(F_i(y_1), F_i(y_2)) - \sum_{i=0}^{n_2-1} d(F_i(x), F_i(y_1)) \geq (n_2 - n_1)\delta.$$

If (x, y_2) forms a mean sensitive pair, then the proof is done. If not, then there exists an integer k_2 with $k_2 > n_2$ such that $\sum_{i=0}^{n-1} d(F_i(x), F_i(y_2)) \leq (n_2 - n_1)\delta$ for all $n \geq k_2$. Again, we can find a neighborhood U_2 of y_2 with $U_2 \subset U_d(y_1, \varepsilon_1)$ such that $\sum_{i=0}^{n_2-1} d(F_i(x), F_i(z)) > (n_2 - n_1)\delta$ for all $z \in U_2$. Thus, there exists $\varepsilon_2 > 0$ such that

$$\overline{U_d(y_2, \varepsilon_2)} \subset U_2.$$

Proceeding inductively, we eventually obtain either the mean sensitive pair (x, y_k) or a sequence $\{y_n\}$ in $U_d(x, \varepsilon)$. It follows from the construction that the sequence $\{y_n\}$ converges to a point y_0. Thus

$$y_0 \in U_d(y_i, \varepsilon_i) \subset \overline{U_d(y_i, \varepsilon_i)} \subset U_i \subset U_d(x, \varepsilon).$$

Hence for each i, we have

$$\sum_{i=0}^{n_i-1} d(F_i(x), F_i(y_0)) > r_i\delta,$$

where

$$r_i = \begin{cases} \sum_{k=1}^{i} (-1)^{k-1} n_k, & i = 2m - 1 \\ \sum_{k=1}^{i} (-1)^k n_k, & i = 2m. \end{cases}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_\infty(F_i(x), F_i(y_0)) > \delta$$

and then $\{f_n\}_{n=1}^{\infty}$ is mean sensitive.

The following example shows that, in general, the converse of Theorem 3.4 is not true.

Example 1. *Let S^1 be a circle. It is known that the Denjoy map $D_\lambda : S^* \to S^*$ is an orientation preserving homeomorphism of the constructed circle S^*. There exists a Cantor set $C_\lambda \subset S^*$ on which D_λ acts minimally. There exists a continuous surjection $h_\lambda : S^* \to S^1$ that semi-conjugates D_λ with R_λ. In [22], the authors show that the system $(\mathbb{K}(C_\lambda), \overline{D}_\lambda)$ is not sensitive. Hence it is not mean sensitive, as the mean sensitivity is stronger than sensitivity.*

Let $f_n = D_\lambda$, $n = 1, 2, \cdots$. Define $i_\lambda : \mathbb{K}(C_\lambda) \to \mathbb{F}(C_\lambda)$ by $i_\lambda(K) = \lambda\chi_K$ for any $K \in \mathbb{K}(C_\lambda)$ and any $\lambda \in (0, 1]$, where χ_K is the characteristic function of K. Hence, $i_\lambda \circ \overline{D}_\lambda = \widehat{D_\lambda} \circ i_\lambda$. Note that i_λ is continuous. We show that the mean sensitivity of D_λ cannot be inherited by $\widehat{D_\lambda}$ as follows.

Since $(\mathbb{K}(C_\lambda), \overline{D}_\lambda)$ is not mean sensitive, for every $\delta > 0$, there exist a nonempty set $A \in \mathbb{K}(C_\lambda)$ and a neighborhood U of A such that for all $B \in U$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_H(\overline{D}_\lambda^n(A), \overline{D}_\lambda^n(B)) \leq \delta. \tag{6}$$

Suppose $u \in e(A)$ (recall that $e(A) = \{u \in \mathbb{F}(C_\lambda) \mid [u]_0 \subseteq A\}$), by continuity of i_λ and (3.5), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_H(\overline{D}_\lambda^n([u]_0), \overline{D}_\lambda^n(B)) \leq \delta \\
&\Rightarrow \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_H(i_\lambda \circ \overline{D}_\lambda^n([u]_0), i_\lambda \circ \overline{D}_\lambda^n(B)) \leq \delta \\
&\Rightarrow \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_\infty(\widehat{D_\lambda}^n \circ i_\lambda([u]_0), \widehat{D_\lambda}^n \circ i_\lambda(B)) \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d_\infty(\widehat{D_\lambda}^n(u), \widehat{D_\lambda}^n(\nu)) \leq \delta,
\end{aligned}$$

where $\nu = i_\lambda(B) \in \mathbb{F}(C_\lambda)$. It follows that $(\mathbb{F}(C_\lambda), \widehat{D}_\lambda)$ is not mean sensitive.

4. Conclusions

In this paper, we introduce the notions of strong sensitivity and mean sensitivity for nonautonomous systems and investigate these two forms of sensitivity in an original nonautonomous system and its connections with the same ones in its fuzzified system. More precisely, we prove that the strong sensitivity of original system and its induced systems, including set-valued system and fuzzified system, are equivalent. The mean sensitivity of induced fuzzy system implies the same one in original nonautonomous system, however, the converse is not true.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (NO. 11601051) and China Scholarship Council Contract (NO. 201608505146).

References

[1] N.C. Bernardes Jr., A. Bonilla, A. Peris, X. Wu. Distributional chaos for operators on Banach spaces. *Journal of Mathematical Analysis and Applications*, 459:797–821, 2018.

[2] N.C. Bernardes Jr., R. M. Vermersch. On the dynamics of induced maps on the space of probability measures. *Transactions of the American Mathematical Society*, 368:7703–7725, 2016.

[3] M. Murillo-Arcila, A. Peris. Mixing properties for nonautonomous linear dynamics and invariant sets. *Applied Mathematics Letters*, 26:215–218, 2013.

[4] Y. Shi, G. Chen. Chaos of time-varying discrete dynamical systems. *Journal of Difference Equations and Applications*, 15: 429–449, 2009.

[5] Jose S. Cánovas. Li-Yorke chaos in a class of nonautonomous discrete systems. *Journal of Difference Equations and Applications*, 17:479–486, 2011.

[6] J. Dvořáková. Chaos in nonautonomous discrete dynamical systems. *Communications in Nonlinear Science and Numerical Simulation*, 17:4649–4652, 2012.

[7] F. Balibrea, P.Oprocha. Weak mixing and chaos in nonautonomous discrete system. *Applied Mathematics Letter*, 25:1135–1141, 2012.

[8] T K Subrahmonian Moothathu. Stronger forms of sensitivity for dynamical systems. *Nonlinearity*, 20:2115–2126, 2007.

[9] Puneet Sharma, Anima Nagar. Inducing sensitivity on hyperspaces. *Topology and its Applications*, 157:2052–2058, 2010.

[10] Risong Li. A note on stronger forms of sensitivity for dynamical systems. *Chaos, Solitons* & *Fractals*, 45:753–758, 2012.

[11] J. Li, S.M. Tu, X.D. Ye. Mean equicontinuity and mean sensitivity. *Ergodic Theory Dynamical Systems*, 35:2587–2612, 2015.

[12] F. Garcia-Ramos, L. Jin. Mean Proximality and Mean Li-Yorke Chaos. *Proceedings of the American Mathematical Society*, 145:2959–2969, 2017.

[13] Q.L. Huang, Y.M. Shi, L.J. Zhang. Sensitivity of non-autonomous discrete dynamical systems. *Applied Mathematics Letters*, 39:31–34, 2015.

[14] H.Román-Flores, Y.Chalco-Cano. Robinson's chaos in set-valued discrete sysyems. *Chaos Solitons & Fractals*, 25:33–42, 2005.

[15] H.Román-Flores, Y.Chalco-Cano. Some chaotic properties of Zadeh's extension. *Chaos Solitons & Fractals*, 35:452–459, 2008.

[16] H.Román-Flores, Laécio C. Barros, Rodney C. Bassanezi. A note on Zadeh's extensions. *Fuzzy Sets and Systems*, 117:327–331, 2001.

[17] P. Diamond, A. Pokrovdkii. Chaos, entropy and a generalized extension principle. *Fuzzy Sets and Systems*, 61:277–283, 1994.

[18] Jiří Kupka. On Devaney chaotic induced fuzzy and set-valued dynamical systems. *Fuzzy Sets and Systems*, 117:34–44, 2011.

[19] Jiří Kupka. On fuzzifications of discrete dynamical systems. *Information Sciences*, 181:2858–2872, 2011.

[20] Jose S.Cánovasa, Jiří Kupka. On fuzzy entropy and topological entropy of fuzzy extensions of dynamical systems. *Fuzzy Sets and Systems*, 309:115–130, 2017.

[21] X.X. Wu, G.R. Chen. Sensitivity and transitivity of fuzzified dynamical systems. *Information Sciences*, 396:14–23, 2017.

[22] H. Liu, E.H. Shi, G.F. Liao. Sensitivity of set-valued discrete systems. *Nonlinear Analysis: Theory, Methods and Applications*, 71:6122–6125, 2009.