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Some Closure Operators and Topologies on a Hyper BCK-algebra

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Abstract. Given a hyper BCK-algebra $(H, *, 0)$, we introduce some subsets of H and use them to generate two closure operators on H. In this paper, we show that each of the two closure operators on H can be utilized to form a base for some topology on H. Moreover, we show that each of the induced topologies coincides with a previously known topology on a hyper BCK-algebra.
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1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [4] in 1966 as a generalization of the concept of set theoretic difference and propositional calculi. The hyperstructure theory (or multialgebras) was introduced in 1934 by F. Marty [6] at the 8th Congress of Scandinavian Mathematicians in 1934. In [5], Y.B. Jun et al. applied the hyperstructures to BCK-algebras and introduced the notion of a hyper BCK-algebra which is a generalization of a BCK-algebra.

By using the sets $L_H(A)$ and $R_H(A)$, we introduce the bases $\mathcal{B}_L(H)$ and $\mathcal{B}_R(H)$ and the induced topologies $\tau_L(H)$ and $\tau_R(H)$ by these sets, respectively, and investigate their related properties [7, 8]. In this paper, we present two closure operators on a hyper BCK-algebra and consider the respective topologies they generate. It is shown that these topologies coincide, respectively, with the topologies generated by $\mathcal{B}_L(H)$ and $\mathcal{B}_R(H)$.

2. Preliminaries

A *hyper BCK-algebra* is a nonempty set H endowed with a hyperoperation "$*$" and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

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(H1) $(x * z) * (y * z) \ll x * y$,

(H2) $(x * y) * z = (x * z) * y$,

(H3) $x * H \ll x$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where for every $A, B \subseteq H$, $A \ll B$ if and only if for each $a \in A$, there exists $b \in B$ such that $0 \in a * b$. In particular, for every $x, y \in H$, $x \ll y$ if and only if $0 \in x * y$. In such case, we call " $\ll$ " the *hyper order* in H.

Throughout this study, $(H_1, *_1, 0_1)$ (or simply H_1) and $(H_2, *_2, 0_2)$ (or simply H_2) are hyper BCK-algebras.

Let H be a hyper BCK-algebra and $A \subseteq H$. The sets $L_H(A)$ and $R_H(A)$ are given as follows:

$$L_H(A) := \{x \in H \mid x \ll a \ \forall a \in A\} = \{x \in H \mid 0 \in x * a \ \forall a \in A\} \quad \text{and}$$
$$R_H(A) := \{x \in H \mid a \ll x \ \forall a \in A\} = \{x \in H \mid 0 \in a * x \ \forall a \in A\}.$$

If $A = \{a\}$, we write $L_H(\{a\}) = L_H(a)$ and $R_H(\{a\}) = R_H(a)$.

Let X be a nonempty set and let $\mathscr{P}(X)$ denote the power set of X. A mapping $\phi : \mathscr{P}(X) \to \mathscr{P}(X)$ is called a *closure operator* on X, if for all $A, B \in \mathscr{P}(X)$, the following properties hold [2]:

(i) $A \subseteq \phi(A)$

(ii) $\phi^2(A) = \phi(A)$

(iii) $A \subseteq B \Rightarrow \phi(A) \subseteq \phi(B)$.

3. Known Results

Proposition 1. [7] *Let H be a hyper BCK-algebra and $A, B \subseteq H$. Then the following hold:*

(i) $R_H(\varnothing) = H$.

(ii) $R_H(A) = \bigcap_{a \in A} R_H(a)$.

(iii) *For any $\varnothing \neq A \subseteq H$ such that $A \neq \{0\}$, $0 \notin R_H(A)$.*

(iv) *If $A \subseteq B$, then $R_H(B) \subseteq R_H(A)$.*

The next result is generated by Albaracin and Vilela.

Proposition 2. [1] *Let A and B be subsets of a hyper BCK-algebra H. Then the following hold:*

(i) $L_H(\varnothing) = H$.

(ii) *If $A \subseteq B$, then $L_H(B) \subseteq L_H(A)$.*

(iii) $L_H(A) = \bigcap_{a\in A} L_H(a)$.

(iv) *For any* $A \subseteq H$, $0 \in L_H(A)$. *If* $0 \in A$, *then* $L_H(A) = \{0\}$.

Theorem 1. [8] *Let* H *be a hyper BCK-algebra. Then* $\mathcal{B}_L(H) = \{L_H(A) : A \subseteq H\}$ *is a basis for some topology on* H.

Denote by $\tau_L(H)$ the topology generated by $\mathcal{B}_L(H)$.

Theorem 2. [7] *Let* H *be a hyper BCK-algebra. Then* $\mathcal{B}_R(H) = \{R_H(A) : \varnothing \neq A \subseteq H\}$ *is a basis for some topology on* H.

Denote by $\tau_R(H)$ the topology generated by $\mathcal{B}_R(H)$.

4. Topology Induced by $R_H L_H$

Let H be any hyper BCK-algebra with $H \neq \{0\}$. By Proposition 1(iii), $0 \notin R_H(H)$. By definition of a closure operator, we have the following remark.

Remark 1. $R_H : \mathscr{P}(H) \to \mathscr{P}(H)$ *is not a closure operator for every hyper BCK-algebra* $H \neq \{0\}$.

Theorem 3. *Let* H *be a hyper BCK-algebra and* $A, B \subseteq H$. *Then the following properties hold:*

(i) $A \subseteq R_H(L_H(A))$.

(ii) *If* $A \subseteq B$, *then* $R_H(L_H(A)) \subseteq R_H(L_H(B))$.

(iii) $[R_H L_H]^2(A) = R_H L_H[R_H(L_H(A))] = R_H(L_H(A))$.

Proof.

(i) Let $A \subseteq H$ and $y \in L_H(A)$. Then $y \ll a$ for all $a \in A$. Pick $x \in A$. Then $y \ll x$ for every $y \in L_H(A)$. This means that $x \in R_H(y)$ for every $y \in L_H(A)$. Thus, by Proposition 1(ii), $x \in \bigcap_{y\in L_H(A)} R_H(y) = R_H L_H(A)$. Therefore, $A \subseteq R_H(L_H(A))$.

(ii) Let A, B be subsets of H. Suppose $A \subseteq B$. By Proposition 2(ii), $L_H(B) \subseteq L_H(A)$. Thus, by Proposition 1(iv), $R_H L_H(A) \subseteq R_H L_H(B)$.

(iii) By (i) and (ii), $R_H L_H(A) \subseteq [R_H L_H]^2(A)$. We are left to prove that $[R_H L_H]^2(A) \subseteq (R_H L_H)(A)$. First, we need to show that $L_H(A) \subseteq L_H[R_H(L_H(A))]$. Let $x \in R_H(L_H(A))$. Then $y \ll x$ for all $y \in L_H(A)$. Choose $z \in L_H(A)$. Then $z \ll x$ for every $x \in R_H(L_H(A))$. This implies that $z \in L_H(x)$ for every $x \in R_H(L_H(A))$. Hence, by Proposition 2(iii), $z \in \bigcap_{x\in R_H(L_H(A))} L_H(x) = L_H[R_H(L_H(A))]$. Consequently, $L_H(A) \subseteq L_H[R_H(L_H(A))]$. Therefore, by Proposition 1(iv), $R_H L_H[R_H(L_H(A))] \subseteq R_H(L_H(A))$.

Theorem 4. *Let H be a hyper BCK-algebra. The function $R_H L_H : \mathscr{P}(H) \to \mathscr{P}(H)$ is a closure operator on H.*

Proof. Since H is a hyper BCK-algebra, $H \neq \varnothing$. Hence, by the definition of a closure operator and Theorem 3, $R_H L_H$ is a closure operator.

Theorem 5. *The family $\mathcal{B}_{RL}(H) = \{A : R_H(L_H(A)) = A, A \subseteq H\}$ is a basis for some topology on H.*

Proof. Since $R_H(L_H(H)) \subseteq H$ and by Theorem 3(i), $R_H L_H(H) = H$, it follows that $H \in \mathcal{B}_{RL}(H)$. Thus, $\mathcal{B}_{RL}(H) \neq \varnothing$. Let $A, B \in \mathcal{B}_{RL}(H)$ and $x \in A \cap B$. Then by Theorem 3(i), we have $A \cap B \subseteq R_H(L_H(A \cap B))$. Also, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $R_H(L_H(A \cap B)) \subseteq R_H(L_H(A))$ and $R_H(L_H(A \cap B)) \subseteq R_H(L_H(B))$, by Theorem 3(ii). Since A and B are elements of $\mathcal{B}_{RL}(H)$, we have $R_H(L_H(A)) = A$ and $R_H(L_H(B)) = B$. Consequently, $R_H(L_H(A \cap B)) \subseteq A \cap B$. Combining the two set inclusions, we have $R_H(L_H(A \cap B)) = A \cap B$, that is, $x \in A \cap B \in \mathcal{B}_{RL}(H)$. Therefore, $\mathcal{B}_{RL}(H)$ is a basis for some topology on H.

Denote by $\tau_{RL}(H)$ the topology generated by $\mathcal{B}_{RL}(H)$.

Example 1. Consider the infinite hyper BCK-algebra $(H, *, 0)$ given by Harizavi in [3], where $H = \{0, 1, 2, \ldots\}$ and "$*$" is defined as follows:

$$x * y = \begin{cases} \{0, x\} & \text{if } x \leq y \\ \{x\} & \text{if } x > y \end{cases}$$

for all $x, y \in H$. Now, let $r \in H$. Then $R_H(L_H(r)) = R_H(\{0, 1, 2, \ldots, r\}) = \{r, r+1, r+2, \ldots\} \neq \{r\}$. Hence, for any $s \in H$, $\{s\} \notin \mathcal{B}_{RL}(H)$. Let $\varnothing \neq A \subseteq H$ be a finite set and let $u = \min A$. Since $R_H(L_H(A)) = R_H L_H(u) = R_H(\{0, 1, 2, \ldots, u\}) = \{u, u+1, u+2, \ldots\} = R_H(u) \neq A$, it follows that for any finite set $A \subseteq H$, $A \notin \mathcal{B}_{RL}(H)$. Next, suppose that $A \subseteq H$ is an infinite set and let $p = \min A$. Then $R_H(L_H(A)) = R_H(p) = \{p, p+1, p+2, \ldots\}$. Thus, $A \in \mathcal{B}_{RL}(H)$ if and only if $A = \{p, p+1, p+2, \ldots\}$. Therefore, $\tau_{RL}(H) = \{\varnothing, H\} \cup \{\{p, p+1, p+2, \ldots\} : p \in H\}$.

In the next example, we extend the set H in Example 1 by adjoining to it the set $\{\frac{1}{n} : n = 2, 3, ...\}$.

Example 2. Consider now the infinite hyper BCK-algebra $(H, \circ, 0)$ given also by Harizavi in [3], where $H = \{0, 1, 2, \ldots\} \cup \{\frac{1}{n} : n = 2, 3, ...\}$ and "$\circ$" is defined as follows:

$$x \circ y = \begin{cases} \{0, x\} & \text{if } x \leq y \\ \{x\} & \text{if } x > y \end{cases}$$

for all $x, y \in H$. For convenience, let $\mathbb{N} = \{1, 2, ...\}$, $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, and let $J_k = \{\frac{1}{m} : m \text{ is a positive integer and } m \geq k\}$. Let $p \in H$. If $p = 0$, then $L_H(p) = \{0\}$ and $R_H(L_H(p)) = H$. If $p \in \mathbb{N}$, then $L_H(p) = J_2 \cup \{0, 1, 2, ..., p\}$. Hence, $R_H(L_H(p)) = R_H(J_2 \cup \{0, 1, 2, \ldots, p\}) = \{p, p+1, p+2, \ldots\}$. If $p = \frac{1}{n}$ for some $n \in \{2, 3, ...\}$, then

$L_H(p) = J_n \cup \{0\} = \{0, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, ...\}$. It follows that $R_H(L_H(p)) = R_H(J_n \cup \{0\}) = \{x \in H : a \leq x \text{ for all } a \in (J_n \cup \{0\})\} = \mathbb{N} \cup \{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}\}$.

Now let $\varnothing \neq A \subseteq H$. If $0 \in A$, then $L_H(A) = \{0\}$ and $R_H(L_H(A)) = H$. Thus, $A \in \mathcal{B}_{RL}(H)$ if and only if $A = H$. Next, suppose that $0 \notin A$. Suppose first that $A \cap J_2 \neq \varnothing$. Suppose that $A \cap J_2$ is infinite and suppose further that there exists $z \in L_H(A)$. Then $z \leq a$ for all $a \in A$. It follows that $z \leq b$ for all $b \in A \cap J_2$, contrary to the assumption that $A \cap J_2$ is infinite. Thus, $L_H(A) = \varnothing$. Next, suppose that $A \cap J_2$ is finite and let $\frac{1}{q} = \min(A \cap J_2)$. Then $L_H(A) = J_q \cup \{0\}$. Hence, $R_H(L_H(A)) = \mathbb{N} \cup \{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{q}\}$. Therefore, $A \in \mathcal{B}_{RL}(H)$ if and only if $A = \mathbb{N} \cup \{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{q}\}$. Suppose now that $A \cap J_2 = \varnothing$. Then $L_H(A) = J_2 \cup \{0, 1, 2, ..., r\}$, where $r = \min A$. Thus, $R_H(L_H(A)) = R_H(J_2 \cup \{0, 1, 2, \ldots, r\}) = \{r, r+1, r+2, \ldots\}$. Thus, $A \in \mathcal{B}_{RL}(H)$ if and only if $A = \{r, r+1, r+2, \ldots\}$. Therefore, $\tau_{RL}(H) = \{\varnothing, H\} \cup \{\{p, p+1, p+2, \ldots\} : p \in \mathbb{N}\} \cup \{\mathbb{N} \cup \{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{k}\} : k = 2, 3, ...\}$.

It can be observed that the topology $\tau_{RL}(H)$ in Example 1 coincides with the topology $\tau_R(H)$ in Example 2.5 [7]. In fact, the next result shows that this equality is true for any hyper BCK-algebra H.

Theorem 6. *Let H be a hyper BCK-algebra. Then the topology $\tau_{RL}(H)$ coincides with the topology $\tau_R(H)$.*

Proof. By Theorem 5, a basis for $\tau_{RL}(H)$ is the family $\mathcal{B}_{RL}(H) = \{A : R_H L_H(A) = A, A \subseteq H\}$ while a basis for $\tau_R(H)$ is the family $\mathcal{B}_R(H) = \{R_H(A) : \varnothing \neq A \subseteq H\}$. Let $A \in \mathcal{B}_{RL}(H)$. Then $R_H L_H(A) = A$. Set $D = L_H(A)$. Then $R_H(D) = A \in \mathcal{B}_R(H)$, showing that $\mathcal{B}_{RL}(H) \subseteq \mathcal{B}_R(H)$. Next, let $U \in \mathcal{B}_R(H)$. Then there exists $B \subseteq H$ such that $R_H(B) = U$. This implies that for every $u \in U$, $b \ll u$ for all $b \in B$. Hence, $B \subseteq L_H(U)$. By Proposition 1(iv), $R_H L_H(U) \subseteq R_H(B) = U$. Also, by Theorem 3(i), we have $U \subseteq R_H L_H(U)$. Thus, $U = R_H L_H(U) \in \mathcal{B}_{RL}(H)$, it shows that $\mathcal{B}_R(H) \subseteq \mathcal{B}_{RL}(H)$. Therefore, $\mathcal{B}_{RL}(H) = \mathcal{B}_R(H)$. Accordingly, $\tau_{RL}(H) = \tau_R(H)$.

5. Topology Induced by $L_H R_H$

Let H be any hyper BCK-algebra with $H \neq \{0\}$. By Proposition 2(iv), $L_H(H) = \{0\}$. By definition of a closure operator, we obtain the following remark.

Remark 2. *$L_H : \mathscr{P}(H) \to \mathscr{P}(H)$ is not a closure operator for every hyper BCK-algebra $H \neq \{0\}$.*

Theorem 7. *Let A, B be subsets of a hyper BCK-algebra H. Then the following properties hold:*

(i) *$A \subseteq L_H R_H(A)$.*

(ii) *If $A \subseteq B$, then $L_H(R_H(A)) \subseteq L_H(R_H(B))$.*

(iii) *$[L_H R_H]^2(A) = L_H R_H[L_H(R_H(A))] = L_H(R_H(A))$.*

Proof.

(i) Let $A \subseteq H$ and $x \in R_H(A)$. Then $a \ll x$ for all $a \in A$. Select $b \in A$. Then $b \ll x$ for every $x \in R_H(A)$. This means that $b \in L_H(x)$ for every $x \in R_H(A)$. Hence, $b \in L_H(R_H(A))$. Therefore, $A \subseteq L_H(R_H(A))$.

(ii) Let $A, B \subseteq H$ such that $A \subseteq B$. By Proposition 1(iv), $R_H(B) \subseteq R_H(A)$. Thus, by Proposition 2(ii), $L_H(R_H(A)) \subseteq L_H(R_H(B))$.

(iii) By (i) and (ii), $L_H(R_H(A)) \subseteq [L_H R_H]^2(A)$. We are left to prove that $[L_H R_H]^2(A) \subseteq L_H(R_H(A))$. First, we need to show that $R_H(A) \subseteq R_H[L_H(R_H(A))]$. Let $x \in L_H(R_H(A))$. Then $x \ll y$ for all $y \in R_H(A)$. Take $z \in R_H(A)$. It follows that $x \ll z$ for every $x \in L_H(R_H(A))$. This implies that $z \in R_H(x)$ for each $x \in L_H(R_H(A))$. Hence, $z \in R_H[L_H(R_H(A))]$. This shows that $R_H(A) \subseteq R_H[L_H(R_H(A))]$. Thus, by Proposition 2(ii), $L_H R_H[L_H(R_H(A))] \subseteq L_H(R_H(A))$. By combining the two set inclusions, we have $L_H R_H[L_H(R_H(A))] = L_H(R_H(A))$.

Theorem 8. *Let H be a hyper BCK-algebra. The function $L_H R_H : \mathscr{P}(H) \to \mathscr{P}(H)$ is a closure operator on H.*

Proof. Since H is a hyper BCK-algebra, $H \neq \varnothing$. Then by the definition of a closure operator and Theorem 7, $L_H R_H$ is a closure operator.

Theorem 9. *The family $\mathcal{B}_{LR}(H) = \{A : \; L_H R_H(A) = A, \; \varnothing \neq A \subseteq H\}$ is a basis for some topology on H.*

Proof. Since $L_H R_H(H) \subseteq H$ and by Theorem 7(i), we have $L_H R_H(H) = H$ and so, $H \in \mathcal{B}_{LR}(H)$. Thus, $\mathcal{B}_{LR}(H) \neq \varnothing$. Next, let $A, B \in \mathcal{B}_{LR}(H)$ and $x \in A \cap B$. Then by Theorem 7(i), we have $A \cap B \subseteq L_H R_H(A \cap B)$. Also, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $L_H R_H(A \cap B) \subseteq L_H R_H(A)$ and $L_H R_H(A \cap B) \subseteq L_H R_H(B)$ by Theorem 7(ii). Since $A, B \in \mathcal{B}_{LR}(H)$, we have $L_H R_H(A) = A$ and $L_H R_H(B) = B$. Hence, $L_H R_H(A \cap B) \subseteq A \cap B$. Accordingly, $L_H R_H(A \cap B) = A \cap B \in \mathcal{B}_{LR}(H)$. This shows that $\mathcal{B}_{LR}(H)$ is a basis for some topology on H.

Denote by $\tau_{LR}(H)$ the topology generated by $\mathcal{B}_{LR}(H)$.

Example 3. Consider the hyper BCK-algebra $(H, *, 0)$ in Example 1. Let $n \in H$. Then $L_H R_H(n) = L_H(\{n, n+1, n+2, \ldots\}) = \{0, 1, 2, \ldots, n\} \neq \{n\}$. Hence, for any $m \in H$, $\{m\} \notin \mathcal{B}_{LR}(H)$. Let A be any infinite subset of H. Then $L_H R_H(A) = L_H(\varnothing) = H \neq A$. It follows that $A \notin \mathcal{B}_{LR}(H)$. Now, suppose that A is any finite set of H such that $A = \{0, 1, 2, \ldots, r\}$. Then $L_H R_H(A) = L_H(\{r, r+1, r+2, \ldots\}) = \{0, 1, 2, \ldots, r\} = A$. Thus, $\{0, 1, 2, \ldots, r\} \in \mathcal{B}_{LR}(H)$. Now, for any finite set $B \neq A$, $L_H R_H(B) \neq B$. Therefore, $\tau_{LR}(H) = \{\varnothing, H\} \cup \{\{0, 1, 2, \ldots, r\} : \; r \in H\}$.

Example 4. Consider the infinite hyper BCK-algebra $(H, \circ, 0)$ given in Example 2. Again, for convenience, let $\mathbb{N} = \{1, 2, ...\}$, $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, and let $J_k = \{\frac{1}{m} : m$ is a positive integer and $m \geq k\}$. Let $p \in H$. If $p = 0$, then $R_H(0) = \{x \in H : 0 \leq x\} = H$ and $L_H(R_H(0)) = L_H(H) = \{0\}$. It follows that $\{0\} \in \mathcal{B}_{LR}(H)$. Let $p \in \mathbb{N}$. Then $R_H(p) = \{x \in H : p \leq x\} = \{p, p+1, p+2, \ldots\}$. It follows that $L_H(R_H(p)) = L_H(\{p, p+1, p+2, \ldots\}) = J_2 \cup \{0, 1, 2, \ldots, p\}$. If $p = \frac{1}{n}$ for some $n \in \{2, 3, ...\}$, then $R_H(p) = \{x \in H : \frac{1}{n} \leq x\} = \mathbb{N} \cup \{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}\}$. Hence, $L_H(R_H(p)) = L_H(\mathbb{N} \cup \{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}\}) = J_n \cup \{0\} = \{0, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, ...\}$.

Next, let $\varnothing \neq A \subseteq H$ with $A \neq \{0\}$. Suppose first that $A \cap \mathbb{N} \neq \varnothing$. If $A \cap \mathbb{N}^0$ is infinite, then $R_H(A) = \varnothing$ and $L_H(R_H(A)) = H$. Suppose $A \cap \mathbb{N}$ is finite and let $s = \max(A \cap \mathbb{N})$. Then $R_H(A) = R_H(s)$ and $L_H(R_H(A)) = J_2 \cup \{0, 1, 2, \ldots, s\}$. Hence, $A \in \mathcal{B}_{LR}(H)$ if and only if $A = H$ or $A = J_2 \cup \{0, 1, 2, \ldots, s\}$.

Suppose now that $A \cap \mathbb{N} = \varnothing$. Since $A \neq \{0\}$, $A \cap J_2 \neq \varnothing$. Let $d = \min\{k \in \{2, 3, ...\} : \frac{1}{k} \in A\}$. Then $R_H(A) = R_H(\frac{1}{d}) = \mathbb{N} \cup \{\frac{1}{2}, \frac{1}{3}, ..., \frac{1}{d}\}$ and $L_H(R_H(A)) = J_d \cup \{0\} = \{0, \frac{1}{d}, \frac{1}{d+1}, \frac{1}{d+2}, ...\}$. Consequently, $A \in \mathcal{B}_{LR}(H)$ if and only if $A = \{0, \frac{1}{d}, \frac{1}{d+1}, \frac{1}{d+2}, ...\}$.

Therefore, $\tau_{LR}(H) = \{\varnothing, \{0\}, H\} \cup \{J_2 \cup \{0, 1, 2, \ldots, p\} : p \in H\} \cup \{\{0, \frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2}, ...\} : k = 2, 3, ...\}$.

It can be seen that the topology $\tau_{LR}(H)$ generated in Example 3 is equal to the topology $\tau_L(H)$ generated in Example 2.5 [8]. The next result says that the equality of these topologies holds for any hyper BCK-algebra H.

Theorem 10. *Let H be a hyper BCK-algebra. The the topology $\tau_{LR}(H)$ coincides with the topology $\tau_L(H)$.*

Proof. By Theorem 9, a basis for $\tau_{LR}(H)$ is given by $\mathcal{B}_{LR}(H) = \{A : L_H R_H(A) = A, \varnothing \neq A \subseteq H\}$ while a basis for $\tau_L(H)$ is the family $\mathcal{B}_L(H) = \{L_H(A) : \varnothing \neq A \subseteq H\}$. Let $A \in \mathcal{B}_{LR}(H)$. Then $L_H R_H(A) = A$. Put $B = R_H(A) \subseteq H$. Then $L_H(B) = A \in \mathcal{B}_L(H)$, showing that $\mathcal{B}_{LR}(H) \subseteq \mathcal{B}_L(H)$. Next, let $V \in \mathcal{B}_L(H)$. Then there exists $D \subseteq H$ such that $L_H(D) = V$. Now, for every $v \in V$, $v \ll d$ for all $d \in D$. Thus, $D \subseteq R_H(V)$. By Proposition 2(ii), $L_H R_H(V) \subseteq L_H(D) = V$. By Theorem 7(i), we have $V \subseteq L_H R_H(V)$. By combining the two set inclusions, we obtain $V = L_H R_H(V) \in \mathcal{B}_{LR}(H)$. Accordingly, $\mathcal{B}_{LR}(H) = \mathcal{B}_L(H)$, showing that $\tau_{LR}(H) = \tau_L(H)$.

Conclusion Right and left applications of a hyper order associated with a hyper BCK-algebra can give rise to two closure operators which , in turn, can be used to generate bases for some topologies on the given hyper structure. The two induced topologies turn out to coincide, respectively, with some previously known topologies on this hyper BCK-algebra.

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