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On Approximation and Generalized Type of Entire Functions of Several Complex Variables

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Abstract. In the present paper, we study the polynomial approximation of entire functions of several complex variables. The characterizations of generalized type of entire functions of several complex variables have been obtained in terms of approximation and interpolation errors.

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1. Introduction

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [4] and Shah [5]. Hence, let L^0 denote the class of

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functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii)
$$\lim_{x\to\infty} \frac{h[\{1+1/\psi(x)\}x]}{h(x)} = 1$$
for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions $h(x)$ satisfying conditions (i) and

(iii)
$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$$
for every $c > 0$, that is $h(x)$ is slowly increasing.

For an entire transcendental function $f(z) = \sum_{n=1}^{\infty} b_n z^n$, define $M(r) = \max_{|z|=r} |f(z)|$. For functions $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$, the generalized order of $f(z)$ is given by

$$\rho(\alpha,\beta,f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r)]}{\beta(\log r)}.$$

Further, for $\alpha(x)$, $\beta^{-1}(x)$ and $\gamma(x) \in L^0$, generalized type of an entire transcendental function $f(z)$ is given as

$$\sigma(\alpha,\beta,\rho,f) = \limsup_{r\to\infty} \frac{\alpha[\log M(r)]}{\beta[\{\gamma(r)\}^{\rho}]}$$

where $0 < \rho < \infty$ is a fixed number.

Let $g : C^N \to C$, $N \geq 1$, be an entire transcendental function. For $z = (z_1, z_2, ..., z_N) \in C^N$, we put $S(r,g) = \sup\{|g(z)| : |z_1|^2 + |z_2|^2 + ... + |z_N|^2 = r^2\}$, $r > 0$. Then we define the generalized order and generalized type of $g(z)$ as

$$\rho(\alpha,\beta,g) = \limsup_{r\to\infty} \frac{\alpha\ [\log S(r,g)]}{\beta\ (\log r)}$$

and

$$\sigma(\alpha,\beta,\rho,g)=\lim_{r\to\infty}\sup\frac{\alpha\,[\log S(r,g)]}{\beta\,[\{\gamma(r)\}^{\rho}]}.$$

Let K be a compact set in C^N and let $||.||_K$ denote the sup norm on K. The function $\Phi_K(z)=\sup\left[|p(z)|^{1/n}: p\text{−polynomial}, \deg p\le n, ||p||_K\le 1, n=1,2,..\text{and}\, z\in C^N\right]$, is called the Siciak extremal function of the compact set K (see [2] and [3]). Given a function f defined and bounded on K, we put for $n=1,2,...$

$$E_n^1(f,K)=||f-t_n||_K;$$
$$E_n^2(f,K)=||f-l_n||_K;$$
$$E_{n+1}^3(f,K)=||l_{n+1}-l_n||_K;$$

where t_n denotes the n^{th} Chebyshev polynomial of the best approximation to f on K and l_n denotes the n^{th} Lagrange interpolation polynomial for f with nodes at extremal points of K (see [2] and [3]).

Janik [1] obtained the characterizations of order of entire functions in terms of the approximation errors defined above. Later he obtained the characterizations of the generalized order [3]. In this note we obtained the characterizations of the generalized type.

For the case $N=1$ this result was obtained by Shah [5].

2. Results

We first prove a lemma.

Lemma 1. *Let K be a compact set in C^N such that Φ_K is locally bounded in C^N. Set $G(x,t,\rho)=\gamma^{-1}\{[\beta^{-1}\{t\alpha(x)\}]^{1/\rho}\}$. Suppose that for all $t, 0<t<\infty$,*

(a) If $\gamma(x)\in\Lambda$ and $\alpha(x)\in\Lambda$, then

$$\frac{d[\log\{G(x,t,\rho)\}]}{d(\log x)}=O(1)$$

as $x \to \infty$.

(b) *If* $\gamma(x) \in (L^0 - \Lambda)$ *or* $\alpha(x) \in (L^0 - \Lambda)$, *then*

$$\lim_{x\to\infty} \frac{d[\log\{G(x,t,\rho)\}]}{d(\log x)} = \frac{1}{\rho}.$$

Let $(p_n)_{n\in N}$ *be a sequence of polynomials in* C^N *such that*

(i) $\deg p_n \le n, n \in N$.

(ii) *there exists* $n_0 \in N$ *such that*

$$||p_n||_K \le e^{n/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\overline{t}}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-n},$$

where $\overline{t} = t + \varepsilon$, *for small* $\varepsilon > 0$.

Then $\sum_{n=0}^{\infty} p_n$ *is an entire function and the generalized type* $\sigma(\alpha,\beta,\rho,\sum_{n=0}^{\infty} p_n)$ *of this entire function satisfies*

$$\sigma(\alpha,\beta,\rho,\sum_{n=0}^{\infty} p_n) \le t$$

provided $\sum_{n=0}^{\infty} p_n$ *is not a polynomial.*

Proof. By assumption, we have

$$||p_n||_K r^n \le r^n e^{n/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\overline{t}}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-n}, n \ge n_0, r > 0.$$

If $\gamma(x) \in \Lambda$ and $\alpha(x) \in \Lambda$, then by assumptions of lemma, there exists a number $b > 0$ such that for $x > a$, we have

$$\left|\frac{d[\log\{G(x,t,\rho)\}]}{d(\log x)}\right| < b.$$

Let us consider the function

$$\phi(x) = r^x e^{x/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\overline{t}}\alpha(x/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-x}.$$

The maximum of $\phi(x)$ is attained for a value of x given by (see e.g. [4])

$$x^*(r) = \rho\alpha^{-1}[\bar{t}\beta\{(\gamma\{re^{1/\rho - a(r)}\})^{\rho}\}],$$

where

$$a(r) = \frac{d[\log\{G(x/\rho, 1/\bar{t}, \rho)\}]}{d(\log x)}.$$

Thus,

$$||p_n||_K r^n \leq \exp(b\rho\alpha^{-1}[\bar{t}\beta(\{\gamma(re^{1/\rho+b})\}^{\rho})]), n \geq n_0, r > 0. \tag{1}$$

Let us write $K_r = \{z \in C^N : \Phi_K(z) < r, r > 1\}$, then for every polynomial p of degree $\leq n$, we have (see e.g. [3] p.323)

$$|p_n(z)| \leq ||p_n||_K \Phi_K^n(z), z \in C^N. \tag{2}$$

So the series $\sum_{n=0}^{\infty} p_n$ is convergent in every $K_r, r > 1$, whence $\sum_{n=0}^{\infty} p_n$ is an entire function. Put

$$M^*(r) = \sup\{||p_n||_K r^n : n \in N, r > 0\}.$$

On account of 1, for every $r > 0$, there exists a positive integer $\nu(r)$ such that

$$M^*(r) = ||p_{\nu(r)}||_K r^{\nu(r)}$$

and

$$M^*(r) > ||p_n||_K r^n, n > \nu(r).$$

It is evident that $\nu(r)$ increases with r. First suppose that $\nu(r) \to \infty$ as $r \to \infty$. Then putting $n = \nu(r)$ in 1 we get for sufficiently large r

$$M^*(r) \leq \exp(b\rho\alpha^{-1}[\bar{t}\beta(\{\gamma(re^{1/\rho+b})\}^{\rho})]). \tag{3}$$

Put

$$F_r = \{z \in C^N : \Phi_K(z) = r\}, r > 1$$

and

$$M(r) = \sup\{|\sum_{n=0}^{\infty} p_n(z)| : z \in F_r\}, r > 1.$$

Now following Janik ([3] p.323), we have for some positive constant k,

$$S\left(r, \sum_{n=0}^{\infty} p_n\right) \leq M(kr) \leq 2M^*(2kr). \tag{4}$$

Combining 3 and 4, we get

$$S\left(r, \sum_{n=0}^{\infty} p_n\right) \leq 2\exp(b\rho\alpha^{-1}[\overline{t}\beta(\{\gamma(2kre^{1/\rho+b})\}^{\rho})])$$

or

$$\frac{\alpha\left(\frac{1}{b\rho}\log\left\{\frac{1}{2}S(r, \sum_{n=0}^{\infty} p_n)\right\}\right)}{\beta(\{\gamma(2kre^{1/\rho+b})\}^{\rho})} \leq \overline{t}.$$

Since $\alpha(x)$ and $\gamma(x) \in \Lambda$, we get on using (iii),

$$\limsup_{r\to\infty} \frac{\alpha\left(\log S(r, \sum_{n=0}^{\infty} p_n)\right)}{\beta(\{\gamma(r)\}^{\rho})} \leq \overline{t}. \tag{5}$$

Now let $\alpha(x) \in (L^0 - \Lambda)$ or $\gamma(x) \in (L^0 - \Lambda)$, then by the assumption of the lemma and as in [4], we have

$$\log r + o(1) = \log F(x/\rho, 1/\overline{t}, \rho).$$

Hence we obtain

$$r\{1 + o(1)\} = F(x/\rho, 1/\overline{t}, \rho).$$

As in [4], the maximum of the function $\phi(x)$ in this case is attained for

$$x^*(r) = \rho\alpha^{-1}[\overline{t}\beta([\gamma(r\{1 + o(1)\})]^{\rho})].$$

Further,

$$||p_n||_K r^n \leq \exp(\{1 + o(1)\}\alpha^{-1}[\overline{t}\beta([\gamma(r\{1 + o(1)\})]^{\rho})]), n \geq n_0, r > 0$$

and in this case we have

$$S\left(r,\sum_{n=0}^{\infty}p_n\right)\leq 2\exp\left(\{1+o(1)\}\alpha^{-1}[\overline{t}\beta(\{\gamma(2kr\{1+o(1)\})\}^{\rho})]\right)$$

or

$$\frac{\alpha\left(\{1+o(1)\}^{-1}\log\left\{\frac{1}{2}S(r,\sum_{n=0}^{\infty}p_n)\right\}\right)}{\beta(\{\gamma(2kr\{1+o(1)\})\}^{\rho})}\leq\overline{t}.$$

Using the properties of the functions α,β and γ and proceeding to limits we again obtain 5. Since $\overline{t}=t+\varepsilon,\varepsilon>0$ being arbitrarily, we finally get

$$\sigma(\alpha,\beta,\rho,\sum_{n=0}^{\infty}p_n)\leq t.$$

In the case when $\nu(r)$ is bounded then $M^{*}(r)$ is also bounded, whence $\sum_{n=0}^{\infty}p_n$ reduces to a polynomial. Hence the Lemma is proved.

Now we give our main result.

Theorem 1. *Let K be a compact set in C^N such that Φ_K is locally bounded in C^N. Set $F(x,t,\rho)=\gamma^{-1}\{[\beta^{-1}\{t\alpha(x)\}]^{1/\rho}\}$. Suppose that for all $t, 0<t<\infty$,*

(a) If $\gamma(x)\in\Lambda$ and $\alpha(x)\in\Lambda$, then

$$\frac{d[\log(F(x,t,\rho))}{d(\log x)}=O(1)$$

as $x\to\infty$.

(b) If $\gamma(x)\in(L^0-\Lambda)$ or $\alpha(x)\in(L^0-\Lambda)$, then

$$\lim_{x\to\infty}\frac{d[\log(F(x,t,\rho))}{d(\log x)}=\frac{1}{\rho}.$$

Then the function f, defined and bounded on K, is the restriction of an entire function g of the generalized type $\sigma(\alpha,\beta,\rho,g)$ if and only if

$$\sigma(\alpha,\beta,\rho,g)=\limsup_{n\to\infty}\frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}[E_n^s(f,K)]^{-1/n})]^{\rho}\}};s=1,2,3.$$

Proof. First we assume that f has an entire function extension g which is of generalized type $\sigma = \sigma(\alpha, \beta, \rho, g)$. We write

$$\eta_s = \limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}[E_n^s]^{-1/n})]^\rho\}}; s = 1, 2, 3.$$

Here E_n^s stands for $E_n^s\left(g|_K, K\right)$, $s = 1, 2, 3$. We show that $\sigma = \eta_s, s = 1, 2, 3$. It is known (see e.g. [6]) that

$$E_n^1 \le E_n^2 \le (n_* + 2)E_n^1, n \ge 0, \tag{6}$$

$$E_n^3 \le 2(n_* + 2)E_{n-1}^1, n \ge 1, \tag{7}$$

where $n_* = \begin{pmatrix} n+N \\ n \end{pmatrix}$. Using Stirling formula for the approximate value of

$$n! \approx e^{-n} n^{n+1/2}\sqrt{2\pi},$$

we get $n_* \approx \frac{n^N}{N!}$ for all large values of n. Hence for all large values of n, we have

$$E_n^1 \le E_n^2 \le \frac{n^N}{N!}[1 + o(1)]E_n^1$$

and

$$E_n^3 \le 2\frac{n^N}{N!}[1 + o(1)]E_n^1.$$

Thus $\eta_3 \le \eta_2 = \eta_1$ and it suffices to prove that $\eta_1 \le \sigma \le \eta_3$. First we prove that $\eta_1 \le \sigma$. Using the definition of the generalized type , for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$S(r, g) \le \exp[\alpha^{-1}\{\overline{\sigma}\beta(\{\gamma(r)\}^\rho)\}],$$

where $\overline{\sigma} = \sigma + \varepsilon$ provided r is sufficiently large. Without loss of generality, we may suppose that

$$K \subset B = \{z \in C^N : |z_1|^2 + |z_2|^2 + ... + |z_N|^2 \le 1\}.$$

Then

$$E_n^1 \le E_n^1(g, B).$$

Now following Janik ([3] p.324), we get

$$E_n^1(g,B) \leq r^{-n} S(r,g), r \geq 2, n \geq 0$$

or

$$E_n^1 \leq r^{-n} \exp[\alpha^{-1}\{\overline{\sigma}\beta(\{\gamma(r)\}^{\rho})\}].$$

Putting

$$r = r(n) = F(n/\rho, 1/\overline{\sigma}, \rho) = \gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\overline{\sigma}}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\},$$

we get

$$E_n^1 \leq e^{n/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\overline{\sigma}}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-n}$$

or

$$[E_n^1]^{-1/n} \geq e^{-1/\rho}\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\overline{\sigma}}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}$$

or

$$\frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}[E_n^1]^{-1/n})]^{\rho}\}} \leq \overline{\sigma}.$$

Taking limits as $n \to \infty$, we get

$$\limsup_{n\to\infty} \frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}[E_n^1]^{-1/n})]^{\rho}\}} \leq \overline{\sigma}.$$

Since $\varepsilon > 0$ is arbitrarily small, therefore finally we get

$$\eta_1 \leq \sigma.$$

Now we will prove that $\sigma \leq \eta_3$. Suppose that $\eta_3 < \sigma$. Then for every $\lambda, \eta_3 < \lambda < \sigma$,

$$\frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}[E_n^3]^{-1/n})]^{\rho}\}} \leq \lambda$$

provided n is sufficiently large. Thus

$$E_n^3 \leq e^{n/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\lambda}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-n}.$$

Also by previous lemma,

$$\sigma \leq \lambda,$$

where $\sigma = \sigma(\alpha, \beta, \rho, g)$ is the generalized type of $g(z)$ as dfefined on page 2.

Since λ has been chosen less than σ, we get a contradiction. Hence

$$\sigma \leq \eta_3.$$

Now let f be a function defined and bounded on K and such that for $s = 1, 2, 3$,

$$\eta_s = \lim_{n\to\infty} \sup \frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}[E_n^s]^{-1/n})]^\rho\}}.$$

So for every $\lambda_1 > \eta_s$ and for sufficiently large n, we have

$$\frac{\alpha(n/\rho)}{\beta\{[\gamma(e^{1/\rho}[E_n^s]^{-1/n})]^\rho\}} \leq \lambda_1$$

or

$$E_n^s \leq e^{n/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\lambda_1}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-n}.$$

Proceeding to limits as $n \to \infty$, we get

$$\lim_{n\to\infty}[E_n^s]^{1/n} \leq 0.$$

Also it is obvious that

$$\lim_{n\to\infty}[E_n^s]^{1/n} \geq 0.$$

Hence finally we get

$$\lim_{n\to\infty}[E_n^s]^{1/n} = 0.$$

So following Janik (see [1], Prop. 3.1), we claim that the function f can be continuously extended to an entire function . Let us put

$$g = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1}),$$

where $\{l_n\}$ is the sequence of Lagrange interpolation polynomials of f as defined earlier.

Now we claim that g is the required continuation of f and $\sigma(\alpha,\beta,\rho,g)=\eta_s$. For every $\lambda_1>\eta_3$ and for sufficiently large n, we have

$$E_n^3 \leq e^{n/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\lambda_1}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-n}$$

or

$$||l_n-l_{n-1}|| \leq e^{n/\rho}\left[\gamma^{-1}\left\{\left(\beta^{-1}\left\{\frac{1}{\lambda_1}\alpha(n/\rho)\right\}\right)^{1/\rho}\right\}\right]^{-n}.$$

So using the Lemma 1, we get

$$\sigma(\alpha,\beta,\rho,g)\leq\lambda_1.$$

Since $\lambda_1>\eta_3$ is arbitrary, so finally we get

$$\sigma(\alpha,\beta,\rho,g)\leq\eta_3.$$

Using the inequalities 6, 7 and the proof of first part given above, we have $\sigma(\alpha,\beta,\rho,g)=\eta_s$, as claimed. This completes the proof of the Theorem.

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References

[1] Adam Janik, On approximation and interpolation of entire functions, Zeszyty Naukowe Universtytetu Jagiellonskiego, 22 (1981), 173-188.

[2] Adam Janik, A characterization of the growth of analytic functions by means of polynomial approximation, Univ. Iagel. Acta Math., 24 (1984), 295-319.

[3] Adam Janik, On approximation of entire functions and generalized order, Univ. Iagel. Acta Math., 24 (1984), 321-326.

[4] M. N. Seremeta, On the connection between the growth of the maximum modulus of an entire function and the moduli of the coefficients of its power series expansion, Amer. Math. Soc. Transl., 88 (2) (1970), 291-301.

[5] S. M. Shah, Polynomial approximation of an entire function and generalized order, J. Approx. Theory, 19 (1977), 315-324.

[6] T. Winiarski, Application of approximation and interpolation methods to the examination of entire functions of n complex variables, Ann. Pol. Math, 28 (1973), 97-121.