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Inf-hesitant fuzzy subalgebras and ideals in BCK/BCI-algebras

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Abstract. In the present paper, we introduce the notions of Inf-hesitant fuzzy subalgebras and Inf-hesitant fuzzy ideals in BCK/BCI-algebras and investigate their relations and properties. In addition, we discuss the characterizations of Inf-hesitant fuzzy subalgebras and Inf-hesitant fuzzy ideals in BCK/BCI-algebras.

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Key Words and Phrases: p-semisimple BCI-algebra; Inf-hesitant fuzzy subalgebras; Inf-hesitant fuzzy ideals.

1. Introduction

The motivation for introducing hesitant fuzzy sets is that it is sometimes difficult to determine the membership of an element into a set and in some circumstances this difficulty is caused by a doubt between a few different values. For example, two experts discuss the membership of x into A, and one wants to assign 0.3 and the other 0.4. So, the uncertainty on the possible values is somehow limited. Torra [25] proposed the concept of hesitant fuzzy sets as a new generalization of fuzzy sets [33], which allows the membership of an element of a set to be represented by several possible values. They also discussed relationships among hesitant fuzzy sets and other generalizations of fuzzy sets such as intuitionistic fuzzy sets, type-2 fuzzy sets, and fuzzy multisets. Some set theoretic operations such as union, intersection and complement on hesitant fuzzy sets have also been proposed by Torra [25]. Hesitant fuzzy sets can be used as an efficient mathematical tool for modeling peoples hesitancy in daily life than the other classical extensions of fuzzy sets. Hesitant fuzzy sets are a very useful to express peoples hesitancy in daily life and a very useful tool to deal with uncertainty, which can be accurately and perfectly described in

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terms of the opinions of decision makers. After the pioneering work of Torra, the hesitant fuzzy has received much attention from many authors in many fields for eg. Xu and Xia [30] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. They investigated the connections of the aforementioned distance measures and further develop a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. A number of research papers have been appeared on hesitant fuzzy set theory in decision making problem etc. (see [23, 27–29, 31]). Fuzzy set theory plays an important role in the development of hesitant fuzzy sets theory. Muhiuddin et al. have applied the fuzzy set theory and related notions to different algebraic structures (see for e.g., [15–18, 18, 19, 19–22]). In recent years, a number of research papers have been devoted to the study of fuzzy sets theory and related concepts on different algebraic structures (see e.g., [5–8, 24]). Recently, hesitant fuzzy sets theory have been applied to different algebraic structures on various aspects viz., Jun et al. have applied the hesitant fuzzy sets theory to BCK/BCI-algebras and semigroups (see [2–4]). Also, Muhiuddin et al. have applied the hesitant fuzzy sets theory to residuated lattices, lattice implication algebras and BCK/BCI-algebras (see [10–14]).

In this paper, we introduce some new types of hesitant fuzzy subalgebras and ideals in BCK/BCI-algebras, and investigate their relations and properties. Finally, we discuss the characterizations of these new types of hesitant fuzzy subalgebras and hesitant fuzzy ideals in BCK/BCI-algebras.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a BCI-algebra if it satisfies the following conditions:

(I) $(\forall x,y,z \in X)\ (((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y \in X)\ ((x*(x*y))*y=0)$,

(III) $(\forall x \in X)\ (x*x=0)$,

(IV) $(\forall x,y \in X)\ (x*y=0, y*x=0 \Rightarrow x=y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0*x=0)$,

then X is called a BCK-algebra. A BCK-algebra X is said to be positive implicative if it satisfies:

$$(\forall x,y,z \in X)\left((x*y)*z=(x*z)*(y*z)\right). \tag{1}$$

A BCK-algebra X is said to be implicative if it satisfies:

$$(\forall x, y \in X)\,(x = x * (y * x))\,. \tag{2}$$

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{3}$$
$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,, \tag{4}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{5}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{6}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions:

$$(\forall x, y, z \in X)\,(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))\,, \tag{7}$$
$$(\forall x, y \in X)\,(0 * (0 * (x * y)) = (0 * y) * (0 * x))\,, \tag{8}$$
$$(\forall x \in X)\,(0 * (0 * (0 * x)) = 0 * x)\,. \tag{9}$$

A BCI-algebra X is said to be p-semisimple (see [1]) if $0 * (0 * x) = x$ for all $x \in X$. Every p-semisimple BCI-algebra X satisfies:

$$(\forall x, y, z \in X)\,((x * z) * (y * z) = x * y)\,. \tag{10}$$

A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X if $x*y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an ideal of X if it satisfies:

$$0 \in A, \tag{11}$$
$$(\forall x \in X)\,(x * y \in A,\; y \in A \;\Rightarrow\; x \in A)\,. \tag{12}$$

A subset A of a BCI-algebra X is called a p-ideal of X (see [32]) if it satisfies (11) and

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A,\; y \in A \;\Rightarrow\; x \in A)\,. \tag{13}$$

Note that every p-ideal is an ideal, but the converse is not true in general (see [32]). Note that an ideal A of a BCI-algebra X is a p-ideal of X if and only if the following assertion is valid:

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A \;\Rightarrow\; x * y \in A)\,. \tag{14}$$

We refer the reader to the books [1, 9] for further information regarding BCK/BCI-algebras.

3. Inf-hesitant fuzzy subalgebras and ideals

Torra [25] introduced a new extension for fuzzy sets to manage those situations in which several values are possible for the definition of a membership function of a fuzzy set.

Definition 1 ([25, 26])**.** *Let X be a reference set. A hesitant fuzzy set on X is defined in terms of a function that when applied to X returns a subset of $[0,1]$, which can be viewed as the following mathematical representation:*

$$\mathcal{H} := \{(x, h(x)) \mid x \in X\}$$

where $h : X \to P([0,1])$.

In what follows, the power set of $[0,1]$ is denoted by $P([0,1])$ and

$$P^*([0,1]) = P([0,1]) \setminus \{\emptyset\}.$$

For any element $D \in P^*([0,1])$, the infimum of D is denoted by $\inf D$. For any hesitant fuzzy set $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ and $D \in P^*([0,1])$, consider the set

$$\mathrm{Inf}[\mathcal{H}; D] := \{x \in X \mid \inf h(x) \geq \inf D\}.$$

Definition 2. *Let X be a BCK/BCI-algebra. Given an element $D \in P^*([0,1])$, a hesitant fuzzy set $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is called an Inf-hesitant fuzzy subalgebra of X related to D (briefly, D-Inf-hesitant fuzzy subalgebra of X if the set* $\mathrm{Inf}[\mathcal{H}; D]$ *is a subalgebra of X whenever it is non-empty. If $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy subalgebra of X for all $D \in P^*([0,1])$ with* $\mathrm{Inf}[\mathcal{H}; D] \neq \emptyset$*, then we say that $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.*

Example 1. *(1) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:*

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} = \{(0, (0.8, 1]), (a, (0.3, 0.5) \cup \{0.9\}), (b, [0.5, 0.7]), (c, (0.3, 0.5) \cup \{0.7\})\}.$$

Since $\inf h(0) = 0.8$, $\inf h(a) = 0.3 = \inf h(c)$ *and* $\inf h(b) = 0.5$*, it is routine to verify that $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.*

(2) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	0
c	c	c	c	0	0
d	d	c	c	a	0

Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} = \{(0, \{0.8, 0.9\}), (a, [0.2, 0.9)), (b, (0.7, 0.8]), (c, \{0.5\} \cup (0.7, 0.9)), (d, [0.1, 0.5])\}.$$

Note that $\inf h(0) = 0.8$, $\inf h(a) = 0.2$, $\inf h(b) = 0.7$, $\inf h(c) = 0.5$ *and* $\inf h(d) = 0.1$. *It is easy to check that $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.*

(3) Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table.

$*$	0	1	a	b	c
0	0	0	c	c	a
1	1	0	c	c	a
a	a	a	0	0	c
b	b	a	1	0	c
c	c	c	a	a	0

Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} = \{(0, [0.8, 0.9]), (1, (0.6, 0.7]), (a, [0.5, 0.6]), (b, [0.5, 0.6]), (c, [0.3, 0.7])\}.$$

Then $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy subalgebra of X with $D_1 :=$ $[0.55, 0.65]$. But it is not a D_2-Inf-hesitant fuzzy subalgebra of X with $D_2 := [0.4, 0.6]$ since $\mathrm{Inf}[\mathcal{H}; D_2] = \{0, 1, a, b\}$ *is not a subalgebra of X.*

(4) Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	a	0	0	b
c	c	b	a	0	c
d	d	d	d	d	0

Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} = \{(0, [0.7, 0.8]), (a, (0.6, 0.7]), (b, [0.3, 0.6]), (c, [0.5, 0.7]), (d, [0.2, 0.4])\}.$$

Then $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is a D_1-Inf-hesitant fuzzy subalgebra of X with $D_1 :=$ $[0.2, 0.4]$. If we take $D_2 := (0.4, 0.6]$, then $\mathrm{Inf}[H; D_2] = \{0, a, c\}$ *which is not a subalgebra of X. Hence $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is not a D_2-Inf-hesitant fuzzy subalgebra of X.*

Theorem 1. *A hesitant fuzzy set $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ on a BCK/BCI-algebra X is an Inf-hesitant fuzzy subalgebra of X if and only if the following assertion is valid:*

$$(\forall x, y \in X)\,(\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\})\,. \tag{15}$$

Proof. Assume that $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X. Assume that there exists $Q \in P^*([0,1])$ such that

$$\inf h(x * y) < \inf Q \leq \min\{\inf h(x), \inf h(y)\}.$$

Then $x, y \in \mathrm{Inf}[\mathcal{H}; D]$ and $x * y \notin \mathrm{Inf}[\mathcal{H}; D]$. This is a contradiction, and so

$$\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\}$$

for all $x, y \in X$.

Conversely, suppose that (15) is valid. Let $D \in P^*([0,1])$ and $x, y \in \mathrm{Inf}[H; D]$. Then $\inf h(x) \geq \inf D$ and $\inf h(y) \geq \inf D$. It follows from (15) that

$$\inf h(x * y) \geq \min\{\inf h(x), \inf h(y)\} \geq \inf D$$

and that $x * y \in \mathrm{Inf}[\mathcal{H}; D]$. Hence the set $\mathrm{Inf}[\mathcal{H}; D]$ is a subalgebra of X, and so $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of X.

Lemma 1. *If $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy subalgebra of a BCK/BCI-algebra X, then*

$$(\forall x \in X)\,(\inf h(0) \geq \inf h(x))\,. \tag{16}$$

Proof. Using (III) and (15), we have

$$\inf h(0) = \inf h(x * x) \geq \min\{\inf h(x), \inf h(x)\} = \inf h(x)$$

for all $x \in X$.

Proposition 1. *Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy subalgebra of a BCK-algebra X. For any elements $a_1, a_2, \cdots, a_n \in X$, if there exists $a_k \in \{a_1, a_2, \cdots, a_n\}$ such that $a_1 = a_k$, then*

$$(\forall x \in X)\,(\inf h((\cdots((a_1 * a_2) * a_3) * \cdots) * a_n) \geq \inf h(x))\,.$$

Proof. Using (5), (III) and (IV), we have $(\cdots((a_1 * a_2) * a_3) * \cdots) * a_n = 0$. Thus the desired result follows from Lemma 1.

Definition 3. *Let X be a BCK/BCI-algebra. Given an element $D \in P^*([0,1])$, a hesitant fuzzy set $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is called an Inf-hesitant fuzzy ideal of X related to D (briefly, D-Inf-hesitant fuzzy ideal of X) if the set* $\mathrm{Inf}[\mathcal{H}; D]$ *is an ideal of X whenever it is non-empty. If $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is a D-Inf-hesitant fuzzy ideal of X for all $D \in P^*([0,1])$ with* $\mathrm{Inf}[\mathcal{H}; D] \neq \emptyset$, *then we say that $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.*

Example 2. *(1) The hesitant fuzzy set $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ in Example 1(1) is an Inf-hesitant fuzzy ideal of X.*

(2) Let $(Y, *, 0)$ be a BCI-algebra and $(\mathbb{Z}, +, 0)$ an additive group of integers. Let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of $(\mathbb{Z}, +, 0)$ and let $X := Y \times \mathbb{Z}$. Then $(X, \otimes, (0, 0))$ is a BCI-algebra where the operation $\otimes$ is given by

$$(\forall (x, m), (y, n) \in X)\,((x, m) \otimes (y, n) = (x * y, m - n))\,.$$

For a subset $A := Y \times \mathbb{N}_0$ of X where $\mathbb{N}_0$ is the set of nonnegative integers, let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} = \{(x, (0.5, 1]), (y, [0.4, 0.9]) \mid x \in A,\ y \in X \setminus A\}\,.$$

Then $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

(3) Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	0
d	d	d	d	d	0

Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$\mathcal{H} = \{(0, [0.8, 1)), (a, [0.4, 0.7]), (b, \{0.3\} \cup (0.4, 0.6]), (c, [0.6, 0.9]), (d, [0.1, 0.5])\}\,.$$

If $D_1 := [0.5, 0.8)$, then $\mathrm{Inf}[\mathcal{H}; D_1] = \{0, c\}$ which is not an ideal of X since $b * c = 0 \in \mathrm{Inf}[\mathcal{H}; D_1]$ but $b \notin \mathrm{Inf}[\mathcal{H}; D_1]$. Thus $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is not a D_1-Inf-hesitant fuzzy ideal of X. We can easily verify that $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is a D_2-Inf-hesitant fuzzy ideal of X with $D_2 = [0.25, 0.5]$.

Theorem 2. *A hesitant fuzzy set $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ on a BCK/BCI-algebra X is an Inf-hesitant fuzzy ideal of X if and only if it satisfies (16) and*

$$(\forall x, y \in X)\,(\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\})\,. \tag{17}$$

Proof. Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be an Inf-hesitant fuzzy ideal of X. If (16) is not valid, then there exists $D \in P^*([0, 1])$ and $a \in X$ such that $\inf h(0) < \inf D \leq \inf h(a)$. It follows that $a \in \mathrm{Inf}[\mathcal{H}; D]$ and $0 \notin \mathrm{Inf}[\mathcal{H}; D]$. This is a contradiction, and so (16) is valid. Now assume that there exist $a, b \in X$ such that $\inf h(a) < \min\{\inf h(a*b), \inf h(b)\}$. Then there exists $K \in P^*([0, 1])$ such that

$$\inf h(a) < \inf K \leq \min\{\inf h(a * b), \inf h(b)\},$$

which implies that $a * b \in \mathrm{Inf}[\mathcal{H}; K]$, $b \in \mathrm{Inf}[\mathcal{H}; K]$ but $a \notin \mathrm{Inf}[\mathcal{H}; K]$. This is a contradiction, and thus (17) holds.

Conversely, suppose that $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ satisfies two conditions (16) and (17). Let $K \in P^*([0,1])$ be such that $\mathrm{Inf}[\mathcal{H}; K] \neq \emptyset$. Obviously, $0 \in \mathrm{Inf}[\mathcal{H}; K]$. Let $x, y \in X$ be such that $x * y \in \mathrm{Inf}[\mathcal{H}; K]$ and $y \in \mathrm{Inf}[\mathcal{H}; K]$. Then $\inf h(x * y) \geq \inf K$ and $\inf h(y) \geq \inf K$. It follows from (17) that

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\} \geq \inf K$$

and that $x \in \mathrm{Inf}[\mathcal{H}; K]$. Hence $\mathrm{Inf}[\mathcal{H}; K]$ is an ideal of X for all $K \in P^*([0,1])$, and therefore $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

Theorem 3. *Let $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ be a hesitant fuzzy set on a BCI-algebra X defined by*

$$\mathcal{H} = \{(x, D), (y, E) \mid x \in B,\ y \in X \setminus B,\ \inf D \geq \inf E\}$$

where $D, E \in P^*([0,1])$ and B is the BCK-part of X. Then $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Inf-hesitant fuzzy ideal of X.

Proof. Since $0 \in B$, we have $\inf h(0) = \inf D \geq \inf h(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in B$, then it is clear that

$$\inf h(x) \geq \min\{\inf h(x * y), \inf h(y)\}.$$

Assume that $x \in X \setminus B$. Since B is an ideal of X, it follows that $x * y \in X \setminus B$ or $y \in X \setminus B$ and that

$$\inf h(x) = \min\{\inf h(x * y), \inf h(y)\}.$$

Therefore $\mathcal{H} := \{(x, h(x)) \mid x \in X\}$ is an Int-hesitant fuzzy ideal of X by Theorem 2.

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