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A-paracompactness and Strongly A-screenability in Topological Groups

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Abstract. A space is said to be strongly A-screenable if there exists a σ-discrete refinement for each open cover. In this article, we have investigated some of the features of A-paracompact and strongly A-screenable spaces in topological and semi topological groups. We predominantly show that (i) Topological direct product of (countably) A-paracompact topological group and a compact topological group is (countably) A-paracompact topological group. (ii) All the left and right cosets of a strongly A-screenable subset H of a semi topological group $(G, *, \tau)$ are strongly A-screenable.

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1. Introduction and Background Results

It is always captivating to dig into relationship of the topological spaces with algebraic structures. To bring out some new results and explore several concepts, many of the mathematicians make a relationship between these two structures by debilitating or strengthening different conditions [3, 19, 20]. J. Dieudonne (1944) and P. Alexandrov (1945) introduced the terms paracompactness and A-paracompactness respectively [5, 11]. L. Ivanovski and V. Kusminov discussed that each bicompact topological group is dyadic. In 1962, J. Kister proved some properties of compactness in topological groups [17]. In 1972, O. T. Alas explored the properties of paracompactness in topological groups [4], and L. G. Brown discussed some properties in topologically complete topological groups [9]. In 1981, A. V. Arhangelskii discussed locally subparacompact, locally paracompact,

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locally strongly paracompact topological groups [8]. In 1989, D. B. Shakhmatov presented strongly and completely paracompactness in topological groups [24]. In 1996, D. Buhagiar and B. Pasynkov discussed uniform paracompactness in topological groups [10]. S. Romaguera and M. Sanchis in 2000 investigated locally compactness in topological groups [23]. In 2007, A. V. Arhangelskii discusses some properties concerning paracompactness in topological groups [6]. Strong realcompactness is discussed in 2012 by M. G. Tkachenko in topological groups [26]. In 2017, H. Juarez-Anguiano discussed strongly paracompactness in topological groups [15].

To extend this work, we have discussed A-paracompactness and strongly A-screenability in topological and semi topological groups. We introduced the term N-capc disjoint sets, and presented the notion semi δ-topological group. Moreover, we prove that, $(R, +, \tau)$ is an ultra-A-paracompact topological group.

2. Preliminaries

A topological space is said to be compact if there is a finite subcover for each open cover [28]. An A-paracompact space is a space which contain a locally finite refinement for every open cover. In paracompact space there exists a locally finite open refinement for each open cover [11]. Unlike paracompact space, in A-paracompact space locally finite refinement need not be necessarily open. Every closed subset of a paracompact space is paracompact [5]. Closed continuous image of a paracompact space is paracompact [22]. Every regular strongly screenable topological space is paracompact [21]. Topological product of metric and compact Hausdorff space is paracompact [25]. Katetov [16] and Dowker [12] introduced countably paracompact spaces. A space in which there exists a locally finite refinement for each countably open cover is said to be countably A-paracompact space [12]. Moreover, in many results countable paracompactness occur with normality [13, 14].

Let $K \leq G$ and $g \in G$, then Kg and gK are said to be the right and left cosets of K in G respectively. Left (right) translation $l_{t_1} : G \to G$ ($r_{t_1} : G \to G$) is defined as $l_{t_1}(t_2) = t_1 * t_2$ ($r_{t_1}(t_2) = t_2 * t_1$). For a group $(G, *)$, the multiplication mapping $m : G \times G \to G$ is defined as $m(x, y) = x \times y = z$, for $x, y, z \in G$. A multiplication mapping is said to be jointly continuous if the defined multiplication mapping is continuous and is separately continuous if the left and the right translations are continuous. For a space τ and a group G, a triplet $(G, *, \tau)$ is said to be a semi topological group if multiplication mapping is separately continuous. A quasi topological group is a semi topological group with continuous inverse mapping. In a paratopological group $(G, *, \tau)$ multiplication mapping is jointly continuous. A paratopological group having continuous inverse mapping is said to be a topological group [7]. In addition, many mathematicians have explored different properties related to compactness [1, 2, 18]. Our notations are standard as used in [13, 27].

3. A-paracompactness and Strongly A-screenability

Definition 1. *A space is said to be strongly A-screenable if there exists a σ-discrete refinement for each open cover.*

Theorem 1. *All the left and right cosets of a strongly A-screenable subset H of a semi topological group $(G, *, \tau)$ are strongly A-screenable.*

Proof. For any $a \in G$, let Ω be an open cover of left coset aH. Then $l_a^{-1}(\Omega)$ is an open cover of subset H. Since H is strongly A-screenable, there exists σ-discrete refinement $\mathcal{U} = \cup_{i=1}^{\infty}\mu_i$. Thus, $l_a(\mathcal{U})$ is σ-discrete refinement of open cover Ω of aH which asseverates that for every $a \in G$, aH is strongly A-screenable. Similarly, all right cosets are strongly A-screenable.

Corollary 1. *A semi topological group $(G, *, \tau)$ is strongly A-screenable if it contains a strongly A-screenable subset H such that $|G|/|H|$ is countable.*

Theorem 2. *In a semi topological group free product of an A-paracompact (countably A-paracompact) subset with any finite subset is A-paracompact (countably A-paracompact).*

Proof. Suppose that $(G, *, \tau)$ is a semi topological group, where S and T are respectively A-paracompact (countably A-paracompact) and finite subsets of G. For $t_1 \in T$, $l_{t_1}(S) = t_1 * S$. Let $\{A_\lambda, \lambda \in \omega\}$ be an open (countably open) cover of $t_1 * S$. Then $\{l_{t_1^{-1}}(A_\lambda), \lambda \in \omega\}$ is an open (countably open) cover of S. So, there is a locally finite refinement $\{l_{t_1^{-1}}(A_\lambda^*), \lambda \in \omega^*\}$ of S. Therefore, $\{A_\lambda^*, \lambda \in \omega^*\}$ is locally finite refinement of $t_1 * S$. Hence, $t_1 * S$ is A-paracompact. Let Ω be an open (countably open) cover of $TS = \cup_{t_i \in T} t_i * S$, then there is an open (countably open) cover $\Omega_1 \subseteq \Omega$ of $t_1 * S$. So, there exists a locally finite refinement Ω_1^* of Ω_1. Therefore, $\Omega^* = \cup\{\Omega_i^*, i = 1, 2, ..., |T|\}$ is locally finite refinement of TS.

Theorem 3. *Let H be a Hausdorff paracompact subset of a semi topological group $(G, *, \tau)$, then each right or left coset of H is a normal space if and only if each pair of closed disjoint singleton subsets of a coset can be separated by its open sets.*

Proof. For $g \in G$, let F_1 and F_2 be a pair of disjoint singleton closed subsets and M_1 is an open set of a coset gH of a set $H \subseteq G$. As the set $U = \{h | l_g(h) \cap F_1 \subseteq M_1\}$ is open in H. Let $W_1 = H \setminus l_g^{-1}(F_1 \cap (gH \setminus M_1))$ and h^* be an arbitrary point such that $l_g(h^*) \cap F_1 \subseteq M_1$, then W_1 is open in H and $h^* \in W_1$, $(l_g(W_1) \cap F_1) \cap (gH \setminus M_1) = \phi$. Hence, $l_g(W_1) \cap F_1 \subseteq M_1$. Thus, the set U is open in H. Moreover, $U_{M_1} = \{h | l_g(h) \cap F_1 \subseteq M_1, l_g(h) \cap F_2 \subseteq gH \setminus Cl(M_1)\}$ is open in H. For each $h^* \in H$, $l_g(h^*) \cap F_1$ and $l_g(h^*) \cap F_2$ are closed and disjoint sets of $l_g(h^*)$. Therefore, there exists two open sets M_1^* and M_2^* of gH such that $l_g(h^*) \cap F_1 \subseteq M_1^*$, $l_g(h^*) \cap F_2 \subseteq M_2^*$ and $M_1^* \cap M_2^* = \phi$. As $Cl(M_1^*) \cap M_2^* = \phi$, M_2^* contained in complement of closure of M_1^* in gH, so $y^* \in U_{M_1}^*$. Hence, $\{U_{M_1}\}$ for all open sets M_1 of gH is open cover of H. Thus, there exists an open locally finite refinements $\{W_{M_1} | M_1 \in \Omega\}$, Ω is collection of open sets of gH such that $Cl(W_{M_1}) \subseteq U_{M_1}$

for each $M_1 \in \Omega$. Suppose, $M_2 = \cup_{M_1\in\Omega}(l_g(W_{M_1}) \cap M_1)$, then M_2 is open in gH and $\{l_g((W_{M_1}) \cap M_1), M_1 \in \Omega\}$ is locally finite. Therefore, $Cl(M_2) = \cup_{M_1\in\Omega}(Cl(l_g(W_{M_1}) \cap M_1)) \subseteq \cup_{M_1\in\Omega}(l_g(Cl(W_{M_1})) \cap Cl(M_1))$. Also, as $l_g(W_{M_1}) \cap F_1 \subseteq l_g(U_{M_1}) \cap F_1 \subseteq M_1$, we have $l_g(W_{M_1}) \cap F_1 \subseteq l_g(W_{M_1}) \cap M_1 \subseteq M_2$. As $\{l_g(W_{M_1})|M_1 \in \Omega\}$ is cover of gH, so $F_1 \subseteq M_2$. Moreover, $(l_g(Cl(W_{M_1})) \cap F_2) \cap Cl(M_1) \subseteq (l_g(U_{M_1}) \cap F_2) \cap Cl(M_1) \subseteq (gH\backslash Cl(M_1))\cap Cl(M_1) = \phi$. Then $F_2\cap Cl(M_2) = \phi$. F_2 contained in open set $gH\backslash Cl(M_2)$. Thus, open sets M_2 and $gH \setminus Cl(M_2)$ separates F_1 and F_2. Hence, any left coset of H is normal. Similarly, it can be prove that, any right coset of H is normal. Conversely, if any left or right coset of H is a normal space, then each pair of disjoint singleton closed subsets of coset can be separated by its open sets.

Theorem 4. *Topological direct product of (countably) A-paracompact topological group and a compact topological group is (countably) A-paracompact topological group.*

Proof. Suppose that X is a (countably) A-paracompact topological group and Y be a compact topological group. Suppose that $\{U_j\}(j = 1, 2, 3, ...)$ is a (countable) covering of $X \times Y$. Let V_i consists of all points x of X satisfying $x \times Y \subseteq \cup_{j\leq i}U_j$. If $x \in V_i$, then each (x, y) of $x \times Y$ has a neighbourhood $N \times M$ contained in open set $\cup_{j\leq i}U_j$. These finite open sets M cover Y. Let N_x be the intersection of corresponding sets N. Then $x \in N_x$, N_x is open and $N_x \times Y \subseteq \cup_{j\leq i}U_j$, and hence $N_x \subseteq V_i$. Therefore, V_i is open. Moreover, for an arbitrary $x \in X$, $x \times Y$ is contained in some finite sets of the covering $\{U_j\}$, because $x \times Y$ is compact. Therefore, x is in some V_i. Thus, $\{V_i\}$ is a covering of X. As $\{V_i\}$ is (countable) open covering and X is (countably) A-paracompact, $\{V_i\}$ possess a locally finite refinement $\mathcal{B}$. For every $W \in \mathcal{B}$, suppose $g(W)$ is the first V_i that contain W and let G_i is the union of all W for which $g(W) = V_i$. Then $G_i \subseteq V_i$ and $\{G_i\}$ is locally finite covering of X.

Let $G_{ij} = (G_i \times Y) \cap U_j$, for $j \leq i$. If (x, y) is an arbitrary point of (X, Y), then for some i, $x \in G_i$, $(x, y) \in G_i \times Y$. Also since $x \in G_i \subseteq V_i$, $(x, y) \in x \times Y \subseteq \cup_{j\leq i}U_j$. Hence, for some $i \geq j$, $(x, y) \in U_j$. It follows that, $(x, y) \in G_{ij}$. Therefore, $\{G_{ij}\}$ is covering of $X \times Y$. Since, $G_{ij} \subseteq U_j$, then $\{G_{ij}\}$ is a refinement of $\{U_j\}$. Also if $(x, y) \in X \times Y$, x belongs to an open set $H(x)$ which meet only a finite sets of $\{G_i\}$. Then $H(x) \times Y$ is an open set containing (x, y) which can meet G_{ij} only if $H(x)$ meet G_i. But for each i there is only finite sets G_{ij}. Hence, $H(x) \times Y$ meets only a finite sets of $\{G_{ij}\}$. So, $\{G_{ij}\}$ is locally finite. Also $X \times Y$ is a topological group. Therefore, $X \times Y$ is (countably) A-paracompact topological group.

Definition 2. *A triplet $(G, *, \tau)$, where τ is a space and G is a group, is called semi δ-topological group if multiplication mapping is separately δ-continuous in $(G, *, \tau)$.*

Definition 3. *A space is called almost A-paracompact if its each open cover has a star-finite refinement.*

Definition 4. *A space is called nearly almost A-paracompact if its every regular open cover has a star-finite refinement.*

Theorem 5. *$(G,*,\tau_s)$ is almost A-paracompact semi topological group if and only if $(G,*,\tau)$ is nearly almost A-paracompact semi δ-topological group.*

Proof. Separate continuity of multiplication mapping in $(G,*,\tau_s)$ is the same as separate δ-continuity of multiplication mapping in $(G,*,\tau)$. Let Ω is a regular open cover of $(G,*,\tau)$. Therefore, for each $\omega \in \Omega$, $\Omega = Int(Cl(\omega))$, so $\{Int(Cl(\omega)), \omega \in \Omega\}$ is an open cover of $(G,*,\tau_s)$. Thus, there is a star finite refinement $\mathcal{U} = \{\mu_\alpha, \alpha \in J\}$ of $(G,*,\tau_s)$. Hence, $\mathcal{U}$ is star finite refinement of Ω. Conversely, every open cover Ω of $(G,*,\tau_s)$ is regular open cover of $(G,*,\tau)$. So, there is a star finite refinement $\mathcal{U} = \{\mu_\alpha, \alpha \in J\}$ of Ω.

Definition 5. *In semi topological group $(G,*,\tau)$ having a closed A-paracompact (countably A-paracompact) subset N. A subset S is said to be $N-$capc disjoint ($N-$ccapc disjoint), if for each $s_1, s_2 \in S$, $s_1 \notin s_2 * N$.*

Theorem 6. *Let $(G,*,\tau)$ be a semi topological group with $B \subseteq G$ and H is closed A-paracompact (countably A-paracompact) subset of G. Then there exists a subset N of H such that B is $N-$capc disjoint ($N-$ccapc disjoint).*

Proof. Since, H is closed A-paracompact (countably A-paracompact) therefore $l_b(H) = b*H$ for every $b \in B$ is A-paracompact (countably A-paracompact). Moreover, $b*H$ being inverse image of closed set under translation $l_{b^{-1}}(b*H) = H$ is closed. Each $a_i \in B$ has an open neighbourhood U_i which intersects finite sets in the refinement of open (countably open) cover of $b*H$. By removing all U_i from $b*H$ a closed A-paracompact (countably A-paracompact) set $N = b*H \setminus U_i$ is obtained such that B is $N-$capc disjoint ($N-$ccapc disjoint).

Definition 6. *A Hausdorff space is called an ultra-A-paracompact if there exits a closed locally finite refinement for each open cover.*

Example 1. *$(R,+,\tau)$ is an ultra-A-paracompact topological group for a usual topology τ.*

Proof. Let Ω be an open cover of $(0,\infty)$. If $\kappa = \sup\{x_\alpha : \alpha < \beta\} < \infty$ for an ordinal β, then let there is a real number x_β satisfying $\kappa < x_\beta$ and $[\kappa, x_\beta] \subseteq U$ for $U \in \Omega$. Then $\{[x_\beta, x_{\beta+1}] : \beta\}$ is a partition into closed sets of $(0,\infty)$ that refines Ω. As R is homeomorphic to $(0,\infty)$ and $(R,+,\tau)$ is a topological group. Hence, $(R,+,\tau)$ is an ultra-A-paracompact topological group.

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