



Some Results on C -retractable Modules

Abdoul Djibril Diallo¹, Papa Cheikhou Diop^{2,*}, Mamadou Barry¹

¹ *Département de Mathématiques et Informatique, Faculté des Sciences et Techniques, Université Cheikh Anta Diop, Dakar, Sénégal*

² *Département de Mathématiques, UFR Sciences et Technologies, Université de Thiès, Thiès, Sénégal*

Abstract. An R -module M is called c -retractable if there exists a nonzero homomorphism from M to any of its nonzero complement submodules. In this paper, we provide some new results of c -retractable modules. It is shown that every projective module over a right SI -ring is c -retractable. A dual Baer c -retractable module is a direct sum of a Z_2 -torsion module and a module which is a direct sum of nonsingular uniform quasi-Baer modules whose endomorphism rings are semi-local quasi-Baer. Conditions are found under which, a c -retractable module is extending, quasi-continuous, quasi-injective and retractable. Also, it is shown that a locally noetherian c -retractable module is homo-related to a direct sum of uniform modules. Finally, rings over which every c -retractable is a C_4 -module are determined.

2020 Mathematics Subject Classifications: 13B10,13C05,13C13

Key Words and Phrases: Retractable modules, complement submodules, c -retractable modules, projective modules

1. Introduction

Throughout all rings are associative with identity and all modules are unitary right module. Let R be a ring. Following [19], we say that an R -module M is retractable if $\text{Hom}_R(M, N) \neq \{0\}$ for any nonzero submodules N of M . It is shown in [19] that every projective module over a right V -ring is retractable. In [19] again, the semisimplicity of retractable modules is studied. M. R. Vedadi [23], introduced the concept of essentially retractable modules and proved that over semiprime right nonsingular rings, a nonsingular essentially retractable module is precisely a module with non-zero dual. In [7], A. Ghorbani and M. R. Vedadi introduced and studied the notion of epi-retractable module, where a module M is called epi-retractable if every submodule of M is a homomorphic image of M . They reveal some applications of projective, nonsingular, injective epi-retractable

*Corresponding author.

DOI: <https://doi.org/10.29020/nybg.ejpam.v13i1.3588>

Email addresses: cheikpapa@yahoo.fr (P. C. Diop),
dialloabdoulaziz58@yahoo.fr (A. D. Diallo), mansabadion1@hotmail.com (M. Barry)

modules regarding the characterization of Bezout, pri, quasi-Frobenius rings. Note that epi-retractable modules are retractable. Earlier, P. F. Smith and A. Tercan [20] introduced C_{11} -module as a generalization of extending modules, where a module M is said to satisfy C_{11} -condition if every submodule of M has a complement which is a direct summand. It is shown in ([20], Theorem 2.7) that a module satisfies (C_{11}) if and only if $M = Z_2(M) \oplus K$ for some (nonsingular) K of M and $Z_2(M)$ and K both satisfy (C_{11}) . Later, the same authors investigated when a direct summand of a C_{11} -module inherits the property [21]. Recently, t -closed submodules of a module M are defined in [2] as closed submodules of M which contain $Z_2(M)$. In [3], S. H. Asgari, A. Haghany and A.R. Rezaei studied the modules M for which C_{11} -condition holds for t -closed submodules (T_{11} -type, for short). They showed among others the following results: *i*) A T_{11} -type module is exactly a direct sum of a Z_2 -torsion module and a nonsingular C_{11} -modules.

ii) A T_{11}^+ -module (modules for which direct summands are T_{11} -type) is precisely a direct sum of Z_2 -torsion and nonsingular C_{11}^+ -module (modules for which direct summands satisfy C_{11}). A. W. Chatters and S. M. Kheuri [4] defined the concept of c -retractable module, where an R -module M is called c -retractable if $Hom_R(M, C) \neq \{0\}$ for any nonzero complement submodules C of M . This notion is a generalization of both the retractable modules and the extending modules. They have shown that if M is a nonsingular c -retractable module such that S_S is extending, then M is extending. But the converse is not true in general. On the other hand it is shown in [22] that if M is a retractable wd -Rickart module, then every indecomposable submodule of M is a simple direct summand. Motivated by the definition of the modules mentioned above and the results on retractable and c -retractable modules, we investigate the c -retractability. Our aim in this paper is to give some new results on c -retractable modules. In general, c -retractable modules need not be projective and vice versa. Connections between projectivity and c -retractability are investigated. Conditions are found under which, a c -retractable module is extending, quasi-continuous, quasi-injective and retractable. With the help of c -retractability, we investigated when the notions of \mathcal{K} -nonsingularity and Baer modules are equivalent. Also, we characterize semisimple artinian rings in terms of c -retractable modules.

Our paper is structured as follows:

In the second section, we are going to give preliminary definitions which we will use throughout this paper.

In the third section, we are going to show among others, the following results:

- (1) Every projective module over a right SI -ring is c -retractable.
- (2) Let M be a wd -Rickart module in which local summands are summand. Then M is uniform-extending and c -retractable if and only if M is extending.
- (3) Let M be a dual Baer c -retractable R -module. Then the following hold:
 - i*) M is a direct sum of uniform submodules.
 - ii*) $M = Z_2(M) \oplus (\oplus_{i \in I} M_i)$ with all M_i nonsingular uniform quasi-Baer and $End(M_i)$ semi-local quasi-Baer.
 - iii*) M is ADS if and only if M is quasi-continuous.
 - iv*) M is auto-invariant if and only if M is quasi-injective.
- (5) The following conditions are equivalent for a ring R :

- (a) R is semisimple artinian.
 (b) Every c -retractable R -module is a C_4 -module.
 (c) Every c -retractable R -module is pseudo-projective.
 (6) Let M be a locally noetherian c -retractable R -module. Then M is homo-related to a direct sum $\bigoplus_{i \in I} U_i$ of uniform submodules of M .

For an R -module M , $S = \text{End}_R(M)$ denotes the endomorphism ring of M . For $\phi \in S$, $\text{Im}\phi$ stands for image of ϕ . The notations $N \leq M$, $N \leq_e M$ and $N \leq^\oplus M$ mean that N is a submodule of M , an essential submodule and a direct summand of M , respectively.

2. Preliminaries

In this section, we are going to give preliminary definitions which we will use throughout this paper.

Definition 1. Let S be a submodule of an R -module M . A submodule C of M is said to be complement to S in M if C is maximal with respect to the property that $C \cap S = \{0\}$.

Definition 2. A submodule C of an R -module is a complement in M ($C \subseteq_c M$, for short) if there exists a submodule S of M such that C is complement to S in M .

Definition 3. 1. An R -module M is called extending module if every complement submodule of M is a direct summand.

2. An R -module M is called continuous if it is extending and satisfies the following condition: (C2) Every submodule of M that is isomorphic to a direct summand M is itself a direct summand of M .

3 An R -module M is called quasi-continuous if it is extending and satisfies the following condition: (C3) If N and K are direct summands of M with $N \cap K = 0$, then $N \oplus K$ is a direct summand of M .

Definition 4. Let M be an R -module, put $Z(M) = \{m \in M : \text{ann}_R(m) \leq_e R\}$. M is called nonsingular if $Z(M) = \{0\}$, and singular if $Z(M) = M$. The Goldie torsion submodule $Z_2(M)$ of M is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. M is Z_2 -torsion if, $Z_2(M) = M$.

Definition 5. A module M has finite uniform dimension n (written $\text{Udim}(M) = n$) if there is an essential submodule $V \leq_e M$ that is a direct sum of n uniform submodules.

3. Main results

Definition 6. An R -module is called c -retractable if $\text{Hom}_R(M, C) \neq 0$ for each $0 \neq C \subseteq_c M$.

Remark 1. Clearly, every retractable module is c -retractable. The converse is not true in general. For example: \mathbb{Q} as a \mathbb{Z} -module is c -retractable while it is not retractable..

Example 1. Every extending module is c -retractable.

Remark 2. If $R = \mathbb{Z}[x]$, then R is c -retractable by ([4], Example 2.4). Clearly, $R \oplus R$ is a c -retractable R -module. However, $R \oplus R$ is not extending. (see [4], Example 2.4).

Remark 3. ([4], Example 3.2)

Let R be the ring of all 2 by 2 upper triangular matrices which have arbitrary real numbers on the diagonal and an arbitrary complex number in the (1, 2)-position and let e_{ij} be the element of R with 1 in the (i, j) -position and 0 elsewhere. Set $P = e_{11}R$ and let K denote the field of real numbers. Hence, P is a nonsingular projective R -module which is not a c -retractable R -module while the R -module $M = R \oplus P$ is c -retractable. This shows that a direct summand (hence a submodule or a factor module) of a c -retractable module need not be c -retractable.

Proposition 1. Let M be a c -retractable R -module. Then M/N is c -retractable for any fully invariant complement submodule $N \leq M$.

Proof.

Let $K/N \subseteq_c M/N$ where $N \leq K \leq M$ and N is a fully invariant complement submodule of M . Then, $K \subseteq_c M$ by Proposition 6.28 in [11]. Thus, there exists a nonzero homomorphism $f : M \rightarrow K$. Now, $f(N) \subseteq N$ by hypothesis, and so $\bar{f} : M/N \rightarrow K/N$ defined by $\bar{f}(m + N) = f(m) + N$ for all $m \in M$ is a nonzero homomorphism.

Proposition 2. Let M be a c -retractable R -module such that $\text{Hom}_R(M/C, C)$ contains a monomorphism for any $C \subseteq_c M$. Then M/C is c -retractable.

Proof.

Let $N/C \subseteq_c M/C$. By the c -retractable condition on M , there is a nonzero homomorphism $g : M \rightarrow N$. From this and by our assumption, $\text{Hom}_R(M/C, N/C) \neq 0$.

Proposition 3. Let M be a c -retractable R -module. If $M = L \oplus N$ such that $\text{Hom}_R(L, N) = 0$, then N is a c -retractable R -module.

Proof.

Note that $\text{End}_R(M) = \begin{bmatrix} \text{End}_R(L) & \text{Hom}_R(N, L) \\ 0 & \text{End}_R(N) \end{bmatrix}$. Hence, $\text{End}_R(M) \begin{bmatrix} L \\ 0 \end{bmatrix} \subseteq \begin{bmatrix} L \\ 0 \end{bmatrix}$.

It follows that $(L \oplus 0)$ is a fully invariant complement submodule of M . Now, an application of Proposition 1 shows that N is c -retractable.

Proposition 4. Let M be a c -retractable R -module and $0 \neq C \subseteq_c M$. If $\text{Hom}_R(M/C, C) = 0$, then C is c -retractable.

Proof.

Let $0 \neq K \subseteq_c C$. Thus, there exists $0 \neq f \in S$ such that $\text{Im} f \subseteq K$.

If $f(C) = 0$, then the rule $m + n \rightarrow m + \text{Ker} f$ yields a nonzero homomorphism $M/C \rightarrow M/\text{Ker} f \cong \text{Im} f$ which is in contradiction with our assumption $\text{Hom}_R(M/C, C) = 0$. Thus, $f(C) \neq 0$, hence $f|_C$ is a nonzero endomorphism of C with image in K .

Proposition 5. *If an arbitrary direct sum of copies of M is c -retractable, then M is c -retractable.*

Proof.

This follows from ([17], Proposition 2.10).

Remark 4. *A projective module need not be c -retractable and vice-versa. In fact a simple is c -retractable but not be projective. Moreover, by Remark 3, there is a projective module which is not c -retractable. In the following, we show that certain classes of projective modules are c -retractable.*

Following [24], we call an R -module SI if every singular module is M -injective. Recall that a ring R is called right SI , if every singular R -module is injective.

Lemma 1. ([24], Proposition 2.2)

Every homomorphic image of a SI -module is a SI -module.

Lemma 2. ([24], Proposition 2.7)

The following conditions are equivalent for a ring R .

- (1) R is a right SI -ring.
- (2) Every R -module is a SI -module.

Theorem 1. *Let R be any ring. Then every projective SI R -module is retractable and hence c -retractable.*

Proof.

Let M be a nonzero projective SI R -module. Let $0 \neq m \in M$. For a given submodule A of mR , there exists a submodule C of mR such that $C \oplus A \leq_e mR$. Thus, $mR/(C \oplus A)$ is singular. Since M is a SI -module, $M/(C \oplus A)$ is a SI -module by Lemma 1. Hence, $mR/(C \oplus A)$ is $M/(C \oplus A)$ -injective and hence a direct summand of $M/(C \oplus A)$. It follows that M has a submodule B such that M/B is isomorphic to $mR/(C \oplus A)$. Hence there exists a nonzero homomorphism $f : M \rightarrow mR/(C \oplus A)$. By the projective condition on M , f can be lifted to a nonzero homomorphism $g : M \rightarrow mR$. Therefore, M is c -retractable.

Corollary 1. *Let R be a right SI -ring. Then every projective R -module is c -retractable.*

Theorem 2. *Let R be a right perfect ring. Then the following statements are equivalent for a hereditary R -module:*

- (1) M is c -retractable.
- (2) $\text{Hom}_R(M, C)$ contains an epimorphism for any $0 \neq C \subseteq_c M$.
- (3) M is extending.

Proof.

(1) \Rightarrow (2) follows a similar argument to the one used in ([14], Theorem 2.2).

(2) \Rightarrow (3). Let $0 \neq C \subseteq_c M$. By (2), there exists an epimorphism $f : M \rightarrow C$. Then $I_C : C \rightarrow C$ can be lifted to a nonzero homomorphism $g : C \rightarrow M$, and hence $C \leq^\oplus M$.

Therefore, M is extending.

(3) \Rightarrow (1) It is easy to see.

Recall that a family $\{N_i\}_I$ of independent submodules of a module M is said to be a local summand, if for any finite subset $A \subset I$, $\bigoplus_A N_\alpha$ is a direct summand of N . An R -module M is called uniform-extending if every uniform submodule is essential in a direct summand of M .

Recall that a module M is called *wd-Rickart* if the image any endomorphism of M contains a nonzero direct summand.

Lemma 3. *If M is an R -module such that every nonzero complement submodule contains a nonzero direct summand, then M is c -retractable.*

Proof.

This is clear.

Lemma 4. *Let M be a wd -Rickart R -module. Then M is c -retractable if and only if every nonzero complement submodule of M contains a nonzero direct summand.*

Proof.

The sufficiency follows from Lemma 3. Conversely, assume that M is any wd -Rickart c -retractable module. Let $0 \neq C \subseteq_c M$. Since M is c -retractable, there is a nonzero endomorphism φ of M such that $Im\varphi \subseteq C$. Thus, the wd -Rickart property of M implies that C contains a nonzero direct summand.

Lemma 5. *If M is any wd -Rickart c -retractable R -module, then every indecomposable complement submodule of M is uniform.*

Proof.

Let M be any wd -Rickart c -retractable module. Let C be an indecomposable complement submodule of M . Let D any nonzero complement submodule of C . Since $D \subseteq_c M$, we infer from Lemma 4 that D contains a nonzero direct summand E of M . As $E \leq C \leq M$ and $E \leq^\oplus M$, $E \leq^\oplus C$. Since C is indecomposable, $C = E = D$. It follows that D is a direct summand of C , and hence C is an extending module. Since C is indecomposable, C is uniform.

Theorem 3. *Let M be a wd -Rickart R -module for which local summands are summand. Then M is uniform-extending and c -retractable if and only if M is extending.*

Proof.

Suppose that M is a uniform-extending c -retractable module. Since local summands of M are summand, M is a direct sum of indecomposable modules (see [14], Theorem 2.17). Thus by Lemma 5, M is direct sum of uniform modules. Therefore, by ([6], 8.5), M is extending. The converse implication is clear.

Corollary 2. *Let M be a wd-Rickart quasi-discrete R -module. Then M is uniform-extending and c -retractable if and only if M is extending.*

Proof.

This follows from Theorem 3 and the fact that any local summand of a quasi-discrete module is a summand (see [6], Corollary 4.13).

Remark 5. *By Lemma 5, an indecomposable wd-Rickart c -retractable module is uniform.*

Recall that a module M is called simple radical, if $M \neq 0$ such that $Rad(M) = M$ and M has no proper nonzero submodules N with $Rad(N) = N$. Hence a simple radical c -retractable module is uniform.

Let M be an R -module and $N \leq M$. Put $D(N) = \{\varphi \in S : Im\varphi \subseteq N\}$. M is called dual Baer if for every $N \leq M$, there is $e^2 = e \in S$ such that $D(N) = eS$.

Recall that an R -module M is said to be ADS if for every decomposition $M = S \oplus T$ and every complement T' of S , we have $M = S \oplus T'$. Recall that an R -module M is called quasi-Baer if, for all fully invariant submodules $N \leq M$, $L_S(N) = Se$, with $e^2 = e \in S$.

Proposition 6. *Let M be a dual Baer c -retractable R -module. Then the following statements hold:*

- (1) M is a direct sum of uniform submodules.
- (2) $M = Z_2(M) \oplus (\oplus_{i \in I} M_i)$ with all M_i nonsingular uniform quasi-Baer and $End(M_i)$ semi-local quasi-Baer.
- (3) If R is a right self-injective ring, then $M = Z_2(M) \oplus M'$ where M' is nonsingular semisimple.

Proof.

(1) Suppose M is dual-Baer c -retractable. By Corollary 2.6(i) in [10], M is a direct sum of indecomposable submodules. By ([24], Theorem 3.1), M is wd-Rickart. Thus, according to Lemma 5, M is a direct sum of uniform submodules.

(2) Suppose M has the stated condition. Then by (1), M is a direct sum of uniform modules. Hence by ([2], Corollary 2.3, Theorems 3.2 and 3.9), $M = Z_2(M) \oplus M'$ where M' is quasi-Baer. Since M is dual Baer, we infer from Corollaries 2.5 and 2.6 in [2] that $M' = \oplus_{i \in I} M_i$ with all M_i indecomposable. Thus, $M = Z_2(M) \oplus (\oplus_{i \in I} M_i)$ where each M_i is indecomposable. Consequently, each M_i is nonsingular uniform by Lemma 5. On the other hand since M' is quasi-Baer, it follows from ([18], Theorem 3.17) that each M_i is quasi-Baer for each $i \in I$. The last part follows from ([10], Corollary 2.5 and Proposition 2.17) and ([18], Theorem 4.1).

(3) By (2), $M = Z_2(M) \oplus (\oplus_{i \in I} M_i)$ with all M_i nonsingular uniform. Let $M' = \oplus_{i \in I} M_i$. Thus, since R is right self-injective, all M_i are simple, proving the result.

Theorem 4. *Let M be a dual Baer R -module. Then the following statements are equivalent:*

- (1) M is ADS and c -retractable.
- (2) M is continuous.
- (3) M is quasi-continuous.

Proof.

(1) \Rightarrow (2) Suppose M is ADS and c -retractable. Since M is dual Baer, we infer from Proposition 6(1) that $M = \bigoplus_{i \in I} M_i$ is a direct sum of uniform modules. Thus, every M_i is quasi-continuous for every $i \in I$. On the other hand since M is ADS, we infer from Lemma 3.1 in [1] that $\bigoplus_{i \neq j \in I} M_j$ is M_i -injective for every $i \in I$. Therefore M is quasi-continuous by ([14], Theorem 2.13). Now, let φ be an essential monomorphism of M . Then $Im\varphi \leq_e M$. Since M is dual Baer, $Im\varphi \leq^\oplus M$. Hence, $Im\varphi = M$. Therefore, according to ([14], Lemma 3.14), M is continuous.

(3) \Rightarrow (1) This implication is clear.

Corollary 3. *Let M be a dual Baer c -retractable R -module such that every nonsingular summand is ADS. Then $M = Z_2(M) \oplus M'$ where M' is nonsingular quasi-continuous.*

Proof.

By Proposition 6(2), $M = Z_2(M) \oplus (\bigoplus_{i \in I} M_i)$ with all M_i nonsingular uniform. Let $M' = \bigoplus_{i \in I} M_i$. Thus, by our assumption, M' is ADS. Therefore, applying the same techniques as in the proof of Theorem 4, one can show easily that M' is quasi-continuous.

Proposition 7. *Let M be a d -Rickart R -module with S is left T -nilpotent. Then the following statements are equivalent:*

- (1) M is ADS and c -retractable.
- (2) M is quasi-continuous.

Proof.

(1) \Rightarrow (2) Since M is d -Rickart and S is left T -nilpotent, it follows from Proposition 3.4.11 in [12] that $M = \bigoplus_i^n M_i$ with all M_i indecomposable. Since d -Rickart modules are wd -Rickart, we infer from Lemma 5 that $M = \bigoplus_i^n M_i$ with all M_i uniform. On the other hand since M is ADS, we infer from Lemma 3.1 in [1] that $\bigoplus_{i \neq j} M_j$ is M_i -injective for every $1 \leq i \leq n$. Thus M is quasi-continuous by ([14], Lemma 2.14).

(2) \Rightarrow (1) This implication is clear.

Let M be an R -module. The left annihilator of $N \leq M$ in $S = End_R(M)$ is denoted by $L_S(N) = \{\phi \in S : \phi N = \{0\}\}$.

Let M be a module. A submodule N of M is said to be an automorphism-invariant submodule if $\varphi N \subseteq N$ for automorphism φ of M . M is called auto-invariant if it is an automorphism-invariant submodule of its injective hull.

Proposition 8. *Let M be a dual Baer R -module. Then M is auto-invariant and c -retractable if and only if M is quasi-injective.*

Proof.

Suppose M is auto-invariant and c -retractable. Since M is dual Baer, we infer from Proposition 6(1) that $M = \bigoplus_{i \in I} M_i$ is a direct sum of extending modules. Thus, by Corollary 15 in [13], M is quasi-injective. The converse implication is clear.

Recall that an R -module M is called C_4 if, whenever A and B are submodules of M with $M = A \oplus B$ and $f : A \rightarrow B$ is an homomorphism with $\text{Ker} f \leq^\oplus A$, we have $\text{Im} f \leq^\oplus B$.

Proposition 9. *If every 2-generated R -module is a C_4 -module, then every dual Baer c -retractable R -module is semisimple.*

Proof.

Let M be any dual Baer c -retractable R -module. Thus, as in the proof of Theorem 4, $M = \bigoplus_{i \in I} M_i$ where each M_i is uniform. Now, we have to show that each M_i is semisimple. For any $0 \neq m \in E(M_i)$, let $0 \neq N \leq mR$ and take $0 \neq n \in N$. By our assumption, $mR \oplus nR$ is a C_4 -module. Consider the inclusion map $i : nR \rightarrow mR$. Thus $i(nR) = nR \leq^\oplus mR$. Since mR is indecomposable, $nR = mR$, and hence $N = mR$. Thus, every cyclic submodule of mR is a direct summand. It follows that mR is semisimple. Hence, $E(M_i)$ is semisimple. Consequently, M_i is semisimple. Therefore, M is semisimple.

Theorem 5. *The following conditions are equivalentes for a ring R :*

- (1) R is semisimple artinian.
- (2) Every c -retractable R -module is a C_4 -module.
- (3) Every c -retractable R -module is pseudo-projective.

Proof.

(1) \Rightarrow (2) is clear.

(2) \Rightarrow (1) Let I be a right ideal of R . Clearly, $I \oplus R$ is c -retractable, and hence a C_4 -module by (2). Consider the inclusion map $i : I \rightarrow R$. Therefore, $i(I) = I \leq^\oplus R$. Hence, R_R is semisimple. Thus, R is semisimple artinian.

(1) \Rightarrow (3) is clear.

(3) \Rightarrow (1) Let S be a simple R -module. Then there is a free R -module F and an epimorphism $f : F \rightarrow S$. Hence, $S \oplus F$ is c -retractable by ([19], Proposition 1.4). By our assumption, $S \oplus F$ is pseudo-projective. Now, Consider the exact sequence

$0 \rightarrow \text{Ker} f \xrightarrow{g} M \xrightarrow{f} 0$. So, by the proof of ([15], Proposition 3.9), this sequence splits. Consequently, $S \leq^\oplus F$, and hence S is projective. Therefore, R is semisimple.

Remark 6. *Theorem 5 shows that the condition "right V -ring" in ([15], Proposition 3.9) is superfluous.*

Recall that an R -module is called Baer if, for all $N \leq M$, $L_S(N) = Se$, with $e^2 = e \in S$. A module M is called \mathcal{K} -nonsingular if, $\forall \varphi \in \text{End}(M)$, $\text{Ker} \varphi \leq_e M$ implies $\varphi = 0$.

Proposition 10. *Let M be a \mathcal{K} -nonsingular c -retractable R -module. Then S is right nonsingular.*

Proof.

See proof of ([16], Proposition 3.6).

Proposition 11. *Let M be a c -retractable R -module such that S_S is extending. Then M is \mathcal{K} -nonsingular if and only if M is Baer.*

Proof.

Suppose M is \mathcal{K} -nonsingular. By Proposition 10, S is right nonsingular. Let N be a submodule of M . Thus, $L_S(N)$ is a complement right ideal in S . Because S_S is extending, then $L_S(N) = S(1 - e)$ for some $e = e^2 \in S$, and hence M is Baer. The converse implication follows from ([18], Lemma 2.15).

Recall that a module is locally noetherian if any of its finitely generated submodules is noetherian. An R -module M is said to be homo-related to an R -module L if there are $\alpha : M \rightarrow L$ and $\beta : L \rightarrow M$ such that $\beta\alpha \neq 0$.

Theorem 6. *Let M be a locally noetherian c-retractable R -module. Then M is homo-related to a direct sum $\bigoplus_{i \in I} U_i$ of uniform submodules of M .*

Proof.

Suppose M is a c-retractable locally noetherian module. Hence, every submodule of M contains a uniform submodule. Thus, by Zorn's Lemma, M contains a maximal local direct summand $N = \bigoplus_{i \in I} U_i$ where each U_i is uniform. Also by the locally noetherian condition on M again, $R/r(m) \cong mR$ is noetherian for any element m in M . Hence, R satisfies ACC on right ideals of the form $r(m)$ where $m \in M$. Thus, according to ([6], 8.1), N is a complement submodule of M . Since M is c-retractable, there exists a nonzero homomorphism $f : M \rightarrow N$. It follows that M is homo-related to N .

Corollary 4. *Let R be a right noetherian ring. Then every c-retractable R -module is homo-related to a direct sum $\bigoplus_{i \in I} U_i$ of uniform submodules of M .*

Theorem 7. *Let M be a nonsingular c-retractable R -module such that every $Udim(mR) < \infty$ for every element $m \in M$. Then M is homo-related to a direct sum $\bigoplus_{i \in I} U_i$ of indecomposable nonsingular submodules of M .*

Proof.

Suppose M has the stated condition. By Zorn's Lemma, M contains a maximal local direct summand $N = \bigoplus_{i \in I} U_i$ where each U_i is indecomposable nonsingular. Let $m \in M$. Then $R/r(m)$ is a nonsingular R -module which has finite uniform dimension. By ([6], Section 5.10), R has ACC on right ideals of the form $r(m)$ where $m \in M$. Thus, according to ([6], 8.1), N is a complement submodule of M . Since M is c-retractable, there exists a nonzero homomorphism $f : M \rightarrow N$. It follows that M is homo-related to N .

Proposition 12. *Let M be a c-retractable R -module with $Udim(M) \geq 2$. Then M is retractable.*

Proof.

Suppose M has the stated condition. Let $0 \neq N \leq M$. Since $Udim(N) < \infty$, N contains a uniform submodule U . After replacing U by an essential closure, we may assume that U is a complement submodule of M . By our assumption, there is a nonzero homomorphism $M \rightarrow U$. Therefore, M is retractable.

Corollary 5. *Let M be an R -module with $Udim(M) \geq 2$. Then M is wd -Rickart c -retractable if and only if M is semisimple.*

Proof.

Suppose M is wd -Rickart c -retractable. Since $Udim(M) \geq 2$, M is a finite direct sum of indecomposable submodules. By Proposition 12, M is retractable. Therefore, according to ([23], Proposition 2.17), M is semisimple. The converse implication is clear.

Proposition 13. *The following statements are equivalent for an R -module M with $udim(M) = n \geq 2$.*

- (1) M is c -retractable.
- (2) $Hom_R(M, U) \neq 0$ for every uniform submodule U of M .
- (3) $Hom_R(M, U) \neq 0$ for every cyclic uniform submodule U of M .
- (4) M is retractable.

Proof.

(1) \Rightarrow (2) follows from Proposition 12.

(2) \Rightarrow (3) Clear.

(3) \Rightarrow (4) Let $0 \neq N \leq M$. Let $0 \neq m \in N$. By hypothesis, mR has finite uniform dimension and hence mR contains a uniform submodule U . Let $0 \neq u \in U$. By (3), $Hom_R(M, uR) \neq 0$. Hence, M is retractable.

(4) \Rightarrow (1) is clear.

Remark 7. *By Proposition 12, every c -retractable module M with $udim(M) = n \geq 2$ is retractable. Note that the condition $udim(M) = n \geq 2$ can not be dropped. In fact, \mathbb{Q} as a \mathbb{Z} -module is c -retractable uniform but it is not retractable.*

References

- [1] A. Alahmadi, S. K Jain and A. Leroy (2012), *ADS modules*, J. Algebra 352:215-22.
- [2] S. Asgari and A. Haghany, *Generalizations of T -extending Modules relative to fully invariant submodules*, J. Korean Math. Soc. 49(2012), No. 3, pp. 503-514.
- [3] Sh. Asgari and A. Haghany and A. R. Rezaei, (2014). *Modules whose t -closed submodule have a summand as a complement*, Comm. Algebra, 42:5299-5318.
- [4] A. W. Chatters and S. M. Kheuri, *Endomorphism rings of modules over nonsingular CS rings*, J. London Math. Soc. 21(1980), 434-444.
- [5] N. Ding, Y. Ibrahim, M. Yousif and Y. Zhou (2017), *$C4$ -Modules*, Comm. Algebra 45(4):1727-1740.
- [6] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer (1994), *Extending modules*, Pitman Research Notes in Mathematics 313. Harlow: Longman.

- [7] A. Ghorbani and M. R. Vedadi, *Epi-retractable modules and some application*, Bull. Iranian Math. Soc. 35 (2009), no. 1, 155-166.
- [8] K. R. Goodearl, *Ring theory, nonsingular rings and modules*. Marcel Dekker, Inc New York and Basel (1976).
- [9] Y. Ibrahim, X. H. Nguyen, M. Yousif and Y. Zhou, *Rings whose cyclics are $C3$ -module*, J. Algebra. Appl. 15 (8) (2016)(18 pages).
- [10] D. Keskin Tutunu and R. Tribak (2010), *On dual Baer modules* 52:261-269.
- [11] T. Y. Lam, *Lectures on modules and rings*, G.T.M.(189), Springer-Verlag, Berlin-Heidelber, New York, 1999.
- [12] G. Lee, *Theory of Rickart modules*, Ph. D. Thesis, M.S., Graduate, School of the Ohio State University (2010).
- [13] T. K Lee and Y. Zhou (2013), *Modules which are invariant under automorphisms injective hulls*, J. Algebra Appl. 12(2):9pp.
- [14] S. H. Mohamed and B. J. Muller, *Continuous and Discrete Modules*, LMS Lecture Note Series, 147. Cambridge University Press, Cambridge, 1990.
- [15] B. M. Pandeya, A. K Chaturvedi and A. J. Gupta, *Applications of epi-retractable modules*, Bull. Iranian Math. Soc. 35 No. 2(2012), pp 469-477.
- [16] S. T. Rizvi and C. S. Roman, *On \mathcal{K} -nonsingular modules and applications*. Comm. Algebra, 35 (2007): 2960-2980.
- [17] S. T. Rizvi and C. S. Roman, *On Direct Sum of Baer Modules*, J. Algebra, 59 (2009): 632-696.
- [18] S. T. Rizvi and C. S. Roman, *Baer and quasi-Baer modules*. Comm. Algebra, 32 (2004): 103-123.
- [19] P. F. Smith, *Modules with many homomorphisms*, J. of Pure and Appl. Algebra, 197: 305-321 (2005).
- [20] P. F. Smith and A. Tercan (1993). *Generalizations of CS -modules*. Comm. Algebra 21:1809-1847.
- [21] P. F. Smith and A. Tercan (1993). *Direct summands of modules which satisfy (C_{11})* . Algebra Colloq. 11:231-237.
- [22] R. Tribak (2015), *On Weakly Dual Rickart Module and Dual Baer Modules*, Comm. Algebra 43:8, 3190-3206.
- [23] M. R. Vedadi, *Essentially retractable modules*, Journal of Science, Islamic Republic of Iran 18(4), 2007, 355-360.
- [24] M. F. Yousif, *SI-Modules*, Math. J. Okayama Univ. 28, 133-146(1986).