# Some new oscillation results for fourth-order neutral differential equations

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Abstract. By employ the Riccati substitution technique, we establish new oscillation criteria for a class of fourth-order neutral differential equations. Our new criteria improves a number of existing ones. An illustrative example is provided.

**Keywords**: Fourth-order differential equations; Neutral delay; Oscillation.

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## 1. Introduction

For several decades, an increasing interest in obtaining sufficient conditions for oscillatory and nonoscillatory behavior of different classes of differential equations has been observed; see, for instance, the monographs [**1**]-[**5**], the papers [**6**]-[**12**], and the references cited therein.

Neutral differential equations are used in numerous applications in technology and natural science. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see Hale [**14**], and therefore their qualitative properties are important.

In this paper, we are concerned with the oscillation of solutions of the fourth-order neutral differential equation

$$\left(r(t)\left(\left(x(t)+p(t)\,x(\tau(t))\right)'''\right)^{\alpha}\right)' + q(t)\,x^{\beta}(\sigma(t)) = 0, \tag{1.1}$$

where $t \geq t_0$. In this work, we assume that $\alpha$ and $\beta$ are quotient of odd positive integers, $r, p, q \in C[t_0, \infty)$, $r(t) > 0$, $r'(t) \geq 0$, $q(t) > 0$, $0 \leq p(t) < p_0 < \infty$, $\tau, \sigma \in C[t_0, \infty)$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$. Moreover, we study (1.1) under the condition that

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s = \infty, \tag{1.2}$$

and we define the function

$$z(t) := x(t) + p(t)\,x(\tau(t)).$$

By a solution of (1.1) we mean a function $x \in C^3[t_y, \infty)$, $t_y \geq t_0$, which has the property $r(t)\left(z'''(t)\right)^{\alpha} \in C^1[t_y, \infty)$, and satisfies (1.1) on $[t_y, \infty)$. We consider only those solutions $x$ of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$, for all $T \geq t_y$.

Definition 1. *A solution $x$ of (1.1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory. The equation itself is termed oscillatory if all its solutions oscillate.*

Let us briefly comment on a number of related results which motivated our study. A number of oscillation results for differential equation

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t) f(x(\tau(t))) = 0,$$

have been established by Baculikova et al. [**8**] under the conditions (1.2) and

$$\int^{\infty} r^{-1/\alpha}(t)\,\mathrm{d}t < \infty.$$

Asymptotic behavior of higher-order quasilinear neutral differential equations of the form

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + q(t) x^{\beta}(\sigma(t)) = 0$$

have been studied by Li and Rogovchenko [**13**]. Agarwal et al. [**6**] investigated the oscillatory behavior of a higher-order differential equation

$$\left(r(t)\left(x^{(n-1)}(t)\right)^{\alpha}\right)' + q(t) x^{\beta}(\tau(t)) = 0,$$

under the condition (1.2).

In order to discuss our main results, we need the following lemmas:

Lemma 1.1. [**5**]*If the function $x$ satisfies $x^{(i)}(t) > 0$, $i = 0, 1, ..., n$, and $x^{(n+1)}(t) < 0$, then*

$$\frac{x(t)}{t^n/n!} \geq \frac{x'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 1.2. [**3**, Lemma 2.2.3]*Let $x \in C^n([t_0,\infty),(0,\infty))$. Assume that $x^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0,\infty)$ and that there exists a $t_1 \geq t_0$ such that $x^{(n-1)}(t)x^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t\to\infty} x(t) \neq 0$, then for every $\mu \in (0,1)$ there exists $t_\mu \geq t_1$ such that*

$$x(t) \geq \frac{\mu}{(n-1)!} t^{n-1} \left|x^{(n-1)}(t)\right| \text{ for } t \geq t_\mu.$$

Lemma 1.3. [**15**]*Let $x(t)$ be a positive and $n$-times differentiable function on an interval $[T,\infty)$ with its $n$th derivative $x^{(n)}(t)$ non-positive on $[T,\infty)$ and not identically zero on any interval of the form $[T',\infty)$, $T' \geq T$ and $x^{(n-1)}(t)x^{(n)}(t) \leq 0$, $t \geq t_x$ then there exist constants $\theta$, $0<\theta<1$ and $N>0$ such that*

$$x'(\theta t) \geq N t^{n-2} x^{(n-1)}(t),$$

*for all sufficient large $t$.*

## 2. One-condition theorems

Lemma 2.1. *Assume that $x$ is an eventually positive solution of (1.1). Then*

$$\left(r(t)\left(z'''(t)\right)^{\alpha}\right)' \leq -q(t)(1-p_0)^{\beta} z^{\beta}(\sigma(t)). \tag{2.1}$$

PROOF. Assume that $x$ be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Since $r'(t) > 0$, we have

$$z(t) > 0,\ z'(t) > 0,\ z'''(t) > 0,\ z^{(4)}(t) < 0 \text{ and } \left(r(t)(z'''(t))^{\alpha}\right)' \leq 0, \tag{2.2}$$

for $t \geq t_1$. From definition of $z$, we get

$$\begin{aligned} x(t) &\geq z(t) - p_0 x(\tau(t)) \geq z(t) - p_0 z(\tau(t)) \\ &\geq (1-p_0) z(t), \end{aligned}$$

which with (1.1) gives

$$\left(r(t)(z'''(t))^{\alpha}\right)' + q(t)(1-p_0)^{\beta} z^{\beta}(\sigma(t)) \leq 0.$$

The proof is complete. □

THEOREM 2.1. *Assume that*

$$\liminf_{t\to\infty} \frac{1}{\widetilde{\Psi}_1(t)} \int_t^{\infty} \Psi_2(s)\, \widetilde{\Psi}_1^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s > \frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}, \tag{2.3}$$

*where*

$$\Psi_1(t) = q(t)(1-p_0)^{\beta} M^{\beta-\alpha}(\sigma(t)),\ \Psi_2(t) = \alpha\varepsilon \frac{\sigma^2(t)\zeta\sigma'(t)}{r^{1/\alpha}(t)} \text{ and } \widetilde{\Psi}_1(t) = \int_t^{\infty} \Psi_1(s)\,\mathrm{d}s$$

*Then, (1.1) is oscillatory.*

PROOF. Assume that $x$ be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Using Lemma 2.1, we obtain that (2.1) holds.
Define $\omega$ as follows

$$\omega(t) := \frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(\zeta\sigma(t))}. \tag{2.4}$$

By differentiating and using (2.1), we obtain

$$\omega'(t) \leq \frac{-q(t)(1-p_0)^{\beta} z^{\beta}(\sigma(t))\,.}{z^{\alpha}(\zeta\sigma(t))} - \alpha \frac{r(t)(z'''(t))^{\alpha} z'(\zeta\sigma(t))\zeta\sigma'(t)}{z^{\alpha+1}(\zeta\sigma(t))}.$$

From Lemma 1.3, we have

$$\omega'(t) \leq -q(t)(1-p_0)^{\beta} z^{\beta-\alpha}(\sigma(t)) - \alpha \frac{r(t)(z'''(t))^{\alpha} \varepsilon\sigma^2(t) z'''(\sigma(t))\zeta\sigma'(t)}{z^{\alpha+1}(\zeta\sigma(t))}.$$

Which is

$$\omega'(t) \leq -q(t)(1-p_0)^{\beta} z^{\beta-\alpha}(\sigma(t)) - \alpha\varepsilon \frac{r(t)\sigma^2(t)\zeta\sigma'(t)(z'''(t))^{\alpha+1}}{z^{\alpha+1}(\zeta\sigma(t))},$$

by using (2.4) we have

$$\omega'(t) \leq -q(t)(1-p_0)^{\beta} z^{\beta-\alpha}(\sigma(t)) - \alpha\varepsilon \frac{\sigma^2(t)\zeta\sigma'(t)}{r^{1/\alpha}(t)} \omega^{(\alpha+1)/\alpha}(t), \tag{2.5}$$

Since $z'(t) > 0$, there exist a $t_2 \geq t_1$ and a constant $M > 0$ such that

$$z(t) > M.$$

Then, (2.5), turn to

$$\omega'(t) \leq -q(t)(1-p_0)^{\beta} M^{\beta-\alpha}(\sigma(t)) - \alpha\varepsilon \frac{\sigma^2(t)\zeta\sigma'(t)}{r^{1/\alpha}(t)} \omega^{(\alpha+1)/\alpha}(t),$$

that is

$$\omega'(t) + \Psi_1(t) + \Psi_2(t)\omega^{(\alpha+1)/\alpha}(t) \leq 0.$$

Integrating the above inequality from $t$ to $l$ , we get

$$\omega(l) - \omega(t) + \int_t^l \Psi_1(s)\,\mathrm{d}s + \int_t^l \Psi_2(s)\omega^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s \leq 0.$$

Letting $l \to \infty$ and using $\omega > 0$ and $\omega' < 0$, we have

$$\omega(t) \geq \widetilde{\Psi}_1(t) + \int_t^{\infty} \Psi_2(s)\omega^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s.$$

This implies

$$\frac{\omega(t)}{\widetilde{\Psi}_1(t)} \geq 1 + \frac{1}{\widetilde{\Psi}_1(t)} \int_t^{\infty} \Psi_2(s)\widetilde{\Psi}_1^{\frac{\alpha+1}{\alpha}}(s) \left(\frac{\omega(s)}{\widetilde{\Psi}_1(s)}\right)^{\frac{\alpha+1}{\alpha}} \mathrm{d}s. \tag{2.6}$$

Let $\lambda = \inf_{t\geq T}\omega(t)/\widetilde{\Psi}_1(t)$ then obviously $\lambda \geq 1$. Thus, from (2.3) and (2.6) we see that

$$\lambda \geq 1 + \alpha\left(\frac{\lambda}{\alpha+1}\right)^{(\alpha+1)/\alpha}$$

or

$$\frac{\lambda}{\alpha+1} \geq \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1}\left(\frac{\lambda}{\alpha+1}\right)^{(\alpha+1)/\alpha},$$

which contradicts the admissible value of $\lambda \geq 1$ and $\alpha > 0$.
Therefore, the proof is complete. □

## 3. Two independent conditions theorems

NOTATION 1. *Here, we introduce Riccati substitutions*

$$\omega(t) := \frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha}(t)} \quad \text{and } w(t) := \frac{z'(t)}{z(t)}. \tag{3.1}$$

*Also, for convenience, we denote that:*

$$R_1(t) \ : \ = \alpha\mu\frac{t^2}{2r^{1/\alpha}(t)},$$

$$Q_1(t) \ : \ = q(t)(1-p_0)^{\beta} M_1^{\beta-\alpha}\left(\frac{\sigma(t)}{t}\right)^{3\beta}$$

*and*

$$Q_2(t) := (1-p_0)^{\beta/\alpha} M_2^{\beta/\alpha-1} \int_t^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty} q(s)\frac{\sigma^{\beta}(s)}{s^{\beta}}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u,$$

*for some* $\mu \in (0,1)$ *and every* $M_1, M_2$ *are positive constants.*

All functional inequalities are assumed to hold eventually, that is, they are assumed to be satisfied for all $t$ sufficiently large. We begin with the following lemma that can be found in [**?**, Lemma 2.1].

LEMMA 3.1. *Assume that (1.2) holds and $x$ is an eventually positive solution of (1.1). Then, $\left(r(t)(z'''(t))^{\alpha}\right)' < 0$ and there are the following two possible cases eventually:*

$$\begin{aligned}(\mathbf{C}_1)\ z(t) &> 0,\ z'(t)>0,\ z''(t)>0,\ z'''(t)>0,\ z^{(4)}(t)<0,\\ (\mathbf{C}_2)\ z(t) &> 0,\ z'(t)>0,\ z''(t)<0,\ z'''(t)>0.\end{aligned}$$

LEMMA 3.2. *Let $x$ is an eventually positive solution of (1.1) and the functions $\omega$ and $w$ are defined as in (3.1).*
$(\mathbf{I}_1)$ *If $x$ satisfies $(\mathbf{C}_1)$, then*

$$\omega'(t)+Q_1(t)+R_1(t)\,\omega^{\frac{\alpha+1}{\alpha}}(t)\leq 0; \tag{3.2}$$

$(\mathbf{I}_2)$ *If $x$ satisfies $(\mathbf{C}_2)$, then*

$$w'(t)+Q_2(t)+w^2(t)\leq 0. \tag{3.3}$$

PROOF. Assume that $x$ be an eventually positive solution of (1.1). Then, there exists a $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t))>0$ for $t\geq t_1$. Using Lemma 2.1, we obtain that (2.1) holds.
In the case $(\mathbf{C}_1)$, by differentiating $\omega$ and using (2.1), we obtain

$$\omega'(t)\leq -q(t)(1-p_0)^{\beta}\frac{z^{\beta}(\sigma(t))}{z^{\alpha}(t)}-\alpha\frac{r(t)(z'''(t))^{\alpha}}{z^{\alpha+1}(t)}z'(t). \tag{3.4}$$

From Lemma 1.1, we have that

$$z(t)\geq \frac{t}{3}z'(t)\ \text{ and hence }\ \frac{z(\sigma(t))}{z(t)}\geq\frac{\sigma^3(t)}{t^3}. \tag{3.5}$$

It follows from Lemma 1.2 that

$$z'(t)\geq\frac{\mu_1}{2}t^2z'''(t), \tag{3.6}$$

for all $\mu_1\in(0,1)$ and every sufficiently large $t$. Since $z'(t)>0$, there exist a $t_2\geq t_1$ and a constant $M>0$ such that

$$z(t)>M, \tag{3.7}$$

for $t\geq t_2$. Thus, by (3.4), (3.5), (3.6) and (3.7), we get

$$\omega'(t)+Q_1(t)+R_1(t)\,\omega^{\frac{\alpha+1}{\alpha}}(t)\leq 0.$$

In the case $(\mathbf{C}_2)$, integrating (2.1) from $t$ to $u$, we obtain

$$r(u)(z'''(u))^{\alpha}-r(t)(z'''(t))^{\alpha}\leq-\int_t^u q(s)(1-p_0)^{\beta}z^{\beta}(\sigma(s))\,\mathrm{d}s. \tag{3.8}$$

From Lemma 1.1, we get that

$$z(t)\geq tz'(t)\ \text{ and hence }\ z(\sigma(t))\geq\frac{\sigma(t)}{t}z(t). \tag{3.9}$$

For (3.8), letting $u\to\infty$ and using (3.9), we see that

$$r(t)(z'''(t))^{\alpha}\geq(1-p_0)^{\beta}z^{\beta}(t)\int_t^{\infty}q(s)\frac{\sigma^{\beta}(s)}{s^{\beta}}\mathrm{d}s.$$

Integrating this inequality again from $t$ to $\infty$, we get

$$z''(t)\leq-(1-p_0)^{\beta/\alpha}z^{\beta/\alpha}(t)\int_t^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty}q(s)\frac{\sigma^{\beta}(s)}{s^{\beta}}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u, \tag{3.10}$$

for all $\mu_2 \in (0,1)$. By differentiating $w$ and using (3.7) and (3.10), we find

$$\begin{aligned} w'(t) &= \frac{z''(t)}{z(t)} - \left(\frac{z'(t)}{z(t)}\right)^2 \\ &\leq -w^2(t) - (1-p_0)^{\beta/\alpha} M^{(\beta/\alpha)-1} \int_t^\infty \left(\frac{1}{r(u)} \int_u^\infty q(s) \frac{\sigma^\beta(s)}{s^\beta} \mathrm{d}s\right)^{1/\alpha} \mathrm{d}u, \end{aligned} \tag{3.11}$$

hence

$$w'(t) + Q_2(t) + w^2(t) \leq 0.$$

The proof is complete. □

THEOREM 3.1. *Assume that*

$$\liminf_{t\to\infty} \frac{1}{\widetilde{Q}_1(t)} \int_t^\infty R_1(s) \widetilde{Q}_1^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s > \frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}} \tag{3.12}$$

*and*

$$\liminf_{t\to\infty} \frac{1}{\widetilde{Q}_2(t)} \int_{t0}^\infty \widetilde{Q}_2^2(s)\,\mathrm{d}s > \frac{1}{4}, \tag{3.13}$$

*where*

$$\widetilde{Q}_1(t) = \int_t^\infty Q_1(s)\,\mathrm{d}s \quad \textit{and} \quad \widetilde{Q}_2(t) = \int_t^\infty Q_2(s)\,\mathrm{d}s. \tag{3.14}$$

*Then, (1.1) is oscillatory.*

PROOF. Assume to the contrary that (1.1) has a nonoscillatory solution in $[t_0,\infty)$. Without loss of generality, we let $x$ be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. From Lemma 3.1 there is two cases.
For case $(\mathbf{C}_1)$. Using Lemma 3.2, we obtain (3.2) holds. Integrating (3.2) from $t$ to $l$ , we get

$$\omega(l) - \omega(t) + \int_t^l Q_1(s)\,\mathrm{d}s + \int_t^l R_1(s)\,\omega^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s \leq 0.$$

Letting $l \to \infty$ and using $\omega > 0$ and $\omega' < 0$, we have

$$\omega(t) \geq \widetilde{Q}_1(t) + \int_t^\infty R_1(s)\,\omega^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s. \tag{3.15}$$

This implies

$$\frac{\omega(t)}{\widetilde{Q}_1(t)} \geq 1 + \frac{1}{\widetilde{Q}_1(t)} \int_t^\infty R_1(s)\,\widetilde{Q}_1^{\frac{\alpha+1}{\alpha}}(s) \left(\frac{\omega(s)}{\widetilde{Q}_1(s)}\right)^{\frac{\alpha+1}{\alpha}} \mathrm{d}s. \tag{3.16}$$

Let $\lambda = \inf_{t\geq T} \omega(t)/\widetilde{Q}_1(t)$ then obviously $\lambda \geq 1$. Thus, from (3.12) and (3.16) we see that

$$\lambda \geq 1 + \alpha \left(\frac{\lambda}{\alpha+1}\right)^{(\alpha+1)/\alpha}$$

or

$$\frac{\lambda}{\alpha+1} \geq \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1}\left(\frac{\lambda}{\alpha+1}\right)^{(\alpha+1)/\alpha},$$

which contradicts the admissible value of $\lambda \geq 1$ and $\alpha > 0$.
The proof of the case where $(\mathbf{C}_2)$ holds is the same as that of case $(\mathbf{C}_1)$. Therefore, the proof is complete. □

Define a sequence of functions $\{u_n(t)\}_{n=0}^{\infty}$ and $\{v_n(t)\}_{n=0}^{\infty}$ as

$$\begin{array}{l} u_0(t) = \widetilde{Q}_1(t), \text{ and } v_0(t) = \widetilde{Q}_2(t), \\ u_n(t) = u_0(t) + \int_t^\infty R_1(t) u_{n-1}^{(\alpha+1)/\alpha}(s)\, \mathrm{d}s, \ n > 1 \\ v_n(t) = v_0(t) + \int_t^\infty v_{n-1}^{(\alpha+1)/\alpha}(s)\, \mathrm{d}s, \ n > 1 \end{array} \tag{3.17}$$

where $\widetilde{Q}_1$ and $\widetilde{Q}_2$ defined as in (3.14). We see by induction that $u_n(t) \leq u_{n+1}(t)$ and $v_n(t) \leq v_{n+1}(t)$ for $t \geq t_0, \ n > 1$.

Theorem 3.2. *Let $u_n(t)$ and $v_n(t)$ be defined as in (3.17). If*

$$\limsup_{t\to\infty} \left( \frac{\mu_1 t^3}{6 r^{1/\alpha}(t)} \right)^\alpha u_n(t) > 1 \tag{3.18}$$

*and*

$$\limsup_{t\to\infty} \lambda t v_n(t) > 1, \tag{3.19}$$

*for some $n$, then (1.1)is oscillatory.*

Proof. Assume to the contrary that (1.1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we let $x$ be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. From Lemma 3.1 there is two cases.
In the case $(C_1)$, proceeding as in the proof of Lemma 3.2, we get that (3.6) holds. It follows from Lemma 1.2 that

$$z(t) \geq \frac{\mu_1}{6} t^3 z'''(t). \tag{3.20}$$

From definition of $\omega(t)$ and (3.20), we have

$$\frac{1}{\omega(t)} = \frac{1}{r(t)} \left( \frac{z(t)}{z'''(t)} \right)^\alpha \geq \frac{1}{r(t)} \left( \frac{\mu_1}{6} t^3 \right)^\alpha.$$

Thus,

$$\omega(t) \left( \frac{\mu_1 t^3}{6 r^{1/\alpha}(t)} \right)^\alpha \leq 1.$$

Therefore,

$$\limsup_{t\to\infty} \omega(t) \left( \frac{\mu_1 t^3}{6 r^{1/\alpha}(t)} \right)^\alpha \leq 1,$$

which contradicts (3.18).
The proof of the case where $(\mathbf{C}_2)$ holds is the same as that of case $(\mathbf{C}_1)$. Therefore, the proof is complete. □

Corollary 3.1. *Let $u_n(t)$ and $v_n(t)$ be defined as in (3.17). If*

$$\int_{t_0}^\infty Q_1(t) \exp\left( \int_{t_0}^t R_1(s) u_n^{1/\alpha}(s)\, \mathrm{d}s \right) \mathrm{d}t = \infty \tag{3.21}$$

*and*

$$\int_{t_0}^\infty Q_2(t) \exp\left( \int_{t_0}^t v_n^{1/\alpha}(s)\, \mathrm{d}s \right) \mathrm{d}t = \infty, \tag{3.22}$$

*for some $n$, then (1.1) is oscillatory.*

PROOF. Assume to the contrary that (1.1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we let $x$ be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. From Lemma 3.1 there is two cases.
In the case $(C_1)$, proceeding as in the proof of Theorem 3.1, we get that (3.15) holds. It follows from (3.15) that $\omega(t) \geq u_0(t)$. Moreover, by induction we can also see that $\omega(t) \geq u_n(t)$ for $t \geq t_0,\ n > 1$. Since the sequence $\{u_n(t)\}_{n=0}^{\infty}$ monotone increasing and bounded above, it converges to $u(t)$. Thus, by using Lebesgue's monotone convergence theorem, we see that

$$u(t) = \lim_{n\to\infty} u_n(t) = \int_t^\infty R_1(t)\, u^{(\alpha+1)/\alpha}(s)\,\mathrm{d}s + u_0(t)$$

and

$$u'(t) = -R_1(t)\, u^{(\alpha+1)/\alpha}(t) - Q_1(t). \tag{3.23}$$

Since $u_n(t) \leq u(t)$, it follows from (3.23) that

$$u'(t) \leq -R_1(t)\, u_n^{1/\alpha}(t)\, u(t) - Q_1(t).$$

Hence, we get

$$u(t) \leq \exp\left(-\int_T^t R_1(s)\, u_n^{1/\alpha}(s)\,\mathrm{d}s\right)\left(u(T) - \int_T^t Q_1(s) \exp\left(\int_T^s R_1(u)\, u_n^{1/\alpha}(u)\,\mathrm{d}u\right)\mathrm{d}s\right).$$

This implies

$$\int_T^t Q_1(s) \exp\left(\int_T^s R_1(u)\, u_n^{1/\alpha}(u)\,\mathrm{d}u\right)\mathrm{d}s \leq u(T) < \infty,$$

which contradicts (3.21). The proof of the case where $(\mathbf{C}_2)$ holds is the same as that of case $(\mathbf{C}_1)$. Therefore, the proof is complete. □

## 4. Further results

LEMMA 4.1. *Assume that $x$ is an eventually positive solution of (1.1) and*

$$p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \geq \left(\frac{\tau^{-1}\left(\tau^{-1}(t)\right)}{\tau^{-1}(t)}\right)^3. \tag{4.1}$$

*Then*

$$\left(r(t)\left(z'''(t)\right)^\alpha\right)' + q(t)\,\widetilde{p}^{\beta}(\sigma(t))\, z^\beta\left(\tau^{-1}(\sigma(t))\right) \leq 0, \tag{4.2}$$

*where*

$$\widetilde{p}(t) := \begin{cases} \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^3}{(\tau^{-1}(t))^3 p(\tau^{-1}(\tau^{-1}(t)))}\right) & \textit{for case } (\mathbf{C}_1); \\ \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{(\tau^{-1}(t)) p(\tau^{-1}(\tau^{-1}(t)))}\right) & \textit{for case } (\mathbf{C}_2). \end{cases} \tag{4.3}$$

PROOF. Proceeding as in the proof of Lemma 2.1, we get that (2.2) holds. It follows from Lemma 3.1 that there exist two possible cases $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$. From the definition of $z(t)$, we see that

$$x(t) = \frac{1}{p(\tau^{-1}(t))}\left(z\left(\tau^{-1}(t)\right) - x\left(\tau^{-1}(t)\right)\right).$$

By repeating the same process, we find that

$$\begin{aligned} x(t) &= \frac{z\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)} - \frac{1}{p\left(\tau^{-1}(t)\right)}\left(\frac{z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)} - \frac{x\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right) \\ &\geq \frac{z\left(\tau^{-1}(t)\right)}{p\left(\tau^{-1}(t)\right)} - \frac{1}{p\left(\tau^{-1}(t)\right)}\frac{z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}. \end{aligned} \tag{4.4}$$

Assume that Case $(\mathbf{C}_1)$ holds. Proceeding as in the proof of Lemma 3.2, we get that (3.5) holds, which with the fact that $\tau(t) \leq t$ gives

$$z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \left(\frac{\tau^{-1}\left(\tau^{-1}(t)\right)}{\tau^{-1}(t)}\right)^3 z\left(\tau^{-1}(t)\right). \tag{4.5}$$

From (4.4) and (4.5), we find that

$$x(t) \geq \frac{1}{p\left(\tau^{-1}(t)\right)}\left(1 - \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)^3}{\left(\tau^{-1}(t)\right)^3 p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right) z\left(\tau^{-1}(t)\right). \tag{4.6}$$

Assume that Case $(\mathbf{C}_2)$ holds. Proceeding as in the proof of $(\mathbf{C}_2)$ in Lemma 3.2, we get that (3.9) holds. Since $\tau^{-1}(t) \leq \tau^{-1}\left(\tau^{-1}(t)\right)$, we obtain

$$\tau^{-1}(t)\, z\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right) \leq \tau^{-1}\left(\tau^{-1}(t)\right) z\left(\tau^{-1}(t)\right). \tag{4.7}$$

From (4.4) and (4.7), we find

$$x(t) \geq \frac{1}{p\left(\tau^{-1}(t)\right)}\left(1 - \frac{\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}{\left(\tau^{-1}(t)\right) p\left(\tau^{-1}\left(\tau^{-1}(t)\right)\right)}\right) z\left(\tau^{-1}(t)\right). \tag{4.8}$$

Next, from (4.6) and (4.8), we get that

$$x(t) \geq \widetilde{p}(t)\, z\left(\tau^{-1}(t)\right),$$

which with (1.1) yields (4.2). Therefore, the proof is complete. □

LEMMA 4.2. *Assume that $\sigma(t) \leq \tau(t)$, $x$ is an eventually positive solution of (1.1) and the functions $\omega$ and $w$ are defined as in (3.1).*
$(\mathbf{I}_3)$ *If $x$ satisfies $(\mathbf{C}_1)$, then*

$$\omega'(t) + Q_3(t) + R_1(t)\,\omega^{\frac{\alpha+1}{\alpha}}(t) \leq 0;$$

$(\mathbf{I}_4)$ *If $x$ satisfies $(\mathbf{C}_2)$, then*

$$w'(t) + Q_4(t) + w^2(t) \leq 0,$$

*where*

$$Q_3(t) = q(t)\,\widetilde{p}^{\beta}(\sigma(t))\, M_3^{\beta-\alpha}\left(\frac{\tau^{-1}(\sigma(t))}{t}\right)^{3\alpha}$$

*and*

$$Q_4(t) = \widetilde{p}^{\beta/\alpha}(\sigma(s))\, M_4^{(\beta/\alpha)-1}\int_t^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty} q(s)\left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^{\beta} \mathrm{d}s\right)^{1/\alpha} \mathrm{d}u.$$

PROOF. Assume that $x$ be an eventually positive solution of (1.1). Then, there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Using Lemma 4.1, we obtain that (4.2) holds.
In the case $(\mathbf{C}_1)$, by differentiating $\omega$ and using (4.2), we obtain

$$\omega'(t) \leq -\frac{q(t)\,\widetilde{p}^{\beta}(\sigma(t))\,z^{\beta}\left(\tau^{-1}(\sigma(t))\right)}{z^{\alpha}(t)} - \alpha\frac{r(t)\,(z'''(t))^{\alpha}}{z^{\alpha+1}(t)}z'(t). \tag{4.9}$$

From Lemma 1.1, we have that

$$z(t) \geq \frac{t}{3}z'(t) \text{ and hence } \frac{z\left(\tau^{-1}(\sigma(t))\right)}{z(t)} \geq \frac{\left(\tau^{-1}(\sigma(t))\right)^{3}}{t^{3}}. \tag{4.10}$$

It follows from Lemma 1.2 that

$$z'(t) \geq \frac{\mu_1}{2}t^2 z'''(t), \tag{4.11}$$

for all $\mu_1 \in (0,1)$ and every sufficiently large $t$. Since $z'(t) > 0$, there exist a $t_2 \geq t_1$ and a constant $M > 0$ such that

$$z(t) > M, \tag{4.12}$$

for $t \geq t_2$. Thus, by (4.9), (4.10), (4.11) and (4.12), we get

$$\omega'(t) + Q_3(t) + R_1(t)\,\omega^{\frac{\alpha+1}{\alpha}}(t) \leq 0.$$

In the case $(\mathbf{C}_2)$, integrating (4.2) from $t$ to $u$, we obtain

$$r(u)\,(z'''(u))^{\alpha} - r(t)\,(z'''(t))^{\alpha} \leq -\int_t^u q(s)\,\widetilde{p}^{\beta}(\sigma(s))\,z^{\beta}\left(\tau^{-1}(\sigma(s))\right)\mathrm{d}s \leq 0. \tag{4.13}$$

From Lemma 1.1, we get that

$$z(t) \geq tz'(t) \text{ and hence } z\left(\tau^{-1}(\sigma(t))\right) \geq \frac{\tau^{-1}(\sigma(t))}{t}z(t). \tag{4.14}$$

For (4.13), letting $u \to \infty$ and using (4.14), we see that

$$r(t)\,(z'''(t))^{\alpha} \geq \widetilde{p}^{\beta}(\sigma(s))\,z^{\beta}(t)\int_t^{\infty} q(s)\left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^{\beta}\mathrm{d}s.$$

Integrating this inequality again from $t$ to $\infty$, we get

$$z''(t) \leq -\widetilde{p}^{\beta/\alpha}(\sigma(s))\,z^{\beta/\alpha}(t)\int_t^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty} q(s)\left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^{\beta}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u, \tag{4.15}$$

for all $\mu_2 \in (0,1)$. By differentiating $w$ and using (3.7) and (4.15), we find

$$\begin{aligned} w'(t) &= \frac{z''(t)}{z(t)} - \left(\frac{z'(t)}{z(t)}\right)^2 \\ &\leq -w^2(t) - \widetilde{p}^{\beta/\alpha}(\sigma(s))\,M^{(\beta/\alpha)-1}\int_t^{\infty}\left(\frac{1}{r(u)}\int_u^{\infty} q(s)\left(\frac{\tau^{-1}(\sigma(s))}{s}\right)^{\beta}\mathrm{d}s\right)^{1/\alpha}\mathrm{d}u, \end{aligned} \tag{4.16}$$

hence

$$w'(t) + Q_4(t) + w^2(t) \leq 0.$$

The proof is complete. □

THEOREM 4.1. *Assume that*

$$\liminf_{t\to\infty}\frac{1}{\widetilde{Q}_3(t)}\int_t^\infty R_1(s)\,\widetilde{Q}_3^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s > \frac{\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}} \tag{4.17}$$

*and*

$$\liminf_{t\to\infty}\frac{1}{\widetilde{Q}_4(t)}\int_t^\infty \widetilde{Q}_4^2(s)\,\mathrm{d}s > \frac{1}{4}, \tag{4.18}$$

*where*

$$\widetilde{Q}_3(t) = \int_t^\infty Q_3(s)\,\mathrm{d}s \quad \text{and} \quad \widetilde{Q}_4(t) = \int_t^\infty Q_4(s)\,\mathrm{d}s.$$

*Then, (1.1) is oscillatory.*

PROOF. Proceeding as in the proof of Theorem 3.1, □

EXAMPLE 4.1. *Consider the differential equation*

$$\left(x(t) + 16x\left(\frac{t}{2}\right)\right)^{(4)} + \frac{q_0}{t^4}x\left(\frac{t}{6}\right) = 0, \tag{4.19}$$

*We note that* $\alpha = \beta = 1$, $r(t) = 1$, $p(t) = 16$, $\tau(t) = t/2$, $\sigma(t) = t/6$ *and* $q(t) = q_0/t^4$. *Hence, it is easy to see that*

$$\widetilde{Q}_3(t) = \frac{q_0}{3^4(32)\,t^3}$$

*and*

$$\widetilde{Q}_4(t) = \frac{7q_0}{3^2(256)\,t}.$$

*Using conditions (4.17) and (4.18), we see that equation (4.19) is oscillatory if* $q_0 >$ 3888.

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