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On Generalized β-Open Sets in Ideal Bitopological Space

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Abstract. In this article, we introduce and study the concepts of γ_{ij}-*semi*-$\mathcal{I}$-open sets and γ_{ij}-$\beta\mathcal{I}$-open sets by generalizing (i,j)-semi-$\mathcal{I}$-open sets and (ij)-$\beta\mathcal{I}$-open sets, respectively, in ideal bitopological spaces with an operation $\gamma : \tau \to P(X)$. Further, we describe and study $(\gamma,\delta)_{ij}$-semi-$\mathcal{I}$-continuous and $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous functions in ideal bitopological spaces and their related notions. In addition, various examples and counterexamples are given for answers to some questions raised in this study.

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Key Words and Phrases: Ideal bitopological space, $\text{Int}_{\gamma_i}(A)$, $Cl_{\gamma_i}(A)$,γ_{ij}-*semi*-$\mathcal{I}$-open sets, γ_{ij}-$\beta\mathcal{I}$-open sets, $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}-$continuous functions.

1. Introduction

Kelly [11] in 1963, introduced the triple (X, τ_1, τ_2) as bitopological space, where X is a nonempty set, τ_1 and τ_2 are topologies on X. Levine [17] in 1963, introduced the notion of semi-open sets in bitopological spaces. Khedr [14] in 1992, defined semi-preopen (β-open)sets in bitopological spaces. K. Kuraowski [15] in 1966, studied and applied the concept of ideals on topological spaces. An ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X having the heredity property (i) if A $\in \mathcal{I}$ and B $\subset$ A then B $\in \mathcal{I}$ and (ii) if A $\in \mathcal{I}$ and B $\in \mathcal{I}$ then A $\cup$ B $\in \mathcal{I}$. Ekici [5] in 2012, studied the concept of semi-I-open sets in ideal topological spaces. If $\mathcal{I}$ is an ideal on X then $(X, \tau_1, \tau_2, \mathcal{I})$ is called an ideal bitopological space. Kasahara.S [10] in 1979 described an operation γ on τ as a mapping $\gamma : \tau \to P(X)$ such that $U \subseteq U^\gamma$, for each $U \in \tau$. Khedr [12] in 1984, extended the operation γ to bitopological space as a mapping $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that for each $U \in \tau_1 \cup \tau_2$, where U^γ denotes the value of γ at U. For example the operations $U^\gamma = U$, $U^\gamma = Cl_i(U), U^\gamma = Int_j(Cl_i(U))$ for $U \in \tau_j$ are operations on $\tau_1 \cup \tau_2$. Caldas [3] in 2013, introduced the notion of β-open sets in ideal bitopological spaces. Csaszar [4] in 1997, defined generalized open sets in generalized topological spaces.

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2. Preliminaries

Throughout the paper, (X, τ_1, τ_2) always mean bitopological space on with no separation axioms are supposed in this space, also $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space. Let A be a subset of X, by $Int_i(A)$ [10] and $Cl_i(A)$ [20] we denote respectively the interior and closure of A with regard to τ_i for $i = 1, 2$.

A subset A of a bitopological space will be called a γ_i-open set if for each $x \in A$, there exists an τ_i-open set U such that $x \in U$ and $U^\gamma \subseteq A$. Let τ_{γ_i} denotes the set of all γ_i-open set in X. Obviously, we have $\tau_{\gamma_i} \subseteq \tau_i$ [13]. Complement of all γ_i-open sets are called γ_i-closed. Assumed $(X, \tau_1, \tau_2, \mathcal{I})$ as an ideal bitopological space and if $P(X)$ is the set of all subsets of X, a set operator $(\cdot)_i^* : P(X) \to P(X)$ named the local function of A [22] with regard to τ_i and τ. The definition of local function is given as: for $A \subset X$, A_i^* $(\tau_i, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$, for all $U \in \tau_i(x)\}$ where, $\tau_i(x) = \{U \in \tau_i | x \in U\}$. Observe additionally that closure operator for $\tau_i^*(\mathcal{I})$ accurate than τ_i is defined by $Cl_i^*(A) = A \cup A_i^*$. The interior of A in $\tau_i^*(\mathcal{I})$ is denoted by $Int_i^*(A)$ and $Int_{\gamma_i}^*(A_i^*)$ denotes the interior of A_i^* with respect to topology τ_i, where $A_i^* = \{x \in X | U \cap A \notin \mathcal{I}\}$, for every $U \in \tau_i$. The interior$_{\gamma_i}$ of A is denoted by $Int_{\gamma_i}(A)$ and described to be the union of all γ_i-open sets of X contained in A. The closure$_{\gamma_i}$ of A is denoted by $Cl_{\gamma i}(A)$ and defined to be the intersection of all γ_i-closed sets containing A. Currently, several results and definitions from [2, 3, 7, 13, 17] are recalled to be used in this article.

Definition 1. *[8] A subset A of a bitopological space (X, τ_1, τ_2) with operation γ on $\tau_1 \cup \tau_2$ is named:*

1. γ_{ij}-*semi*-open set if $A \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(A))$, where $i \neq j$ and $i, j = 1, 2$.
2. γ_{ij}-β-open set if $A \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl_{\gamma_j}(A)))$, where $i \neq j$ and $i, j = 1, 2$.

Definition 2. *[3] A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is called*

1. (i, j)-*semi*-$\mathcal{I}-$open set if $A \subseteq Cl_j^*(Int_i(A))$, where $i \neq j$ and $i, j = 1, 2$.
2. (i, j)-$\beta\mathcal{I}$-open set if $A \subseteq Cl_j(Int_i(Cl_j^*(A)))$, where $i \neq j$ and $i, j = 1, 2$.

Definition 3. *[16] Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with an operation γ on $\tau_1 \cup \tau_2$. The γ-local function of A with regard to γ and $\mathcal{I}$ is described as giving, for $A \subset X$, $A_{\gamma_i}^*(\gamma, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$, for every $U \in \tau_{\gamma_i}(x)\}$ where $\tau_{\gamma_i}(x) = \{U \in \tau_{\gamma_i} | x \in U\}$.*

In the case of no ambiguity, we will replace $A_{\gamma_i}^*(\gamma, \mathcal{I})$ by $A_{\gamma_i}^*$.

Definition 4. *[22] Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with an operation γ and (Y, σ_1, σ_2) be a bitopological space with an operation δ. Then a function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is called pairwise $(\gamma, \delta)_i$-continuous function if $f^{-1}(V)$ is γ_i-open in X for all δ_i-open set V in Y, for $i = 1, 2$.*

Definition 5. *[3] A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is called to be (i, j)-semi-$\mathcal{I}$-continuous function (resp.(i, j)-$\beta\mathcal{I}-$continuous) if $f^{-1}(V)$ is (i, j)-semi-$\mathcal{I}$-open(resp.(i, j)-$\beta\mathcal{I}$-open) in X for all γ_i-open set V in Y, where $i \neq j$ and $i, j = 1, 2$.*

Throughout the article, we suppose that $i \neq j$,and $i, j = 1, 2$.

3. γ_{ij}-$\beta\mathcal{I}-$Open Sets

This section deals with the concept of γ_{ij}-$\beta\mathcal{I}$-open sets and some of their characterizations in an ideal bitopological space.

Definition 6. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, with an operation γ on $\tau_1 \cup \tau_2$, is said to be γ_{ij}-semi-$\mathcal{I}$-open set if $A \subseteq Cl^*_{\gamma_j}(Int_{\gamma_i}(A))$.*

Example 1. *Let $X = \{a, b, c, d\}$ and (X, τ_1, τ_2) be a bitopological space with $\tau_1 = \{\varnothing, X, \{b\}, \{c, d\}, \{b, c, d\}\}$, $\tau_2 = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$ and let $U^\gamma = Cl_j(U)$ for $U \in \tau_i$. Then we have, γ_{12}-semi-$\mathcal{I}$-open sets are $\varnothing$, X, $\{b\}$, $\{c, d\}$, $\{b, c, d\}$.*

Definition 7. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be γ_{ij}-$\beta\mathcal{I}$-open set if $A \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A)))$. The set consisting of all γ_{ij}-$\beta\mathcal{I}$-open sets in X will be denoted by γ_{ij}-$\beta\mathcal{I}O(X)$.*

Definition 8. *A subset A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is called γ_{ij}-$\beta\mathcal{I}$-closed set if the complement A^c is a γ_{ij}-$\beta\mathcal{I}$-open set. Equivalently, A is called γ_{ij}-$\beta\mathcal{I}$-closed set if $A \supseteq Int_{\gamma_i}(Cl_{\gamma_j}(Int^*_{\gamma_i}(A)))$. The set consisting of all γ_{ij}-$\beta\mathcal{I}$-closed sets in X will be denoted by γ_{ij}-$\beta\mathcal{I}C(X)$.*

Theorem 1. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space,*

(i) Every γ_{ij}-semi-$\mathcal{I}$-open set is γ_{ij}-$\beta\mathcal{I}$-open.

(ii) Every γ_{ij}-$\beta\mathcal{I}$-open set is γ_{ij}-β-open.

Proof.

(i) Let A be a subset of X. If A is γ_{ij}-*semi*-$\mathcal{I}$-open, then

$$\begin{aligned} A &\subseteq Cl^*_{\gamma_j}(Int_{\gamma_i}(A)) \subseteq Int_{\gamma_i}(A) \cup (Int_{\gamma_i}(A))^*_{\gamma_j} \\ &\subseteq (Int_{\gamma_i}(A)) \cup Cl_{\gamma_j}(Int_{\gamma_i}(A)) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(A)) \\ &\subseteq Cl_{\gamma_j}(Int_{\gamma_i}(A \cup A^*_{\gamma_j})) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A))). \end{aligned}$$

Therefore, A is a γ_{ij}-$\beta\mathcal{I}$-open set.

(ii) Let A be a subset of X. If A is γ_{ij}-$\beta\mathcal{I}$-open, then

$$\begin{aligned} A &\subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A))) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(A^*_{\gamma_j} \cup A)) \\ &\subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl_{\gamma_j}(A) \cup A)) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl_{\gamma_j}(A))). \end{aligned}$$

Therefore, A is a γ_{ij}-β-open set.

But generally the convers of this theorem is not true as giving in the next example.

Example 2. *From example 1, let* $A = \{c\}$ *or* $A = \{b, d\}$. *Calculations show that* A *is* γ_{12}*-*$\beta\mathcal{I}$*-open, however, it is not* γ_{12}*-semi-*$\mathcal{I}$*-open.*

Conclusion 1. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be an ideal bitopological space. Then every* γ_{ij}*-semi-*$\mathcal{I}$*-open set is* γ_{ij}*-*β*-open.*

Theorem 2. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be an ideal bitopological space. Then*

(i) The union of any γ_{ij}*-*$\beta\mathcal{I}$*-open sets is* γ_{ij}*-*$\beta\mathcal{I}$*-open set.*

(ii) The intersection of any γ_{ij}*-*$\beta\mathcal{I}$*-closed sets is* γ_{ij}*-*$\beta\mathcal{I}$*-closed set.*

Proof.

(i) Let $A_\alpha \in \gamma_{ij}$-$\beta\mathcal{I}O(X)$ for each $\alpha \in \Lambda$, where Λ is an index set. Then

$$A_\alpha \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A_\alpha))).$$

Therefore,

$$\begin{aligned} \cup_{\alpha\in\Lambda} A_\alpha &\subseteq \cup_{\alpha\in\Lambda}\{Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A_\alpha)))\} \\ &\subseteq \{Cl_{\gamma_j}(Int_{\gamma_i}(\cup_{\alpha\in\Lambda} Cl^*_{\gamma_j}(A_\alpha)))\} \\ &\subseteq \{Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(\cup_{\alpha\in\Lambda} A_\alpha)))\}. \end{aligned}$$

Then $\cup_{\alpha\in\Lambda} A_\alpha$ is γ_{ij}-$\beta\mathcal{I}$-open.

(ii) The proof follows by using (i) and taking complement.

The intersection of any two γ_{ij}-$\beta\mathcal{I}$-open sets may not be an γ_{ij}-$\beta\mathcal{I}$-open set as showing in the next example.

Example 3. *Let* $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\}$, $\tau_2 = \{\emptyset, X\}$ *and* $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. *Let define an operation* $\gamma : \tau_1 \cup \tau_2 \to P(x)$ *such that* $U^\gamma = U$ *for all* $U \in \tau_i$. *Then we have* $\{a, c\}$ *and* $\{c, d\}$ *are* γ_{12}*-*$\beta\mathcal{I}$*-open sets but* $\{c\}$ *is not* γ_{12}*-*$\beta\mathcal{I}$*-open.*

Definition 9. *Let* $(X, \tau_1, \tau_2, \mathcal{I})$ *be an ideal bitopological space with an operation* γ, $A \subset X$ *and* x *be a point of* X. *Then*

(i) x *is called an* $\beta\mathcal{I}$*-interior*$_{\gamma_{ij}}$ *point of* A *if there exists any* $U \in \gamma_{ij}$*-*$\beta\mathcal{I}O(X)$ *such that* $x \in U \subset A$.

(ii) The set of all $\beta\mathcal{I}$*-interior*$_{\gamma_{ij}}$ *points of* A *is called* γ_{ij}*-*$\beta\mathcal{I}$*-interior of* A *and is represented by* $\beta\mathcal{I}$*-*$Int_{\gamma_{ij}}(A)$.

Theorem 3. *Let* A *and* B *be subsets of* $(X, \tau_1, \tau_2, \mathcal{I})$. *Then the following properties hold:*

1) $\beta\mathcal{I}$-$Int_{\gamma_{ij}}(A) = \cup\{U : U \subset A$ *and* $U \in \gamma_{ij}$-$\beta\mathcal{I}O(X)\}$.

2) $\beta\mathcal{I}$-$Int_{\gamma_{ij}}(A)$ is the largest γ_{ij}-$\beta\mathcal{I}$-open subset of X contained in A.

3) A is γ_{ij}-$\beta\mathcal{I}$-open if and only if $A = \beta\mathcal{I}$-$Int_{\gamma_{ij}}(A)$.

The proof will be obtained directly from the definition and thus the proof is omitted.

Definition 10. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with an operation γ, $A \subset X$ and x be a point of X. Then,*

1. *x is called an γ_{ij}-$\beta\mathcal{I}$-cluster point of A if $U \cap A \neq \varnothing$ for every $U \in \gamma_{ij}$-$\beta\mathcal{I}O(X)$ such that $x \in U$.*
2. *The set of all γ_{ij}-$\beta\mathcal{I}$-cluster points of A is called γ_{ij}-$\beta\mathcal{I}$-cluster of A and is represented by $\beta\mathcal{I}$-$Cl_{\gamma_{ij}}(A)$.*

Theorem 4. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

1) $\beta\mathcal{I}$-$Cl_{\gamma_{ij}}(A) = \cap\{V : A \subset V$ and $V \in \gamma_{ij}$-$\beta\mathcal{I}C(A)$.

2) $\beta\mathcal{I}$-$Cl_{\gamma_{ij}}(A)$ is the smallest γ_{ij}-$\beta\mathcal{I}$-closed subset of X containing A.

3) A is γ_{ij}-$\beta\mathcal{I}$-closed if and only if $A = \beta\mathcal{I}$-$Cl_{\gamma_{ij}}(A)$

The proof will be obtained directly from the definition and thus the proof is omitted.

Theorem 5. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with an operation γ and $A \subset X$. Then,*

(i) If $\mathcal{I} = \{\varnothing\}$, then A is γ_{ij}-$\beta\mathcal{I}$-open if and only if A is γ_{ij}-β-open.

(ii) If $\mathcal{I} = P(X)$, then A is γ_{ij}-$\beta\mathcal{I}$-open if and only if A is γ_{ij}-semi-open.

Proof.

(i) We have just to show that if $\mathcal{I} = \{\varnothing\}$ and A is γ_{ij}-β-open, then A is γ_{ij}-$\beta\mathcal{I}$-open. If $\mathcal{I} = \{\varnothing\}$, then $A^*_{\gamma_j} = Cl_{\gamma_j}(A)$ for all subset A of X. Assumed A to be γ_{ij}-β-open set, then

$$\begin{aligned} A &\subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl_{\gamma_j}(A))) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(A^*_{\gamma_j})) \\ &\subseteq Cl_{\gamma_j}(Int_{\gamma_i}(A^*_{\gamma_j} \cup A)) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A))). \end{aligned}$$

Therefore, A is γ_{ij}-$\beta\mathcal{I}$-open.

(ii) Let $\mathcal{I} = P(X)$, then $A^*_{\gamma_j} = \{\varnothing\}$ for any subset A of X. Let A be γ_{ij}-*semi*-open.

Then $A \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(A)) = Cl_{\gamma_j}(Int_{\gamma_i}(A \cup A^*_{\gamma_j})) = Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A)))$. Therefore, A is γ_{ij}-$\beta\mathcal{I}$ -open.

Theorem 6. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with an operation γ and $A \subset X$. Then A is γ_{ij}-$\beta\mathcal{I}$-open if and only if $Cl_{\gamma_j}(A) = Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A)))$.*

Proof. Let A be an γ_{ij}-$\beta\mathcal{I}$-open subset of X. Then $A \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A)))$. Hence $Cl_{\gamma_j}(A) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A)))$. Since $A^*_{\gamma_j} \cup A \subseteq Cl_{\gamma_j}(A)$, then we have,

$$Cl_{\gamma_j}(A) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A))) \subseteq Cl_{\gamma_j}(Int_{\gamma_i}(Cl_{\gamma_j}(A))) \subseteq Cl_{\gamma_j}(A).$$

Therefore, $Cl_{\gamma_j}(A) = Cl_{\gamma_j}(Int_{\gamma_i}(Cl^*_{\gamma_j}(A)))$.
The convers is obvious.

4. $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-Continuous Functions

In this section the concept of $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous function in ideal bitopological spaces are introduced along with some characterizations via related notions.

Throughout this section, let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with an operation γ, and (Y, σ_1, σ_2) be a bitopological space with an operation δ.

Definition 11. *A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is called $(\gamma, \delta)_{ij}$-semi-$\mathcal{I}$-continuous function (resp. $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$ continuous) if $f^{-1}(V)$ is γ_{ij}-semi-$\mathcal{I}$-open (resp. γ_{ij}-$\beta\mathcal{I}$ open) in X for all δ_i-open set V in Y.*

Generally every $(\gamma, \delta)_{ij}$-*semi*-$\mathcal{I}$-continuous function is $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous, but the convers is not true as giving in next example.

Example 4. *Let $X = \{a, b, c, d\}$ be a set and $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space with*
$\tau_1 = \{\varnothing, X, \{b\}, \{c, d\}, \{b, c, d\}\}$,
$\tau_2 = \{\varnothing, X, \{a\}, \{a, b\}, \{a, c, d\}\}$,
$\mathcal{I} = \{\varnothing, \{b\}\}$,
$U^\gamma = Cl_j(U)$ *for* $U \in \tau_i$.
Let $Y = \{p, q, r, s\}$ be a set and (Y, σ_1, σ_2) be a bitopological space with
$\sigma_1 = \{\varnothing, Y, \{q\}, \{r, s\}, \{q, r, s\}\}$,
$\sigma_2 = \{\varnothing, Y, \{p\}, \{p, q\}, \{p, r, s\}\}$,
$V^\delta = V$ *for* $V \in \sigma_i$.
Let define $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ such that $f(a) = p$, $f(b) = q$, $f(c) = r$ and $f(d) = s$. Then f is $(\gamma, \delta)_{12}$-$\beta\mathcal{I}$-continuous but not $(\gamma, \delta)_{12}$-semi-$\mathcal{I}$-continuous because $\{p\}$ is δ_i-open set and $f^{-1}(\{p\}) = \{a\}$ which is γ_{12}-$\beta\mathcal{I}$-open in X but not γ_{12}-semi-$\mathcal{I}$-open in X.

Theorem 7. *For any function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$, the next properties are equivalent,*

1) *f is $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous*

2) *For all $x \in X$ and every δ_i-open set V in Y containing $f(x)$, there exists a γ_{ij}-$\beta\mathcal{I}$-open set U of X containing x such that $f(U) \subset V$.*

Proof.

(1 ⇒ 2) Let V is δ_i-open in Y such that $f(x) \in V$. Since f is $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous, $f^{-1}(V)$ is γ_{ij}-$\beta\mathcal{I}$-open set in X. Let $U = f^{-1}(V)$. Then $f(x) \in f(U) \subset V$.

(2 ⇒ 1) Let V be δ_i-open set in Y and $x \in f^{-1}(V)$. Then V is δ_i-open set in Y and $f(x) \in V$. From the hypothesis, there exists an γ_{ij}-$\beta\mathcal{I}$-open set U in X containing x such that $f(U) \subset V$. Then $x \in U \in f^{-1}(V)$, i.e. $f^{-1}(V)$ is γ_{ij}-$\beta\mathcal{I}$-open set in X. Therefore, f is $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous.

Theorem 8. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ be a $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous function. Then the next properties are equivalent:*

1) *The inverse image of every δ_i-closed set in Y is γ_{ij}-$\beta\mathcal{I}$-closed set in X,*
2) *$f(\beta\mathcal{I}\text{-}Cl_{\gamma_{ij}}(U)) \subset Cl_{\delta_j}(f(U))$, for all subset U of X,*
3) *$\beta\mathcal{I}\text{-}Cl_{\gamma_{ij}}(f^{-1}(V)) \subset f^{-1}(Cl_{\delta_j}(V))$, for each subset V of Y.*

Proof.

(1 ⇒ 2) Let $U \subset X$. Since $Cl_{\delta_j}(f(U))$ is an δ_i-closed set in Y, from the hypothesis, we have $f^{-1}(Cl_{\delta_j}(f(U)))$ is γ_{ij}-$\beta\mathcal{I}$-closed set in X. Also $U \subset f^{-1}(Cl_{\delta_j}(f(U)))$ and $\beta\mathcal{I}\text{-}Cl_{\delta_i}(U)$ is the smallest γ_{ij}-$\beta\mathcal{I}$-closed set containing U. Therefore,

$$\beta\mathcal{I} - Cl_{\delta_i}(U) \subset f^{-1}(Cl_{\delta_i}(f(U))).$$

This implies that $f(\beta\mathcal{I}\text{-}Cl_{\gamma_{ij}}(U)) \subset Cl_{\delta_i}(f(U))$.

(2 ⇒ 3) Let $V \subset Y$. Then $f^{-1}(V) \subset X$. From the hypothesis,

$$f(\beta\mathcal{I} - Cl_{\gamma_{ij}}(f^{-1}(V))) \subset Cl_{\delta_j}(f(f^{-1}(V))) \subset Cl_{\delta_j}(V).$$

Hence $\beta\mathcal{I}\text{-}Cl_{\gamma_{ij}}(f^{-1}(V)) \subset f^{-1}(Cl_{\delta_j}(V))$.

(3 ⇒ 1) Let V be a δ_i-closed set in Y. From the hypothesis,

$$\beta\mathcal{I} - Cl_{\delta_i}(f^{-1}(V)) \subset f^{-1}(Cl_{\delta_j}(V)) = f^{-1}(V).$$

Therefore, $f^{-1}(V) = \beta\mathcal{I}\text{-}Cl_{\gamma_{ij}}(f^{-1}(V))$ and so $f^{-1}(V)$ is γ_{ij}-$\beta\mathcal{I}$-closed set in X.

Theorem 9. *The function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous function if and only if*

$$f^{-1}(Int_{\delta_i}(V)) \subset \beta\mathcal{I} - Int_{\gamma_{ij}} f^{-1}(V)$$

for all δ_i-open set of Y.

Proof. Let f is a $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous function and V be an δ_i-open set in Y. Then $f^{-1}(Int_{\delta_i}(V))$ is a γ_{ij}-$\beta\mathcal{I}$-open set in X. Therefore,

$$f^{-1}(Int_{\delta_i}(V)) \subset \beta\mathcal{I} - Int_{\gamma_{ij}} f^{-1}(Int_{\delta_i}(V)) \subset \beta\mathcal{I} - Int_{\gamma_{ij}} f^{-1}((V)).$$

If $f^{-1}(Int_{\delta_i}(V)) \subset \beta\mathcal{I}\text{-}Int_{\gamma_{ij}} f^{-1}((V))$ and V be a δ_i-open set of Y, then

$$f^{-1}(V) = f^{-1}(Int_{\delta_i}(V)) \subset \beta\mathcal{I} - Int_{\gamma_{ij}} f^{-1}((V)).$$

Therefore, $f^{-1}(V)$ is γ_{ij}-$\beta\mathcal{I}$-open set in X and so f is a $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous function.

Note that, the composition of two $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous functions need not to be $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous, in general.

Example 5. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ and $(X,\sigma_1,\sigma_2,\zeta)$ be two ideal bitopological spaces such that $X=\{a,b,c\}, \tau_1=\{\varnothing,X,\{a,b\}\}, \tau_2=\{\varnothing,X,\{a\},\{a,b\}\}, \mathcal{I}=\{\varnothing,\{b\}\}$,*

$$U^{\gamma} = \begin{cases} Cl_j(U) \ for \ U\in\tau_i, & b\notin U \\ U \quad , & b\in U \end{cases}$$

and $\sigma_1=\{\varnothing,X,\{b\},\{b,c\}\}, \sigma_2=\{\varnothing,X,\{b,c\}\}, \zeta=\{\varnothing,\{a\}\}, V^{\delta}=V$ for $V\in\sigma_i$. Let define $f:(X,\tau_1,\tau_2,\mathcal{I})\to(X,\sigma_1,\sigma_2)$ such that $f(a)=b, f(b)=a$ and $f(c)=c$ and let $(X,\varsigma_1,\varsigma_2)$ be a bitopological space such that $\varsigma_1=\{\varnothing,X,\{a\},\{c\}\{a,c\}\}, \varsigma_2=\{\varnothing,X,\{b,c\}\}$,

$$W^{\xi} = \begin{cases} Cl_j(W) \ for \ W\in\tau_i, & c\notin W \\ W \quad , & c\in W \end{cases}$$

and define $g:(X,\tau_1,\tau_2,\mathcal{I})\to(X,\varsigma_1,\varsigma_2)$ such that $g(a)=b$, $g(b)=c$ and $g(c)=a$. Then f is $(\gamma,\delta)_{12}$-$\beta\mathcal{I}$-continuous function and g is $(\delta,\xi)_{12}$-$\beta\mathcal{I}$-continuous function but the composition $g\circ f$ is not $(\gamma,\xi)_{12}$-$\beta\mathcal{I}-$continuous function because $\{a\}$ is ξ_1-open set and $(g\circ f)^{-1}(\{a\})=\{c\}\notin\gamma_{12}$-$\beta\mathcal{I}O(X)$.

Definition 12. *A function $(X,\tau_1,\tau_2,\mathcal{I})\to(Y,\sigma_1,\sigma_2)$ is said to be pairwise $(\gamma,\delta)_i$-$\beta\mathcal{I}$-continuous function if $f^{-1}(V)$ is γ_i-$\beta\mathcal{I}$-open in X for every δ_i-open set V in Y.*

Note that the notion of pairwise $(\gamma,\delta)_i$-$\beta\mathcal{I}$-continuous and $(\gamma,\delta)_{ij}$-$\beta\mathcal{I}$-continuous are independent.

Example 6. *Let $X=\{a,b,c\}, \tau_1=\{\varnothing,X,\{b\},\{b,c\}\}, \tau_2=\{\varnothing,X,\{b,c\}\}, \mathcal{I}=\{\varnothing,\{a\}\}$ with operation $U^{\gamma}=U$ for $U\in\tau_i$ and let (X,σ_1,σ_2) be a bitopological space such that $\sigma_1=\{\varnothing,X,\{a\},\{c\},\{a,c\}\}, \sigma_2=\{\varnothing,X,\{b,c\}\}$ with operation*

$$V^{\delta} = \begin{cases} Cl_j(V) \ for \ V\in\sigma_j, & c\notin V \\ V \quad , & c\in V \end{cases}$$

and define $f:(X,\tau_1,\tau_2,\mathcal{I})\to(X,\sigma_1,\sigma_2)$ such that $f(a)=b, f(b)=c$ and $f(c)=a$ Then f is $(\gamma,\delta)_{12}$-$\beta\mathcal{I}$-continuous function but it is not $(\gamma,\delta)_1$-$\beta\mathcal{I}$-continuous function since $\{a\}\in\delta_1$-open and $f^{-1}(\{a\})=\{c\}$ which is not $(\gamma,\delta)_1$-$\beta\mathcal{I}$-open.

Theorem 10. *Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ and $g : (X, \tau_1, \tau_2, \mathcal{I}) \to (Z, \ell_1, \ell_2)$. Then $g \circ f$ is $(\gamma, \xi)_{ij}$-$\beta\mathcal{I}$-continuous if f is $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous and g is pairwise $(\delta, \xi)_i$-$\beta\mathcal{I}$-continuous.*

Proof. Let $W \in \xi_i$-open set in Z. Since g is pairwise $(\delta, \xi)_i$-continuous, then $g^{-1}(W) \in \delta_i$-open set in Y. On the other hand, since f is $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous, $f^{-1}(g^{-1}(W)) \in \gamma_{ij}$-$\beta\mathcal{I}O(X)$. Therefore, we obtain that $g \circ f$ is $(\gamma, \xi)_{ij}$-$\beta\mathcal{I}$-continuous.

5. Conclusion

In this study, we defined the notion of γ_{ij}-*semi*-$\mathcal{I}$-open sets and γ_{ij}-$\beta\mathcal{I}$-open sets by generalizing (i, j)-semi-$\mathcal{I}$-open sets and (ij)-$\beta\mathcal{I}$-open sets in ideal bitopological spaces with an operation $\gamma : \tau \to P(X)$. We show that every γ_{ij}-*semi*-$\mathcal{I}$-open set is a γ_{ij}-$\beta\mathcal{I}$-open but the converse is not always true. Then we described the notions γ_{ij}-$\beta\mathcal{I}$-interior and γ_{ij}-$\beta\mathcal{I}$-cluster of a set A. Finally we characterized $(\gamma, \delta)_{ij}$-semi-$\mathcal{I}$-continuous and $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous functions and showed that any $(\gamma, \delta)_{ij}$-semi-$\mathcal{I}$-continuous function is a $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous but the converse is not always true. Also it is shown that the composition of two $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous functions need not to be $(\gamma, \delta)_{ij}$-$\beta\mathcal{I}$-continuous. Consequently the following diagrams are true:

$$\gamma_{ij}\text{-semi-}\mathcal{I}\text{-open} \longrightarrow \gamma_{ij}\text{-}\beta\mathcal{I}\text{-open} \longrightarrow \gamma_{ij}\text{-}\beta\text{-open}$$

$$\gamma_{ij}\text{-}\beta\mathcal{I}\text{-open} \longleftrightarrow \gamma_{ij}\text{-}\beta\text{-open} \quad (\mathcal{I} = \{\varnothing\})$$

$$\gamma_{ij}\text{-}\beta\mathcal{I}\text{-open} \longleftrightarrow \gamma_{ij}\text{-semi-open} \quad (\mathcal{I} = P(X))$$

$$(\gamma, \delta)_{ij}\text{-semi-}\mathcal{I}\text{-continuous} \longrightarrow (\gamma, \delta)_{ij}\text{-}\beta\mathcal{I}\text{-continuous}$$

These notations, defined in this study, can be extended to other practicable research fields of topology such as fuzzy topology, soft topology, intuitionistic topology and so on. Also generalized seperation axioms can be introduced by the concept of generalized β-open set.

References

[1] B. Ahmad and S. Hussain. γ-Semi-Open Sets in Topological Spaces II. Southeast Asian Bull. Math 34, no. 6 ,997-1008, 2010.

[2] B. Bhattacharya and A. Paul. A New Approach of γ-open Sets in Bitopological Spaces, Gen. Math. Notes 20(2), 95-110, 2014.

[3] M. Caldas, S. Jafari and N. Rajesh. Some Fundamental Properties of β-open Sets in Ideal Bitopological Spaces. European Journal of Pure and Applied Mathematics, 6(2), 247-255, 2013.

[4] A.Csaszar. Generalized open sets. Acta Mathematica Hungarica, 75(1-2), 65-87, 1997.

[5] E. Ekici. On pre-I-open Sets, semi-I-open Sets and bI-open Sets in Ideal Topological Spaces. Acta Universitatis Apulensis, 30, 293-303, 2012.

[6] S. D. Jyoti. bI-open sets in ideal bitopological space. International Journal of Pure and Applied Mathematics 105(1), 7-18, (2015).

[7] S. Jafari and N. Rajesh. On qI-open sets in ideal bitopological spaces.University of Bacau, Faculty of Sciences, Scientific Studies and Research, Series Mathematics and Informatics 20(2), 29-38, 2010.

[8] M. Ilkhan, M. Akyigit, and E. E. Kara, On new types of sets VIA γ-open sets in bitopological spaces, Communications Series A1 Mathematics and Statistics, 67(1), 225-234, (2017).

[9] M. Kar, S. Thakur, S. Rana, and J. Maitra. I-Continuous Functions in Ideal Bitopological Spaces. American Journal of Engineering Research, 3(3), 51-55, 2014.

[10] S. Kasahara. Operation-Compact spaces, Math. Japon, 24, 97-105, 1979.

[11] J. Kelly. Bitopological Spaces. Proceedings of the London Mathematical Society, 3(1), 71-89, 1963.

[12] F. Khedr. Operation on Bitopologies, Delta J. Sci, 8(1), 309-320, 1984.

[13] F. Khedr and K. Abdelhakiem. Operations on Bitopological Spaces. Fasciculi Mathematici, (45), 47-57, 2010.

[14] F. Khedr, S. Al-Areefi and T. Noiri. Precontinuity and Semi-Precontinuity in Bitopological Spaces. Indian Journal of Pure and Applied Mathematics, 23,625-625, 1992.

[15] K. Kuratowski. Topology, Academic Press, New York. 1966.

[16] J. K. Maitra and H. K. Tripathi, Local function in Generalized Ideal Topological Spaces, VISLESANA, 11(1), 191- 195, (2014).

[17] N. Levine. Semi-open sets and semi continuity in topological spaces, Amer. Math.70, 36-41, 1963.

[18] T. Noiri, M. Rajamani and M. Maheswari.. A decomposition of pairwise continuity via ideals. Boletim da Sociedade Paranaense de Matemtica, 34(1), 141-149,(2016).

[19] S. Maheshwari, and R. Prasad. Semi-open sets and semi-continuous function in bitopological spaces, 26, 29-37, 1977.

[20] H. Ogata. Operations on Topological Spaces and Associated Topology. Math. Japon. 36, 175-184, 1991.

[21] S. Tahiliani. Operation approach to β-open sets and applications. Mathematical Communications 16, no. 2, 577-591, 2011.

[22] R. Vaidyanathaswamy. The Localisation Theory in Set Topology, Proceedings of the Indian Acadamic of Sciences, 20, 51-61, 1945.