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Bipolar Soft Topological Spaces

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Abstract. Bipolar soft set theory is a mathematical tool associates between bipolarity and soft set theory. It is defined by two soft sets; one of them gives us the positive information where the other gives us the negative. The goal of our paper is to define another concept of bipolar soft topological space; this new concept is defined on a bipolar soft set. Then, we investigate the concepts of bipolar soft interior, bipolar soft closure, bipolar soft exterior, bipolar soft boundary on our new bipolar soft topological space and establish some important properties of them. Some relations between them are also discussed. Moreover, the notions of bipolar soft point, bipolar soft limit point and the derived set of a bipolar soft set are discussed. In addition, examples are presented to illustrate our work.

Key Words and Phrases: Bipolar soft topology, bipolar soft interior, bipolar soft closure, bipolar soft exterior, bipolar soft boundary, bipolar soft point, bipolar soft limit point, derived set of a bipolar soft set

1. Introduction

Many problems of our lives in decision making, engineering, computer sciences and economics have various uncertainties. Therefore, traditional methods fail to solve them. Motivated by that, many theories have been established to solve these problems. Molodtsov [15], presented the notion of soft set theory which is a bright tool used for dealing with uncertainty. Later, many authors discussed the properties, operations and applications of soft set theory [2, 4, 13].

Because of the importance of topology and its great applications especially in physics, economics and computer sciences, researchers interested in the topological structure of soft sets. Two definitions of soft topological spaces were introduced. The first was introduced by Shabir and Naz [19], they defined the notion of soft topological space on a universe set. While, Çağman et al. [5] demonstrated the definition of soft topological space on a soft

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set. After that, many researchers worked on soft topological spaces [1, 3, 8, 10, 14, 17, 21].

Shabir and Naz [20], investigated the concept of bipolar soft sets depending on the role of bipolarity which was introduced by Dubois and Prade [6]. This role was defined by saying that human decision making is based on two sides positive and negative, and we choose according to which one is stronger. To define bipolar soft set on a universal set S we need a set of parameters W, then we get a set called the not set of parameters $\neg W$. In this case, the bipolar soft set on S consists of two soft sets one of them has W as a set of parameters and represents the positive side while the other has $\neg W$ as a set of parameters and represents the negative side. For example, if $W = \{w_1, w_2\} = \{\text{expensive, new}\}$, then $\neg W = \{\neg w_1, \neg w_2\} = \{\text{cheap, old}\}$. For a bipolar soft set on S whose set of parameters is W we define two soft sets $J^+ : W \to P(S)$ and $J^- : \neg W \to P(S)$ where, $J^+(w) \cap J^-(\neg w) = \emptyset$, $\forall w \in W$. If we talk about cars and have the parameter $w_1 =$ expensive, then the expensive cars will belong to $J^+(w_1)$ while the cheap cars will belong to $J^-(\neg w_1) = J^-(\text{cheap})$ and the cars which are neither expensive nor cheap will not belong to $J^+(w_1)$ or $J^-(\neg w_1)$. Shabir and Naz [20], also represented some applications of this bipolar soft set in decision making problems. Later, Karaaslan and Karatas [12] gave another definition of bipolar soft set along with some operations and application of it. Hayat and Mahmood [9], discussed some algebraic structure of bipolar soft sets. In [11], another definition of bipolar soft sets was introduced and some algebraic structure on it were presented.

The topological structure of bipolar soft sets was firstly investigated by Shabir and Bakhtawar [18]. They defined the bipolar soft topological space on a universal set along with a discussion on compactness and connectedness. It was followed by Öztürk [16], who demonstrated the notions of bipolar soft interior, bipolar soft closure, bipolar soft basis and bipolar soft subspace. In [7], the concepts of bipolar soft separation axioms were established along with a discussion on the hereditary of them on a bipolar soft subspace topology.

Motivated by these studies, we define the notion of bipolar soft topological space on a bipolar soft set which can be considered as a generalization of the definition was given in [18]. We also represent some topological concepts and properties of it. Our paper is structured as the following way: Section 2 contains crucial concepts, properties and operations related to bipolar soft set which are required in our work. In section 3, we introduce the notion of bipolar soft topological space on a bipolar soft set and investigate the concepts of bipolar soft interior and bipolar soft closure. In section 4, the notions of bipolar soft exterior and bipolar soft boundary are introduced associated with some of their properties. Moreover, relations between all the previous notions are investigated along with some illustrative examples. In this section too, the notions of bipolar soft point, bipolar soft limit point and the derived set of the bipolar soft set are established with some properties of it. While, the conclusion is included in Section 5.

2. Preliminaries

We recall the concept of bipolar soft sets with some crucial definitions, properties and operations which are required for our work. Before that we will fix some notions. The symbol S stands for the universal set, $P(S)$ is the power set of S, W represents the set of parameters and K, H and R are non empty subsets of W.

Definition 1 ([13]). *Let* $W = \{w_l : l = 1, 2, .., n\}$ *be a set of parameters. The set* $\neg W = \{\neg w_l : l = 1, 2, .., n\}$, *where* $\neg w_l =$ *not* w_l, $\forall l$ *is called the not set of* W.

Definition 2 ([20]). *Let* J^+ *and* J^- *be two mappings, defined as* $J^+ : K \to P(S)$ *and* $J^- : \neg K \to P(S)$ *where* $J^+(w) \cap J^-(\neg w) = \emptyset$, $\forall w \in K$. *Then, the triple* (J^+, J^-, K) *is called a bipolar soft set on* S.

Throughout this paper, the notion $BS(S)$ denotes the set of all bipolar soft sets on S. While, the bipolar soft set $(J^+, J^-, K) \in BS(S)$ will be represented as,

$$(J^+, J^-, K) = \{(w, J^+(w), J^-(\neg w)) : w \in K, \neg w \in \neg K\}.$$

In this present segment, some concepts related to bipolar soft sets will be defined along with a review of the definitions and properties of complement, union and intersection of bipolar soft sets.

Definition 3 ([20]). *Let* $(J^+, J^-, K), (I^+, I^-, H) \in BS(S)$. *Then,*

(i) (J^+, J^-, K) *is said to be a bipolar soft subset of* (I^+, I^-, H) *denoted by* $(J^+, J^-, K)\tilde{\subseteq}(I^+, I^-, H)$ *if,* $K \subseteq H$, $J^+(w) \subseteq I^+(w)$ *and* $I^-(\neg w) \subseteq J^-(\neg w)$, $\forall w \in K$.

(ii) (J^+, J^-, K) *and* (I^+, I^-, H) *are equal if* $(J^+, J^-, K)\tilde{\subseteq}(I^+, I^-, H)$ *and* $(I^+, I^-, H)\tilde{\subseteq}(J^+, J^-, K)$.

(iii) The complement of (J^+, J^-, K) *is defined by* $(J^+, J^-, K)^c = ((J^+)^c, (J^-)^c, K)$ *where the two mapping* $(J^+)^c$ *and* $(J^-)^c$ *are defined by* $(J^+)^c(w) = J^-(\neg w)$ *and* $(J^-)^c(\neg w) = J^+(w)$, $\forall w \in K$.

(iv) (J^+, J^-, K) *is called an absolute bipolar soft set if* $\forall w \in K$, $J^+(w) = S$ *and* $\forall \neg w \in \neg K$, $J^-(\neg w) = \emptyset$, *we write it as* $(\tilde{S}, \Phi, K)$.

(v) (J^+, J^-, K) *is called a null bipolar soft set if* $\forall w \in K$, $J^+(w) = \emptyset$ *and* $\forall \neg w \in \neg K$, $J^-(\neg w) = S$, *we write it as* $(\Phi, \tilde{S}, K)$.
Clearly, $(\Phi, \tilde{S}, K)^c = (\tilde{S}, \Phi, K)$.

(vi) The union (intersection) of (J^+, J^-, K) *and* (I^+, I^-, H) *is the bipolar soft set* (O^+, O^-, R) *on* S *where* $R = K \cup H$, *denoted by* $(J^+, J^-, K)\tilde{\cup}(\tilde{\cap})(I^+, I^-, H) = (O^+, O^-, R)$, *is defined as*

$$O^+(w) = \begin{cases} J^+(w), & if w \in K \setminus H \\ I^+(w), & if w \in H \setminus K \\ J^+(w) \cup (\cap) I^+(w), & if w \in K \cap H \end{cases}$$

$$O^-(\neg w) = \begin{cases} J^-(\neg w), & \textit{if} \neg w \in (\neg K) \setminus (\neg H) \\ I^-(\neg w), & \textit{if} \neg w \in (\neg H) \setminus (\neg K) \\ J^-(\neg w) \cap (\cup) I^-(\neg w), & \textit{if} \neg w \in (\neg K) \cap (\neg H). \end{cases}$$

Union (intersection) is reflexive and associative. Moreover, union and intersection satisfy De Morgan's Laws.

Proposition 1 ([18, 20])**.** *Let* $(J^+, J^-, K), (O^+, O^-, K)$ *and* $(I^+, I^-, H) \in BS(S)$*. Then,*

(i) $[(J^+, J^-, K)^c]^c = (J^+, J^-, K)$.

(ii) $(J^+, J^-, K)\tilde{\subseteq}(O^+, O^-, K) \Rightarrow (O^+, O^-, K)^c\tilde{\subseteq}(J^+, J^-, K)^c$.

(iii) $(\Phi, \tilde{S}, K)\tilde{\subseteq}(J^+, J^-, K)\tilde{\cap}(J^+, J^-, K)^c\tilde{\subseteq}(J^+, J^-, K)\tilde{\cup}(J^+, J^-, K)^c\tilde{\subseteq}(\tilde{S}, \Phi, K)$.

(iv) $(I^+, I^-, H)\tilde{\cap}((J^+, J^-, K)\tilde{\cup}(O^+, O^-, K)) = ((I^+, I^-, H)\tilde{\cap}(J^+, J^-, K))\tilde{\cup}$
. $((I^+, I^-, H)\tilde{\cap}(O^+, O^-, K))$.

Proposition 2. *Let* (J^+, J^-, K), $(I^+, I^-, K) \in BS(S)$ *and* $(J^+, J^-, K)\tilde{\subseteq}(I^+, I^-, K)$*. Then,*

(i) $(I^+, I^-, K)\tilde{\cap}(J^+, J^-, K) = (J^+, J^-, K)$.

(ii) $(I^+, I^-, K)\tilde{\cup}(J^+, J^-, K) = (I^+, I^-, K)$.

Proof. Let (J^+, J^-, K), $(I^+, I^-, K) \in BS(S)$ and $(J^+, J^-, K)\tilde{\subseteq}(I^+, I^-, K)$. Then, $J^+(w) \subseteq I^+(w)$, $\forall w \in K$ and $I^-(\neg w) \subseteq J^-(\neg w)$, $\forall \neg w \in \neg K$.

(i) Suppose $(J^+, J^-, K)\tilde{\cap}(I^+, I^-, K) = (O^+, O^-, H)$. Then, $H = K \cup K = K$, $J^+(w) \cap I^+(w) = J^+(w)$, $\forall w \in K$ and $J^-(\neg w) \cup I^-(\neg w) = J^-(\neg w)$, $\forall \neg w \in \neg K$. Thus, $(O^+, O^-, H) = (J^+, J^-, K)$.

(ii) Suppose $(J^+, J^-, K)\tilde{\cup}(I^+, I^-, K) = (O^+, O^-, H)$. Then, $H = K \cup K = K$, $J^+(w) \cup I^+(w) = I^+(w)$, $\forall w \in K$ and $J^-(\neg w) \cap I^-(\neg w) = I^-(\neg w)$, $\forall \neg w \in \neg K$. Therefore, $(O^+, O^-, H) = (I^+, I^-, K)$.

Now, we suggest the definition of the difference between two bipolar soft sets.

Definition 4. *Let* (J^+, J^-, K), $(I^+, I^-, H) \in BS(S)$*. The difference between* (J^+, J^-, K) *and* (I^+, I^-, H) *is the bipolar soft set* (O^+, O^-, R) *on* S*, where* $R = K \cup H$*, is defined as*

$$(O^+, O^-, R) = (J^+, J^-, K) \setminus (I^+, I^-, H) = (J^+, J^-, K)\tilde{\cap}(I^+, I^-, H)^c.$$

Definition 5 ([18])**.** *Let* $(J^+, J^-, K), (I^+, I^-, K) \in BS(S)$*. Then,* (J^+, J^-, K) *and* (I^+, I^-, K) *are called bipolar soft disjoint sets if* $J^+(w) \cap I^+(w) = \emptyset$*,* $\forall w \in K$*.*

Remark 1. *Let* (J^+, J^-, K), $(I^+, I^-, K) \in BS(S)$ *be two disjoint bipolar soft sets. Then, we only care about* $J^+(w) \cap I^+(w) = \emptyset$*,* $\forall w \in K$ *and no matter what* $J^-(\neg w) \cup I^-(\neg w)$ *equals. So, we will use the notation* $\tilde{\Phi}_K$ *to denote a bipolar soft set* (O^+, O^-, K) *where,* $O^+(w) = \emptyset$*,* $\forall w \in K$ *and* $O^-(\neg w) \subseteq S$*,* $\forall \neg w \in \neg K$*.*

Now, we can write $(J^+, J^-, K)\tilde{\cap}(I^+, I^-, K) = \tilde{\Phi}_K$ if (J^+, J^-, K) and (I^+, I^-, K) are two disjoint bipolar soft sets on S.

3. Bipolar Soft Topological Spaces

In this section, we define the bipolar soft topological space on a bipolar soft set. Then, we investigate the concepts of bipolar soft interior and bipolar soft closure. Moreover, some properties and relations on them are discussed along with some examples.

Next, we define the bipolar soft topological space on a bipolar soft set.

Definition 6. *Let $(J^+, J^-, K) \in BS(S)$ and τ be a collection of bipolar soft subset from (J^+, J^-, K) whose set of parameters is K. τ is called a bipolar soft topology on (J^+, J^-, K) if*

(i) (J^+, J^-, K), $(\Phi, \tilde{S}, K) \in \tau$,

(ii) If $\{(J_l^+, J_l^-, K)\tilde{\subseteq}(J^+, J^-, K), l \in I\} \subseteq \tau$, *then* $\tilde{\bigcup}_{l\in I}(J_l^+, J_l^-, K) \in \tau$,

(iii) $\{(J_l^+, J_l^-, K)\tilde{\subseteq}(J^+, J^-, K), 1 \leq l \leq n, n \in \mathbb{N}\} \subseteq \tau$, *then* $\tilde{\bigcap}_{l=1}^{n}(J_l^+, J_l^-, K) \in \tau$.

Then, $(J^+, \tau, K, \neg K)$ *is said to be a bipolar soft topological space (BSTS).*

Remark 2. *If $(J^+, J^-, K) = (\tilde{S}, \Phi, K)$ in Definition 6, then we get the bipolar soft topology which was defined by Shabir and Bakhtawar [18]. Therefore, our definition of bipolar soft topological space on a bipolar soft set can be considered as a generalization of the definition which was defined in [18].*

Definition 7. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(M^+, M^-, K) \in BS(S)$. Then, (M^+, M^-, K) is said to be*

(i) a bipolar soft open set if it belongs to τ,

(ii) a bipolar soft closed set if $(M^+, M^-, K)^c$ belongs to τ,

(iii) bipolar soft clopen set if (M^+, M^-, K) and $(M^+, M^-, K)^c$ are members of τ.

Theorem 1. *Let $(J^+, \tau, K, \neg K)$ be a BSTS. Then, all the following conditions are satisfied:*

(i) $(\tilde{S}, \Phi, K), (J^+, J^-, K)^c$ are bipolar soft closed sets,

(ii) arbitrary intersection of bipolar soft closed sets is a bipolar soft closed set,

(iii) finite union of bipolar soft closed sets is a bipolar soft closed set.

Proof.

(i) $(\tilde{S}, \Phi, K), (J^+, J^-, K)^c$ are bipolar soft closed sets since their complements $(\Phi, \tilde{S}, K)$, (J^+, J^-, K), respectively, are in τ.

(ii) Let $\Omega = \{(M_l{}^+, M_l{}^-, K) : (M_l{}^+, M_l{}^-, K)^c \in \tau, l \in I\}$. Then,

$$\left(\tilde{\bigcap}_{l\in I}(M_l{}^+, M_l{}^-, K)\right)^c = \tilde{\bigcup}_{l\in I}(M_l{}^+, M_l{}^-, K)^c \in \tau.$$

Thus, $\tilde{\bigcap}_{l\in I}(M_l{}^+, M_l{}^-, K)$ is a bipolar soft closed set.

(iii) Let $\Omega = \{(M_l{}^+, M_l{}^-, K) : (M_l{}^+, M_l{}^-, K)^c \in \tau, 1 \leq l \leq n, n \in \mathbb{N}\}$. Then,

$$\left(\tilde{\bigcup}_{l=1}^{n}(M_l{}^+, M_l{}^-, K)\right)^c = \tilde{\bigcap}_{l=1}^{n}(M_l{}^+, M_l{}^-, K)^c \in \tau.$$

Therefore, $\tilde{\bigcup}_{l=1}^{n}(M_l{}^+, M_l{}^-, K)$ is a bipolar soft closed set.

The notion of bipolar soft interior will be introduced along with some properties of it.

Definition 8. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(D^+, D^-, K) \in BS(S)$. Then, the bipolar soft interior of (D^+, D^-, K), denoted by $(D^+, D^-, K)^\circ$, is the union of all bipolar soft open subsets of (D^+, D^-, K).*

Theorem 2. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and (D^+, D^-, K), $(R^+, R^-, K) \in BS(S)$. Then,*

(i) $(D^+, D^-, K)^\circ \tilde{\subseteq} (D^+, D^-, K)$.

(ii) (D^+, D^-, K) *is a bipolar soft open set* $\Leftrightarrow (D^+, D^-, K) = (D^+, D^-, K)^\circ$.

(iii) $((D^+, D^-, K)^\circ)^\circ = (D^+, D^-, K)^\circ$.

(iv) $(D^+, D^-, K) \tilde{\subseteq} (R^+, R^-, K) \Rightarrow (D^+, D^-, K)^\circ \tilde{\subseteq} (R^+, R^-, K)^\circ$.

(v) $(D^+, D^-, K)^\circ \tilde{\cap} (R^+, R^-, K)^\circ = [(D^+, D^-, K) \tilde{\cap} (R^+, R^-, K)]^\circ$.

(vi) $(D^+, D^-, K)^\circ \tilde{\cup} (R^+, R^-, K)^\circ \tilde{\subseteq} [(D^+, D^-, K) \tilde{\cup} (R^+, R^-, K)]^\circ$.

Proof.

(i) Obvious from the definition.

(ii) Let (D^+, D^-, K) be a bipolar soft open set. Then, $(D^+, D^-, K) \tilde{\subseteq} (D^+, D^-, K)^\circ$ since $(D^+, D^-, K)^\circ$ is the largest bipolar soft open set contained in (D^+, D^-, K). But from (i), $(D^+, D^-, K)^\circ \tilde{\subseteq} (D^+, D^-, K)$. Therefore, we get $(D^+, D^-, K) = (D^+, D^-, K)^\circ$. The converse is obvious.

(iii) $(D^+, D^-, K)^\circ$ is a bipolar soft open set. Thus, by part (ii) it is equal to its interior. Thus, $(D^+, D^-, K)^\circ = ((D^+, D^-, K)^\circ)^\circ$.

(iv) Suppose $(D^+, D^-, K) \tilde{\subseteq} (R^+, R^-, K)$. From (i), $(D^+, D^-, K)^\circ \tilde{\subseteq} (D^+, D^-, K)$. Therefore, $(D^+, D^-, K)^\circ \tilde{\subseteq} (R^+, R^-, K)$. Now, $(D^+, D^-, K)^\circ$ is a bipolar soft open set contained in (R^+, R^-, K) so it is contained in its interior since $(R^+, R^-, K)^\circ$ is the largest bipolar soft open set contained in (R^+, R^-, K). Therefore, $(D^+, D^-, K)^\circ \tilde{\subseteq} (R^+, R^-, K)^\circ$.

(v) From (i), $(D^+, D^-, K)^\circ \tilde{\subseteq} (D^+, D^-, K)$ and $(R^+, R^-, K)^\circ \tilde{\subseteq} (R^+, R^-, K)$. Therefore, $(D^+, D^-, K)^\circ \tilde{\cap} (R^+, R^-, K)^\circ \tilde{\subseteq} (D^+, D^-, K) \tilde{\cap} (R^+, R^-, K)$. But,$(D^+, D^-, K)^\circ \tilde{\cap}$ $(R^+, R^-, K)^\circ$ is a bipolar soft open set contained in $(D^+, D^-, K) \tilde{\cap} (R^+, R^-, K)$ so it is contained in its interior which is the largest bipolar soft open set contained in $(D^+, D^-, K) \tilde{\cap} (R^+, R^-, K)$. Thus,

$$(D^+, D^-, K)^\circ \tilde{\cap} (R^+, R^-, K)^\circ \tilde{\subseteq} [(D^+, D^-, K) \tilde{\cap} (R^+, R^-, K)]^\circ.$$

Now,

$$\begin{aligned} (D^+, D^-, K) \tilde{\cap} (R^+, R^-, K) &\tilde{\subseteq} (D^+, D^-, K), \\ (D^+, D^-, K) \tilde{\cap} (R^+, R^-, K) &\tilde{\subseteq} (R^+, R^-, K). \end{aligned}$$

From (iv),

$$\begin{aligned} ((D^+, D^-, K) \tilde{\cap} (R^+, R^-, K))^\circ &\tilde{\subseteq} (D^+, D^-, K)^\circ, \\ ((D^+, D^-, K) \tilde{\cap} (R^+, R^-, K))^\circ &\tilde{\subseteq} (R^+, R^-, K)^\circ. \end{aligned}$$

Thus,

$$[(D^+, D^-, K) \tilde{\cap} (R^+, R^-, K)]^\circ \tilde{\subseteq} (D^+, D^-, K)^\circ \tilde{\cap} (D^+, D^-, K)^\circ.$$

Now we have, $(D^+, D^-, K)^\circ \tilde{\cap} (R^+, R^-, K)^\circ = [(D^+, D^-, K) \tilde{\cap} (R^+, R^-, K)]^\circ$.

(vi) We know that, $(D^+, D^-, K) \tilde{\subseteq} (D^+, D^-, K) \tilde{\cup} (R^+, R^-, K)$, and $(R^+, R^-, K) \tilde{\subseteq}$ $(D^+, D^-, K) \tilde{\cup} (R^+, R^-, K)$. By (iv), $(D^+, D^-, K)^\circ \tilde{\subseteq} ((D^+, D^-, K) \tilde{\cup} (R^+, R^-, K))^\circ$ and $(R^+, R^-, K)^\circ \tilde{\subseteq} ((D^+, D^-, K) \tilde{\cup} (R^+, R^-, K))^\circ$. So, $(D^+, D^-, K)^\circ \tilde{\cup} (R^+, R^-, K)^\circ$ $\tilde{\subseteq} [(D^+, D^-, K) \tilde{\cup} (R^+, R^-, K)]^\circ$.

In the next example, we will see that the equality in (vi) does not hold.

Example 1. *Let* $S = \{s_1, s_2, s_3, s_4\}$, $W = \{w_1, w_2, w_3, w_4\}$, $K = \{w_3, w_4\}$, $(J^+, J^-, K) = \{(w_3, \{s_1, s_3, s_4\}, \{s_2\}), (w_4, \{s_2, s_3, s_4\}, \emptyset)\}$ *and* $\tau = \{(J^+, J^-, K), (\Phi, \tilde{S}, K), (J_1^+, J_1^-, K),$ $(J_2^+, J_2^-, K), (J_3^+, J_3^-, K), (J_4^+, J_4^-, K)\}$ *be a bipolar soft topology on* (J^+, J^-, K), *where*

$$\begin{aligned} (J_1^+, J_1^-, K) &= \{(w_3, \{s_1, s_4\}, \{s_2\}), (w_4, \{s_4\}, \{s_1, s_3\})\}, \\ (J_2^+, J_2^-, K) &= \{(w_3, \{s_3\}, \{s_1, s_2\}), (w_4, \{s_2, s_3, s_4\}, \{s_1\})\}, \\ (J_3^+, J_3^-, K) &= \{(w_3, \{s_1, s_3, s_4\}, \{s_2\}), (w_4, \{s_2, s_3, s_4\}, \{s_1\})\}, \\ (J_4^+, J_4^-, K) &= \{(w_3, \emptyset, \{s_1, s_2\}), (w_4, \{s_4\}, \{s_1, s_3\})\}. \end{aligned}$$

Let,

$$\begin{aligned} (U^+, U^-, K) &= \{(w_3, \{s_1, s_4\}, \{s_2\}), (w_4, \{s_2, s_4\}, \{s_1\})\}, \\ (I^+, I^-, K) &= \{(w_3, \{s_3\}, \{s_1, s_2\}), (w_4, \{s_2, s_3, s_4\}, \emptyset)\}. \end{aligned}$$

Then,

$$(U^+, U^-, K)^\circ = (J_1^+, J_1^-, K) \qquad \textit{and} \qquad (I^+, I^-, K)^\circ = (J_2^+, J_2^-, K).$$

Thus,

$$(U^+, U^-, K)^\circ \tilde{\cup} (I^+, I^-, K)^\circ = (J_3^+, J_3^-, K).$$

Now,

$$(U^+, U^-, K) \tilde{\cup} (I^+, I^-, K) = \{(w_3, \{s_1, s_3, s_4\}, \{s_2\}), (w_4, \{s_2, s_3, s_4\}, \emptyset)\}.$$

Therefore,

$$\begin{aligned} [(U^+, U^-, K) \tilde{\cup} (I^+, I^-, K)]^\circ &= (J^+, J^-, K) \\ &\neq (U^+, U^-, K)^\circ \tilde{\cup} (I^+, I^-, K)^\circ. \end{aligned}$$

Next, we will define the bipolar soft closure followed by an important properties of it.

Definition 9. *Let $(J^+, \tau, K, \neg K)$ be a $BSTS$ and $(P^+, P^-, K) \in BS(S)$. The bipolar soft closure of (P^+, P^-, K), denoted by $\overline{(P^+, P^-, K)}$, is the intersection of all bipolar soft closed sets containing (P^+, P^-, K).*

Theorem 3. *Let $(J^+, \tau, K, \neg K)$ be a $BSTS$ and $(P^+, P^-, K), (R^+, R^-, K) \in BS(S)$. Then,*

(i) $(P^+, P^-, K) \tilde{\subseteq} \overline{(P^+, P^-, K)}$.

(ii) (P^+, P^-, K) *is a bipolar soft closed set* $\Leftrightarrow (P^+, P^-, K) = \overline{(P^+, P^-, K)}$.

(iii) $\overline{(\overline{(P^+, P^-, K)})} = \overline{(P^+, P^-, K)}$.

(iv) $(P^+, P^-, K) \tilde{\subseteq} (R^+, R^-, K) \Rightarrow \overline{(P^+, P^-, K)} \tilde{\subseteq} \overline{(R^+, R^-, K)}$.

(v) $\overline{(P^+, P^-, K) \tilde{\cap} (R^+, R^-, K)} \tilde{\subseteq} \overline{(P^+, P^-, K)} \tilde{\cap} \overline{(R^+, R^-, K)}$.

(vi) $\overline{(P^+, P^-, K) \tilde{\cup} (R^+, R^-, K)} = \overline{(P^+, P^-, K)} \tilde{\cup} \overline{(R^+, R^-, K)}$.

Proof.

(i) Obvious from the definition.

(ii) Let (P^+, P^-, K) be a bipolar soft closed set then $\overline{(P^+, P^-, K)} \tilde{\subseteq} (P^+, P^-, K)$ since $\overline{(P^+, P^-, K)}$ is the smallest bipolar soft closed set containing (P^+, P^-, K). But from (i), $(P^+, P^-, K) \tilde{\subseteq} \overline{(P^+, P^-, K)}$. Thus, $(P^+, P^-, K) = \overline{(P^+, P^-, K)}$. The converse is obvious.

(iii) $\overline{(P^+, P^-, K)}$ is a bipolar soft closed set. Therefore, by (ii) it is equal to its closure. Therefore, $\overline{(P^+, P^-, K)} = \overline{(\overline{(P^+, P^-, K)})}$.

(iv) Assume that $(P^+,P^-,K)\tilde{\subseteq}(R^+,R^-,K)$. From (i), $(R^+,R^-,K)\tilde{\subseteq}\overline{(R^+,R^-,K)}$. So, $(P^+,P^-,K)\tilde{\subseteq}\overline{(R^+,R^-,K)}$. Now, $\overline{(R^+,R^-,K)}$ is a bipolar soft closed set containing (P^+,P^-,K) so it is containing its closure since $\overline{(P^+,P^-,K)}$ is the smallest bipolar soft closed set containing (P^+,P^-,K). Therefore, $\overline{(P^+,P^-,K)}\tilde{\subseteq}\overline{(R^+,R^-,K)}$.

(v) Since, $(P^+,P^-,K)\tilde{\cap}(R^+,R^-,K)\tilde{\subseteq}(P^+,P^-,K)$ and $(P^+,P^-,K)\tilde{\cap}(R^+,R^-,K)\tilde{\subseteq}(R^+,R^-,K)$. From (iv), $\overline{(P^+,P^-,K)\tilde{\cap}(R^+,R^-,K)}\tilde{\subseteq}\overline{(P^+,P^-,K)}$, and $\overline{(P^+,P^-,K)\tilde{\cap}(R^+,R^-,K)}\tilde{\subseteq}\overline{(R^+,R^-,K)}$. Thus,

$$\overline{(P^+,P^-,K)\tilde{\cap}(R^+,R^-,K)}\tilde{\subseteq}\overline{(P^+,P^-,K)}\tilde{\cap}\overline{(R^+,R^-,K)}.$$

(vi) Since,

$$\begin{aligned}(P^+,P^-,K) &\tilde{\subseteq} (P^+,P^-,K)\tilde{\cup}(R^+,R^-,K),\\ (R^+,R^-,K) &\tilde{\subseteq} (P^+,P^-,K)\tilde{\cup}(R^+,R^-,K).\end{aligned}$$

By(iv),

$$\begin{aligned}\overline{(P^+,P^-,K)} &\tilde{\subseteq} \overline{(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)},\\ \overline{(R^+,R^-,K)} &\tilde{\subseteq} \overline{(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)}.\end{aligned}$$

Therefore,

$$\overline{(P^+,P^-,K)}\tilde{\cup}\overline{(R^+,R^-,K)}\tilde{\subseteq}\overline{(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)}.$$

Now from (i) we get, $(P^+,P^-,K)\tilde{\subseteq}\overline{(P^+,P^-,K)}$ and $(R^+,R^-,K)\tilde{\subseteq}\overline{(R^+,R^-,K)}$. Therefore, $(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)\tilde{\subseteq}\overline{(P^+,P^-,K)}\tilde{\cup}\overline{(R^+,R^-,K)}$. But, $\overline{(P^+,P^-,K)}\tilde{\cup}\overline{(R^+,R^-,K)}$ is a bipolar soft closed set containing $(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)$ so it is containing its closure which is the smallest bipolar soft closed set containing $(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)$. Therefore,

$$\overline{(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)}\tilde{\subseteq}\overline{(P^+,P^-,K)}\tilde{\cup}\overline{(R^+,R^-,K)}.$$

Now we obtain, $\overline{(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)}=\overline{(P^+,P^-,K)}\tilde{\cup}\overline{(R^+,R^-,K)}$.

In the next example we will see that the equality in (v) does not hold.

Example 2. *Let* $S=\{s_1,s_2,s_3\}$, $W=\{w_1,w_2,w_3,w_4\}, K=\{w_3,w_4\}$, $(J^+,J^-,K)=\{(w_3,S,\emptyset),(w_4,S,\emptyset)\}=(\tilde{S},\Phi,K)$ *and* $\tau=\{(J^+,J^-,K),(\Phi,\tilde{S},K),(J_1^+,J_1^-,K),$ $(J_2^+,J_2^-,K),(J_3^+,J_3^-,K),(J_4^+,J_4^-,K)\}$ *is a bipolar soft topology on* (J^+,J^-,K) *where,*

$$\begin{aligned}(J_1^+,J_1^-,K) &= \{(w_3,\{s_1,s_2\},\{s_3\}),(w_4,\{s_1,s_3\},\{s_2\})\},\\ (J_2^+,J_2^-,K) &= \{(w_3,\{s_2,s_3\},\emptyset),(w_4,\{s_1\},\{s_2,s_3\})\},\\ (J_3^+,J_3^-,K) &= \{(w_3,\{s_2\},\{s_3\}),(w_4,\{s_1\},\{s_2,s_3\})\},\end{aligned}$$

$$(J_4^+, J_4^-, K) = \{(w_3, S, \emptyset), (w_4, \{s_1, s_3\}, \{s_2\})\}.$$

Let,

$$\begin{aligned}(V^+, V^-, K) &= \{(w_3, \emptyset, \{s_1, s_2\}), (w_4, \{s_2\}, \{s_1, s_3\})\},\\ (Z^+, Z^-, K) &= \{(w_3, \emptyset, \{s_2, s_3\}), (w_4, \{s_3\}, \{s_1, s_2\})\}.\end{aligned}$$

Then, $\overline{(V^+, V^-, K)} = (J_1^+, J_1^-, K)^c$ *and* $\overline{(Z^+, Z^-, K)} = (J_2^+, J_2^-, K)^c$.
Therefore,

$$\overline{(V^+, V^-, K)} \tilde{\cap} \overline{(Z^+, Z^-, K)} = (J_4^+, J_4^-, K)^c.$$

Next,

$$(V^+, V^-, K) \tilde{\cap} (Z^+, Z^-, K) = \{(w_3, \emptyset, S), (w_4, \emptyset, S)\} = (\Phi, \tilde{S}, K).$$

Thus,

$$\begin{aligned}\overline{(V^+, V^-, K) \tilde{\cap} (Z^+, Z^-, K)} &= (\Phi, \tilde{S}, K)\\ &\neq \overline{(V^+, V^-, K)} \tilde{\cap} \overline{(Z^+, Z^-, K)}.\end{aligned}$$

The relations between bipolar soft interior and bipolar soft closure of a bipolar soft set will be discussed in the following theorem.

Theorem 4. *Let* $(J^+, \tau, K, \neg K)$ *be a BSTS and* $(D^+, D^-, K) \in BS(S)$. *Then,*

(i) $\left[\overline{(D^+, D^-, K)}\right]^c = [(D^+, D^-, K)^c]^\circ$

(ii) $\overline{[(D^+, D^-, K)^c]} = [(D^+, D^-, K)^\circ]^c$

Proof.

(i) Let $\Omega = \{(D_l^+, D_l^-, K) : (D^+, D^-, K) \tilde{\subseteq} (D_l^+, D_l^-, K), (D_l^+, D_l^-, K)^c \in \tau, l \in I\}$,

$$\begin{aligned}\left[\overline{(D^+, D^-, K)}\right]^c &= \left[\tilde{\bigcap}_{l \in I} (D_l^+, D_l^-, K)\right]^c\\ &= \tilde{\bigcup}_{l \in I} (D_l^+, D_l^-, K)^c\\ &= [(D^+, D^-, K)^c]^\circ.\end{aligned}$$

(ii) Similar to (i).

4. Bipolar Soft Exterior, Bipolar Soft Boundary and the Derived Set of a Bipolar Soft Set

In this section, we define the notions of bipolar soft exterior, bipolar soft boundary and the derived set of a bipolar soft set supported with some properties, relations and examples on them.

Next, bipolar soft exterior associated with an example and some properties on it will be discussed.

Definition 10. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(P^+, P^-, K) \in BS(S)$. Then, the bipolar soft exterior of (P^+, P^-, K), denoted by $(P^+, P^-, K)^e$, is the interior of the bipolar soft complement of (P^+, P^-, K).*

In other words, $(P^+, P^-, K)^e = [(P^+, P^-, K)^c]^\circ$.

Example 3. *Consider τ in Example 2. Let $(T^+, T^-, K) = \{(w_3, \{s_2, s_3\}, \emptyset), (w_4, \{s_1\}, \{s_2\})\}$. Then, $(T^+, T^-, K)^e = (\Phi, \tilde{S}, K)$.*

Theorem 5. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(P^+, P^-, K), (R^+, R^-, K) \in BS(S)$. Then,*

(i) $(P^+, P^-, K)\tilde{\subseteq}(R^+, R^-, K) \Rightarrow (R^+, R^-, K)^e\tilde{\subseteq}(P^+, P^-, K)^e$.

(ii) $(P^+, P^-, K)^e\tilde{\cap}(R^+, R^-, K)^e = [(P^+, P^-, K)\tilde{\cup}(R^+, R^-, K)]^e$.

(iii) $(P^+, P^-, K)^e\tilde{\cup}(R^+, R^-, K)^e\tilde{\subseteq}[(P^+, P^-, K)\tilde{\cap}(R^+, R^-, K)]^e$.

Proof. (i) Suppose, $(P^+, P^-, K)\tilde{\subseteq}(R^+, R^-, K)$. Then, $(R^+, R^-, K)^c\tilde{\subseteq}(P^+, P^-, K)^c$. From Theorem 2(iv), $[(R^+, R^-, K)^c]^\circ\tilde{\subseteq}[(P^+, P^-, K)^c]^\circ$. Therefore, $(R^+, R^-, K)^e\tilde{\subseteq}(P^+, P^-, K)^e$.

$$\begin{aligned}(ii)[(P^+, P^-, K)\tilde{\cup}(R^+, R^-, K)]^e &= \left[[(P^+, P^-, K)\tilde{\cup}(R^+, R^-, K)]^c\right]^\circ \\ &= [(P^+, P^-, K)^c\tilde{\cap}(R^+, R^-, K)^c]^\circ \\ &= [(P^+, P^-, K)^c]^\circ\tilde{\cap}[(R^+, R^-, K)^c]^\circ \text{ by Theorem2}(v) \\ &= (P^+, P^-, K)^e\tilde{\cap}(R^+, R^-, K)^e.\end{aligned}$$

$$\begin{aligned}(iii)[(P^+, P^-, K)\tilde{\cap}(R^+, R^-, K)]^e &= \left[[(P^+, P^-, K)\tilde{\cap}(R^+, R^-, K)]^c\right]^\circ \\ &= [(P^+, P^-, K)^c\tilde{\cup}(R^+, R^-, K)^c]^\circ \\ &\tilde{\supseteq} [(P^+, P^-, K)^c]^\circ\tilde{\cup}[(R^+, R^-, K)^c]^\circ \text{ by Theorem2}(vi) \\ &= (P^+, P^-, K)^e\tilde{\cup}(R^+, R^-, K)^e.\square\end{aligned}$$

Now, we introduce the concept of bipolar soft boundary followed by some relations between it, bipolar soft interior, bipolar soft closure and bipolar soft exterior.

Definition 11. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(D^+, D^-, K) \in BS(S)$. Then, the bipolar soft boundary of (D^+, D^-, K), denoted by $(D^+, D^-, K)^b$, is defined as*

$$(D^+, D^-, K)^b = \overline{(D^+, D^-, K)} \tilde{\cap} \overline{(D^+, D^-, K)^c}$$

It is clear that, $(D^+, D^-, K)^b = [(D^+, D^-, K)^c]^b$.

Theorem 6. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(D^+, D^-, K) \in BS(S)$. Then,*

(i) $(D^+, D^-, K)^b = \overline{(D^+, D^-, K)} \setminus (D^+, D^-, K)^\circ$.

(ii) $[(D^+, D^-, K)^b]^c = (D^+, D^-, K)^\circ \tilde{\cup} (D^+, D^-, K)^e$.

(iii) $(D^+, D^-, K)^\circ \tilde{\subseteq} (D^+, D^-, K) \setminus (D^+, D^-, K)^b$.

Proof.

$$\begin{aligned}(i)(D^+, D^-, K)^b &= \overline{(D^+, D^-, K)} \tilde{\cap} \overline{(D^+, D^-, K)^c} \\ &= \overline{(D^+, D^-, K)} \tilde{\cap} [(D^+, D^-, K)^\circ]^c \quad \text{by Theorem } 4(ii) \\ &= \overline{(D^+, D^-, K)} \setminus (D^+, D^-, K)^\circ.\end{aligned}$$

$$\begin{aligned}(ii)[(D^+, D^-, K)^b]^c &= \left[\overline{(D^+, D^-, K)} \tilde{\cap} \overline{(D^+, D^-, K)^c}\right]^c \\ &= \left[\overline{(D^+, D^-, K)}\right]^c \tilde{\cup} \left[\overline{(D^+, D^-, K)^c}\right]^c \\ &= [(D^+, D^-, K)^c]^\circ \tilde{\cup} (D^+, D^-, K)^\circ \quad \text{by Theorem } 4(i) \\ &= (D^+, D^-, K)^e \tilde{\cup} (D^+, D^-, K)^\circ.\end{aligned}$$

$$\begin{aligned}(iii)(D^+, D^-, K) \setminus (D^+, D^-, K)^b &= (D^+, D^-, K) \tilde{\cap} [(D^+, D^-, K)^b]^c \\ &= (D^+, D^-, K) \tilde{\cap} [(D^+, D^-, K)^\circ \tilde{\cup} (D^+, D^-, K)^e] \\ &= [(D^+, D^-, K) \tilde{\cap} (D^+, D^-, K)^\circ] \\ &\tilde{\cup} \; [(D^+, D^-, K) \tilde{\cap} (D^+, D^-, K)^e] \\ &= (D^+, D^-, K)^\circ \tilde{\cup} \tilde{\Phi}_K \\ &\tilde{\supseteq} (D^+, D^-, K)^\circ.\end{aligned}$$

According to the following remark, we can see that there is a difference between the relations between interior, exterior, closure and boundary in bipolar soft topology, and soft topology [1].

Remark 3. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(D^+, D^-, K) \in BS(S)$. Then, in general*

(i) $\overline{(D^+, D^-, K)} \neq (D^+, D^-, K)^\circ \tilde{\cup} (D^+, D^-, K)^b$.

(ii) $(J^+, J^-, K) \neq (D^+, D^-, K)^\circ \tilde{\cup} (D^+, D^-, K)^e \tilde{\cup} (D^+, D^-, K)^b$.

The next example will explain the previous remark.

Example 4. *Consider τ in Example 2 and $(T^+, T^-, K) = \{(w_3, \{s_2, s_3\}, \emptyset),$ $(w_4, \{s_1\}, \{s_2\})\}$ in Example 3. Then,*

$$\begin{aligned} (T^+, T^-, K)^e &= (\Phi, \tilde{S}, K), \\ (T^+, T^-, K)^\circ &= \{(w_3, \{s_2, s_3\}, \emptyset), (w_4, \{s_1\}, \{s_2, s_3\})\}, \\ \overline{(T^+, T^-, K)} &= (\tilde{S}, \Phi, K), \\ (T^+, T^-, K)^b &= \{(w_3, \emptyset, \{s_2, s_3\}), (w_4, \{s_2, s_3\}, \{s_1)\}. \end{aligned}$$

Now,

$$(T^+, T^-, K)^\circ \tilde{\cup} (T^+, T^-, K)^b = \{(w_3, \{s_2, s_3\}, \emptyset), (w_4, S, \emptyset)\} \neq \overline{(T^+, T^-, K)}.$$

Also,

$$\begin{aligned} (T^+, T^-, K)^\circ \tilde{\cup} (T^+, T^-, K)^e \tilde{\cup} (T^+, T^-, K)^b &= \{(w_3, \{s_2, s_3\}, \emptyset), (w_4, S, \emptyset)\} \\ &\neq (J^+, J^-, K). \end{aligned}$$

Next, we will discuss the relations between bipolar soft open set, bipolar soft closed set, bipolar soft clopen set and their bipolar soft boundary set.

Theorem 7. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(D^+, D^-, K) \in BS(S)$. If (D^+, D^-, K) is a bipolar soft open set, then (D^+, D^-, K) and $(D^+, D^-, K)^b$ are disjoint bipolar soft sets.*

Proof. Let (D^+, D^-, K) be a bipolar soft open set. By Theorem 6 (ii) $(D^+, D^-, K)^\circ \tilde{\subseteq} ((D^+, D^-, K)^b)^c$. But, $(D^+, D^-, K) = (D^+, D^-, K)^\circ$ since (D^+, D^-, K) is a bipolar soft open set. Therefore, $(D^+, D^-, K) \tilde{\subseteq} [(D^+, D^-, K)^b]^c$. Which implies that, (D^+, D^-, K) and $(D^+, D^-, K)^b$ are disjoint bipolar soft sets.

The opposite side of the previous theorem is not true and the next example shows that.

Example 5. *Consider τ in Example 2 and $(T^+, T^-, K) = \{(w_3, \{s_2, s_3\}, \emptyset),$ $(w_4, \{s_1\}, \{s_2\})\}$ in Example 4. The bipolar soft boundary of (T^+, T^-, K) is*

$$(T^+, T^-, K)^b = \{(w_3, \emptyset, \{s_2, s_3\}), (w_4, \{s_2, s_3\}, \{s_1\})\}.$$

Now,

$$(T^+, T^-, K) \tilde{\cap} (T^+, T^-, K)^b = \{(w_3, \emptyset, \{s_2, s_3\}), (w_4, \emptyset, \{s_1, s_2\})\}.$$

Therefore, $(T^+, T^-, K), (T^+, T^-, K)^b$ are two disjoint bipolar soft sets but (T^+, T^-, K) is not a bipolar soft open set.

Theorem 8. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(D^+, D^-, K) \in BS(S)$. If (D^+, D^-, K) is a bipolar soft closed set, then $(D^+, D^-, K)^b \tilde{\subseteq} (D^+, D^-, K)$.*

Proof. Let (D^+, D^-, K) be a bipolar soft closed set. From the definition of bipolar soft boundary $(D^+, D^-, K)^b \tilde{\subseteq} \overline{(D^+, D^-, K)}$. But $(D^+, D^-, K) = \overline{(D^+, D^-, K)}$ since (D^+, D^-, K) is a bipolar soft closed set. So, $(D^+, D^-, K)^b \tilde{\subseteq} (D^+, D^-, K)$.

The next example illustrates that the opposite side of Theorem 8 is not true.

Example 6. *Let $S = \{s_1, s_2, s_3\}$, $W = \{w_1, w_2, w_3, w_4\}, K = \{w_3, w_4\}$, $(J^+, J^-, K) = (\tilde{S}, \Phi, K)$ and $\tau = \{(J^+, J^-, K), (\Phi, \tilde{S}, K), (J_1^+, J_1^-, K), (J_2^+, J_2^-, K), (J_3^+, J_3^-, K)\}$ be a bipolar soft topology on (J^+, J^-, K) where,*

$$\begin{aligned}(J_1^+, J_1^-, K) &= \{(w_3, \{s_3\}, \{s_1, s_2\}), (w_4, \{s_3\}, \{s_1\})\},\\ (J_2^+, J_2^-, K) &= \{(w_3, \{s_1\}, \{s_3\}), (w_4, \emptyset, \{s_2, s_3\})\},\\ (J_3^+, J_3^-, K) &= \{(w_3, \{s_1, s_3\}, \emptyset), (w_4, \{s_3\}, \emptyset)\}.\end{aligned}$$

Let,

$$(O^+, O^-, K) = \{(w_3, \{s_1\}, \{s_3\}), (w_4, \{s_1\}, \{s_3\})\}.$$

Then,

$$(O^+, O^-, K)^b = \{(w_3, \emptyset, \{s_1, s_3\}), (w_4, \emptyset, \{s_3\})\} = \tilde{\Phi}_K.$$

Note that, $(O^+, O^-, K)^b \tilde{\subseteq} (O^+, O^-, K)$ but (O^+, O^-, K) is not a bipolar soft closed set.

Theorem 9. *Let $(J^+, \tau, K, \neg K)$ be a $BSTS$ and $(D^+, D^-, K) \in BS(S)$. If (D^+, D^-, K) is a bipolar soft clopen set, then $(D^+, D^-, K)^b = \tilde{\Phi}_K$.*

Proof. Suppose (D^+, D^-, K) is a bipolar soft clopen set. Then, (D^+, D^-, K) is a bipolar soft open set. By Theorem 7, (D^+, D^-, K) and $(D^+, D^-, K)^b$ are disjoint bipolar soft sets. So, $(D^+, D^-, K) \tilde{\cap} (D^+, D^-, K)^b = \tilde{\Phi}_K$. Now, (D^+, D^-, K) is a bipolar soft closed set. Using Theorem 8, $(D^+, D^-, K)^b \tilde{\subseteq} (D^+, D^-, K)$. Thus, $(D^+, D^-, K) \tilde{\cap} (D^+, D^-, K)^b = (D^+, D^-, K)^b$. Therefore, $(D^+, D^-, K)^b = \tilde{\Phi}_K$.

The opposite side of Theorem 9 is not true and the next example shows that.

Example 7. *Consider τ and (O^+, O^-, K) in Example 6. We found that, $(O^+, O^-, K)^b = \tilde{\Phi}_K$ but (O^+, O^-, K) is not a bipolar soft clopen set.*

Theorem 10. *Let $(J^+, \tau, K, \neg K)$ be a $BSTS$ and $(D^+, D^-, K) \in BS(S)$. Then,*

(i) $(D^+, D^-, K)^\circ \tilde{\cap} (D^+, D^-, K)^b = \tilde{\Phi}_K$.

(ii) $(D^+, D^-, K)^e \tilde{\cap} (D^+, D^-, K)^b = \tilde{\Phi}_K$.

Proof.

$$\begin{aligned}(i)\ (D^+, D^-, K)^\circ \tilde{\cap} (D^+, D^-, K)^b &= (D^+, D^-, K)^\circ \tilde{\cap} \left[\overline{(D^+, D^-, K)} \tilde{\cap} \overline{(D^+, D^-, K)^c}\right]\\ &= (D^+, D^-, K)^\circ \tilde{\cap} \overline{(D^+, D^-, K)} \tilde{\cap} [(D^+, D^-, K)^\circ]^c\end{aligned}$$

$$= \tilde{\Phi}_K.$$

$$
\begin{aligned}
(ii)\ (D^+,D^-,K)^e\tilde{\cap}(D^+,D^-,K)^b &= [(D^+,D^-,K)^c]^\circ\tilde{\cap}\left[\overline{(D^+,D^-,K)}\tilde{\cap}\overline{(D^+,D^-,K)^c}\right] \\
&= \left[\overline{(D^+,D^-,K)}\right]^c\tilde{\cap}\overline{(D^+,D^-,K)}\tilde{\cap}\overline{(D^+,D^-,K)^c} \\
&= \tilde{\Phi}_K.
\end{aligned}
$$

Now, we propose the concept of bipolar soft point.

Definition 12. *Let w be a parameter in K. Then, α is called a bipolar soft point if $\{\alpha\} = (J^+, J^-, \{w\})$, where $(J^+, J^-, \{w\}) \in BS(S)$ and $J^+(w) \neq \emptyset$.*

Definition 13. *Let $(J^+, J^-, K) \in BS(S)$ and α be a bipolar soft point. We said that α belongs to (J^+, J^-, K), denoted by $\alpha\tilde{\in}(J^+, J^-, K)$, if $\{\alpha\}\tilde{\subseteq}(J^+, J^-, K)$.*

The set of all bipolar soft open sets containing α will be denoted by $\mathcal{N}(\alpha)$ and defined as

$$\mathcal{N}(\alpha) = \{(Q^+, Q^-, K) : \alpha\tilde{\in}(Q^+, Q^-, K), (Q^+, Q^-, K) \in \tau\}.$$

In the following definition, the notion of bipolar soft limit point and the derived set of a bipolar soft set will be introduced.

Definition 14. *Let $(J^+, \tau, K, \neg K)$ be a $BSTS$, $\alpha\tilde{\in}(J^+, J^-, K)$ and $(P^+, P^-, K) \in BS(S)$. Then, α is a bipolar soft limit point of (P^+, P^-, K) if for every bipolar soft open set (Q^+, Q^-, K) containing α, the two bipolar soft sets (Q^+, Q^-, K) and $(P^+, P^-, K) \setminus \{\alpha\}$ are not disjoint bipolar soft sets. The set of all bipolar soft limit points of (P^+, P^-, K) is called the derived set of the bipolar soft set (P^+, P^-, K) and denoted by $(P^+, P^-, K)'$.*

In other words,

$$\alpha \in (P^+, P^-, K)' \Leftrightarrow \forall (Q^+, Q^-, K) \in \mathcal{N}(\alpha), (Q^+, Q^-, K)\tilde{\cap}(P^+, P^-, K) \setminus \{\alpha\} \neq \tilde{\Phi}_K.$$

In our definition, $(P^+, P^-, K)'$ is a crisp set and it is not a bipolar soft set. After the following example we will illustrate why it is not suitable to define $(P^+, P^-, K)'$ as a bipolar soft set.

Example 8. *Consider τ in Example 1. Let $(M^+, M^-, K) = \{(w_3, \{s_4\}, \{s_2\}),$ $(w_4, \{s_2, s_3\}, \{s_1\})\}$. Then, $\alpha = (w_3, \{s_1, s_3, s_4\}, \{s_2\}) \in (M^+, M^-, K)'$. Since,*

$$
\begin{aligned}
(J^+, J^-, K)\tilde{\cap}(M^+, M^-, K) \setminus \{\alpha\} &= \{(w_3, \emptyset, S), (w_4, \{s_2, s_3\}, \{s_1\})\} \neq \tilde{\Phi}_K \text{ and} \\
(J_3^+, J_3^-, K)\tilde{\cap}(M^+, M^-, K) \setminus \{\alpha\} &= \{(w_3, \emptyset, S), (w_4, \{s_2, s_3\}, \{s_1\})\} \neq \tilde{\Phi}_K.
\end{aligned}
$$

The following two propositions justify why

$$\tilde{\bigcup}\{\{\alpha\} : \alpha \text{ is a bipolar soft limit point of } (P^+, P^-, K)\}$$

is not accepted as a definition of the derived set of the bipolar soft set (P^+, P^-, K).

Proposition 3. *Let $(J^+, \tau, K, \neg K)$ be a BSTS, $\alpha, \alpha_1 \tilde{\in} (J^+, J^-, K)$ where $\{\alpha_1\} \tilde{\subseteq} \{\alpha\}$ and $(P^+, P^-, K) \in BS(S)$. If α is a bipolar soft limit point of (P^+, P^-, K), then α_1 need not to be a bipolar soft limit point of (P^+, P^-, K).*

Proof. Consider τ in Example 1. We found in Example 8, that $\alpha = (w_3, \{s_1, s_3, s_4\}, \{s_2\}) \in (M^+, M^-, K)'$. Let $\alpha_1 = (w_3, \{s_1\}, \{s_2\})$, it is clear that $\{\alpha_1\} \tilde{\subseteq} \{\alpha\}$. But, $\alpha_1 \notin (M^+, M^-, K)'$ since, $\alpha_1 \tilde{\in} (J_1^+, J_1^-, K)$ and

$$(J_1^+, J_1^-, K) \tilde{\cap} (M^+, M^-, K) \setminus \{\alpha_1\} = \{(w_3, \emptyset, \{s_1, s_2\}), (w_4, \emptyset, \{s_1, s_3\} = \tilde{\Phi}_K.$$

Proposition 4. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(P^+, P^-, K) \in BS(S)$. If $\alpha_1, \alpha_2 \in (P^+, P^-, K)'$ and α_3 is a bipolar soft point in (J^+, J^-, K) where $\{\alpha_3\} = \{\alpha_1\} \tilde{\cup} \{\alpha_2\}$, then α_3 need not to be a bipolar soft limit point of (P^+, P^-, K).*

Proof. Consider τ in Example 1 and (M^+, M^-, K) in Example 8. One can see that,

$$\begin{aligned} \alpha_1 = (w_4, \{s_2\}, \{s_1, s_3\}) \quad &\in \quad (M^+, M^-, K)' \text{ and} \\ \alpha_2 = (w_4, \{s_4\}, \{s_1, s_2\}) \quad &\in \quad (M^+, M^-, K)' \end{aligned}$$

But, the bipolar soft point α_3 where $\{\alpha_3\} = \{\alpha_1\} \tilde{\cup} \{\alpha_2\} = \{(w_4, \{s_2, s_4\}, \{s_1\})\}$ is not a bipolar soft limit point of (M^+, M^-, K), since

$$(J_2^+, J_2^-, K) \tilde{\cap} (M^+, M^-, K) \setminus \{\alpha_3\} = \{(w_3, \emptyset, \{s_1, s_2\}), (w_4, \emptyset, \{s_1, s_2, s_4\} = \tilde{\Phi}_K.$$

Next, we discuss some properties of the derived set of a bipolar soft set.

Theorem 11. *Let $(J^+, \tau, K, \neg K)$ be a BSTS and $(P^+, P^-, K), (R^+, R^-, K) \in BS(S)$. Then,*

(i) $(P^+, P^-, K) \tilde{\subseteq} (R^+, R^-, K) \Rightarrow (P^+, P^-, K)' \subseteq (R^+, R^-, K)'$.

(ii) $[(P^+, P^-, K) \tilde{\cap} (R^+, R^-, K)]' \subseteq (P^+, P^-, K)' \cap (R^+, R^-, K)'$.

(iii) $(P^+, P^-, K)' \cup (R^+, R^-, K)' = [(P^+, P^-, K) \tilde{\cup} (R^+, R^-, K)]'$.

Proof.

(i) Let $\alpha \in (P^+, P^-, K)'$ and $(Q^+, Q^-, K) \in \mathcal{N}(\alpha)$. Then, $(Q^+, Q^-, K) \tilde{\cap} (P^+, P^-, K) \setminus \{\alpha\} \neq \tilde{\Phi}_K$. But, $(P^+, P^-, K) \tilde{\subseteq} (R^+, R^-, K)$. So, $(Q^+, Q^-, K) \tilde{\cap} (R^+, R^-, K) \setminus \{\alpha\} \neq \tilde{\Phi}_K$. Thus, $\alpha \in (R^+, R^-, K)'$. Therefore, $(P^+, P^-, K)' \subseteq (R^+, R^-, K)'$.

(ii) Since,

$$\begin{aligned} (P^+, P^-, K) \tilde{\cap} (R^+, R^-, K) \quad &\tilde{\subseteq} \quad (P^+, P^-, K), \\ (P^+, P^-, K) \tilde{\cap} (R^+, R^-, K) \quad &\tilde{\subseteq} \quad (R^+, R^-, K). \end{aligned}$$

From (i) we obtain,

$$\begin{aligned} ((P^+,P^-,K)\tilde{\cap}(R^+,R^-,K))' &\subseteq (P^+,P^-,K)', \\ ((P^+,P^-,K)\tilde{\cap}(R^+,R^-,K))' &\subseteq (R^+,R^-,K)'. \end{aligned}$$

Therefore, $[(P^+,P^-,K)\tilde{\cap}(R^+,R^-,K)]' \subseteq (P^+,P^-,K)' \cap (R^+,R^-,K)'$.

(iii) We know that,

$$\begin{aligned} (P^+,P^-,K) &\tilde{\subseteq} (P^+,P^-,K)\tilde{\cup}(R^+,R^-,K), \\ (R^+,R^-,K) &\tilde{\subseteq} (P^+,P^-,K)\tilde{\cup}(R^+,R^-,K). \end{aligned}$$

This yields from (i),

$$\begin{aligned} (P^+,P^-,K)' \subseteq ((P^+,P^-,K)\tilde{\cup}(R^+,R^-,K))', \\ (R^+,R^-,K)' \subseteq ((P^+,P^-,K)\tilde{\cup}(R^+,R^-,K))'. \end{aligned}$$

Therefore, $(P^+,P^-,K)' \cup (R^+,R^-,K)' \subseteq ((P^+,P^-,K)\tilde{\cup}(R^+,R^-,K))'$.
Now, let $\alpha \in [(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)]'$, and $(Q^+,Q^-,K) \in \mathcal{N}(\alpha)$. Then,

$$(Q^+,Q^-,K)\tilde{\cap}[(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)] \setminus \{\alpha\} \neq \tilde{\Phi}_K.$$

Therefore,

$$\begin{aligned} & (Q^+,Q^-,K)\tilde{\cap}(P^+,P^-,K) \setminus \{\alpha\} \neq \tilde{\Phi}_K \\ \text{or} \quad & (Q^+,Q^-,K)\tilde{\cap}(R^+,R^-,K) \setminus \{\alpha\} \neq \tilde{\Phi}_K. \end{aligned}$$

Therefore,

$$\alpha \in (P^+,P^-,K)' \quad \text{or} \quad \alpha \in (R^+,R^-,K)'.$$

Thus,

$$[(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)]' \subseteq (P^+,P^-,K)' \cup (R^+,R^-,K)'.$$

Now we have, $(P^+,P^-,K)' \cup (R^+,R^-,K)' = [(P^+,P^-,K)\tilde{\cup}(R^+,R^-,K)]'$.

Finally, we provide an example to show that the equality in (ii) does not hold true.

Example 9. *Consider τ in Example 1 and (M^+,M^-,K) in Example 8. Let, $(V^+,V^-,K) = \{(w_3,\{s_1\},\{s_2\}),(w_4,\{s_4\},\emptyset)\}$. Then, $\alpha = (w_3,\{s_1,s_3,s_4\},\{s_2\}) \in (V^+,V^-,K)'$ since,*

$$\begin{aligned} & (J^+,J^-,K)\tilde{\cap}(V^+,V^-,K) \setminus \{\alpha\} = \{(w_3,\emptyset,S),(w_4,\{s_4\},\emptyset)\} \neq \tilde{\Phi}_K, \\ \textit{and} \quad & (J_3^+,J_3^-,K)\tilde{\cap}(V^+,V^-,K) \setminus \{\alpha\} = \{(w_3,\emptyset,S),(w_4,\{s_4\},\{s_1\})\} \neq \tilde{\Phi}_K. \end{aligned}$$

So,

$$\alpha \in (M^+, M^-, K)' \cap (V^+, V^-, K)'.$$

But,

$$\alpha \notin [(M^+, M^-, K)\tilde{\cap}(V^+, V^-, K)]'.$$

Since,

$$(M^+, M^-, K)\tilde{\cap}(V^+, V^-, K) = \{(w_3, \emptyset, \{s_2\}), (w_4, \emptyset, \{s_1\})\}$$

and

$$\begin{aligned}(J_3^+, J_3^-, K)\tilde{\cap}\Big([(M^+, M^-, K)\tilde{\cap}(V^+, V^-, K)] \setminus \{\alpha\}\Big) &= \{(w_3, \emptyset, S), (w_4, \emptyset, \{s_1\})\} \\ &= \tilde{\Phi}_K.\end{aligned}$$

5. Conclusion

This paper devoted for introducing the notion of bipolar soft topological space on a bipolar soft set along with some definitions, properties and relations. The concepts of bipolar soft interior, bipolar soft closure and bipolar soft exterior of a bipolar soft set were discussed. In addition, we introduced the concept of bipolar soft boundary and found interesting relations between it and other notions differ from the relations on soft topological spaces. Relations between different concepts were demonstrated along with some illustrative examples. Moreover, we investigated the definitions of bipolar soft point, bipolar soft limit point and the derived set of a bipolar soft set followed by some properties of it.

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