EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 2, 2020, 303-313
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

McShane Integrability Using Variational Measure

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Abstract. If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then f is McShane integrable on every Lebesgue measurable subset of $[a, b]$. However, integrability of a real-valued function on $[a, b]$ does not imply McShane integrability on any $E \subseteq [a, b]$. In this paper, we give a characterization for the McShane integrability of $f : [a, b] \to \mathbb{R}$ over $E \subseteq [a, b]$ using concept of variational measure.

2020 Mathematics Subject Classifications: 26A39, 26A42

Key Words and Phrases: McShane integral, integrable set, McShane δ-variation, McShane variational measure, variation zero, Cauchy extension.

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1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be a function. It is well-known that f is McShane integrable on $[a, b]$ if and only if f is Lebesgue integrable on $[a, b]$ and the values of the integrals are equal, see [5, Theorem 10.11]. If f is McShane integrable on $[a, b]$, then f is Henstock-Kurzweil integrable on $[a, b]$ (with the same value of integrals) but the converse is not true, as seen in Example 1 below.

If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then f is McShane integrable on any sub-interval $[c, d]$ of $[a, b]$, see [5, Theorem 10.4], [6], [7] and [10]. It is shown in [8] that $f : [a, b] \to \mathbb{R}$ is Henstock-Kurzweil integrable on $[a, b]$ if and only if for each $c \in (a, b)$ the function $f \cdot \chi_{[a,c]}$ is Henstock-Kurzweil integrable on $[a, c]$ and

$$\lim_{c \to b^-} \int_a^c f \text{ exists.}$$

In this case,

$$\lim_{c \to b^-} \int_a^c f = \int_a^b f.$$

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DOI: https://doi.org/10.29020/nybg.ejpam.v13i2.3659

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This is known as the *Cauchy Extension*. Example 1 also shows that Cauchy extension does not hold for McShane integral. Indeed, f is McShane integrable on $[\varepsilon, 1]$ for every $0 < \varepsilon < 1$ and

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} f = \lim_{\varepsilon \to 0^+} (F(1) - F(\varepsilon)) = \sin 1,$$

but f is not McShane integrable on $[0, 1]$.

Example 1. Consider the function $F : [0, 1] \to \mathbb{R}$ defined as follows:

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{, if } x \neq 0; \\ 0 & \text{, if } x = 0. \end{cases}$$

Let $f : [0, 1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} F'(x) & \text{, if } 0 < x \leq 1; \\ 0 & \text{, if } x = 0, \end{cases}$$

where

$$F'(x) = 2x \sin \tfrac{1}{x^2} - \tfrac{2}{x} \cos \tfrac{1}{x^2}, \quad \text{for } 0 < x \leq 1.$$

We observe that F is continuous on $[0, 1]$. Now, we show that F is not of bounded variation on $[0, 1]$; that is, $V(F; [0, 1]) = \infty$, where

$$V(F; [0, 1]) = \sup_{D \in \mathcal{D}} (D) \sum |F(v) - F(u)|$$

and $\mathcal{D}$ is the class of all partition $D = \{[u, v]\}$ of $[0, 1]$.

Note that $F(x) = \pm x^2$ if and only if $x^2 = \frac{2}{2n\pi \pm \pi}$, where n is a positive integer. For each $n \in \mathbb{N}$, let $x_n = \sqrt{\frac{2}{2n\pi + \pi}}$. Then for each $n \in \mathbb{N}$

$$F(x_n) = \frac{2 \cdot (-1)^n}{2n\pi + \pi}.$$

Thus,

$$\begin{aligned} \sum_{n=0}^{\infty} |F(x_n) - F(x_{n+1})| &= \sum_{n=0}^{\infty} \left| \frac{2 \cdot (-1)^n}{2n\pi + \pi} - \frac{2 \cdot (-1)^{n+1}}{2n\pi + 3\pi} \right| = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{4n + 4}{4n^2 + 8n + 3} \\ &\geq \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n + 1} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} = \infty \end{aligned}$$

that is, $\sum_{n=0}^{\infty} |F(x_n) - F(x_{n+1})| = \infty$. Hence,

$$V(F; [0, 1]) \geq \sum_{n=0}^{\infty} |F(x_n) - F(x_{n+1})|,$$

and so,

$$V(F;[0,1]) = \infty.$$

Thus, F is not of bounded variation. Recall that in [4, p.19], if $f:[a,b]\to\mathbb{R}$ is McShane integrable on $[a,b]$, then the primitive of f is of bounded variation. Hence, f is not McShane integrable on $[0,1]$. Moreover, for each $\varepsilon\in(0,1)$, f is continuous on $[\varepsilon,1]$ which implies that f is McShane integrable on every closed interval $[\varepsilon,1]$. However, by Theorem 16 of [11], f is Henstock integrable on $[0,1]$. □

If f is Henstock-Kurzweil integrable on every measurable subset of $[a,b]$, then f is Lebesgue integrable on $[a,b]$, see [5, Theorem 9.13]. In particular, if f is McShane integrable on every measurable subset of $[a,b]$, then f is McShane integrable on $[a,b]$.

It was pointed out in [3] that if $f:[a,b]\to\mathbb{R}$ is McShane integrable on $[a,b]$, then f is McShane integrable on $[a,b]$ on every integrable (equivalently, Lebesgue measurable) subset of $[a,b]$. If f is Henstock-Kurzweil integrable on $[a,b]$, then $[a,b]$ contains a subinterval on which f is Lebesgue (McShane) integrable, see [5, Corollary 9.19] and [8].

Thus, a natural problem is:

> *If $f:[a,b]\to\mathbb{R}$ is McShane integrable on $[a,b]$ and $X\subseteq[a,b]$, find a condition satisfied by X which is necessary and sufficient for the McShane integrability of f on X.*

In this paper, we give a characterization for the McShane integrability of $f:[a,b]\to\mathbb{R}$ on $X\subseteq[a,b]$ using concept of McShane variational measure.

2. Preliminary Concepts and Known Results

A *gauge* on $[a,b]$ is a positive function $\delta:[a,b]\to\mathbb{R}^+$. A *Henstock δ-fine division* of $[a,b]$ is a finite collection $D=\{([x_{i-1},x_i],\xi_i)\}_{i=1}^n$ of non-overlapping interval-point pairs such that for all $i=1,2,\ldots,n$

$$\xi_i\in[x_{i-1},x_i]\subseteq(\xi_i-\delta(\xi_i),\xi_i+\delta(\xi_i)) \text{ and } \bigcup_{i=1}^n[x_{i-1},x_i]=[a,b].$$

We say that $D=\{([x_{i-1},x_i],\xi_i)\}_{i=1}^n$ is a *McShane δ-fine division* of $[a,b]$ if for all $i=1,2,\ldots,n$

$$[x_{i-1},x_i]\subseteq(\xi_i-\delta(\xi_i),\xi_i+\delta(\xi_i)),\ \xi_i\in[a,b],\quad\text{and}\quad\bigcup_{i=1}^n[x_{i-1},x_i]=[a,b].$$

This means that every Henstock δ-fine divisions of $[a,b]$ are McShane δ-fine. For brevity, we use $([u,v],\xi)$ to represent a typical interval-point pair $([x_{i-1},x_i],\xi_i)\in D$. A finite collection $P=\{([u,v],\xi)\}$ of interval-point pairs is a *partial division* of $[a,b]$ if $\cup[u,v]\subseteq[a,b]$. Interested readers may refer to [5–7, 10] for more details on the basic concepts introduced.

Lemma 1 (Cousin's Lemma)**.** [6] *If δ is a gauge on $[a, b]$, then there exists a δ-fine division of $[a, b]$.*

Definition 1. [5–7, 10] A function $f : [a, b] \to \mathbb{R}$ is said to be *McShane integrable* to a real number A on $[a, b]$ if for any $\varepsilon > 0$, there exists a gauge $\delta : [a, b] \to \mathbb{R}^+$ such that for any McShane δ-fine division $D = \{([u, v], \xi)\}$ of $[a, b]$, we have

$$\left|(D)\sum f(\xi)(v-u) - A\right| < \varepsilon.$$

If $f : [a, b] \to \mathbb{R}$ is McShane integrable to A on $[a, b]$, then we write

$$A = \int_a^b f.$$

For $E \subseteq [a, b]$, we say that $f : [a, b] \to \mathbb{R}$ is *McShane integrable on E* if $f \cdot \chi_E$ is McShane integrable on $[a, b]$, where χ_E is the characteristic function of E and write

$$\int_E f = \int_a^b (f \cdot \chi_E).$$

The next theorem tells us that the McShane integral is an absolute integral.

Theorem 1. [7] *If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then so is $|f|$.*

If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f, \quad \text{for all } x \in [a, b]$$

is called the *primitive* of f. Almost all significant results in the Henstock Integration Theory rely on the following very important Theorem involving the primitive, called the Henstock's Lemma.

Theorem 2 (Henstock's Lemma)**.** [6] *If $f : [a, b] \to \mathbb{R}$ is McShane (resp., Henstock) integrable on $[a, b]$ with primitive F, then for each $\varepsilon > 0$, there exist $\delta : [a, b] \to \mathbb{R}^+$ such that whenever $D = \{([u, v], \xi)\}$ is a McShane (resp., Henstock) δ-fine division of $[a, b]$, we have*

$$(D)\sum \left|f(\xi)(v-u) - F(v) + F(u)\right| < \varepsilon.$$

The following concept was introduced by Yang [12].

Definition 2. A set $E \subseteq \mathbb{R}$ is *integrable* if $\chi_{E\cap[a,b]}$ is McShane integrable on $[a, b]$, for all $[a, b] \subseteq \mathbb{R}$.

Clearly, $\varnothing$ and $\mathbb{R}$ are integrable subsets of $\mathbb{R}$. Moreover, it can be seen that the collection $\mathcal{I}$ of all integrable subsets of $\mathbb{R}$ is a σ-algebra. In [9], Quindala and Benitez showed that the set-function $\mu : \mathcal{I} \to [0, \infty]$ defined by

$$\mu(E) = \int_{-\infty}^{\infty} \chi_E \text{ , for all } E \in \mathcal{I}$$

is a measure. In the same paper, for a bounded integrable set $E \subseteq \mathbb{R}$

$$\mu(E) = m^*(E)$$

where m^* is the Lebesgue outer measure.

See [1, p.85] for the proof of the following Lemma which utilizes the Heine-Borel Covering Theorem.

Lemma 2. [1] *If* $E \subseteq [a, b]$ *and* χ_E *is McShane integrable on* $[a, b]$*, then for all* $\varepsilon > 0$ *there exists an open set* $O \subseteq [a, b]$ *such that* $E \subseteq O$ *and*

$$\int_a^b \chi_{O \setminus E} < \varepsilon.$$

The following is a characterization of an integrable set $E \subseteq [a, b]$.

Lemma 3. [12] *Let* $E \subseteq [a, b]$*. The following statements are equivalent:*

(i) χ_E *is McShane integrable on* $[a, b]$.

(ii) E *is integrable.*

Definition 3. An integrable set $E \subseteq [a, b]$ is said to have *variation zero* if

$$\int_a^b \chi_E = 0.$$

It is worth noting that a subset of a set of variation zero is again of variation zero.

Definition 4. A property is said to hold *almost everywhere* (abbreviated *a.e.*) on A if the set of points in A where it fails to hold is an integrable set of variation zero.

It was proved in [9] that the notions of McShane integrable set and Lebesgue measurable set are equivalent. This implies that integrable sets of variation zero are exactly those subsets of $[a, b]$ with zero Lebesgue measure. Furthermore, the concept of "almost everywhere" in the sense of McShane, introduced in Definition 4, coincides with the corresponding concept from Lebesgue theory. This equivalences do not diminish the relevance of the notion of McShane integrable set.

If functions possess certain properties almost everywhere, then some properties of the Henstock integral are preserved. In particular, if two functions are equal a.e. and one of the functions is McShane integrable, then the other function is also McShane integrable and their integral values coincide. This is precisely stated in Theorem 3. The proof of this result is standard and one may follow the proof of Theorem 9.5 in [5] or Theorem 10 in [11].

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be McShane integrable on $[a, b]$. If $g = f$ a.e. on $[a, b]$, then g is McShane integrable on $[a, b]$, and*

$$\int_a^b g(x)\ dx = \int_a^b f(x)\ dx.$$

In view of Theorem 3, the condition "$f_n(x) \to f(x)$ on $[a, b]$" in all convergence theorems for the McShane integral (see [5]) can now be replaced by "$f_n(x) \to f(x)$ *a.e.* on $[a, b]$".

Theorem 4. [4] *If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of step functions such that $\varphi_n \to f$ a.e. on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b |\varphi_n - f| = 0.$$

In the following, if $f : [a, b] \to \mathbb{R}$ is a function and $c \in \mathbb{R}$, then we denote

$$E(f < c) = \{x \in [a, b] : f(x) < c\}.$$

Theorem 5. [2] *Let c be any real number. If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then the characteristic function $\chi_{E(f<c)}$ is McShane integrable on $[a, b]$.*

Since

$$E(f \geq c) = \{x \in [a, b] : f(x) \geq c\} = [a, b] \setminus \{x \in [a, b] : f(x) < c\} = [a, b] \setminus E(f < c),$$

$\chi_{E(f\geq c)}$ is also McShane integrable. Similarly, $\chi_{E(f\leq c)}$ is McShane integrable.

Furthermore, if a set A is a countable union of sets of the form:

$$E(f < c),\ E(f \geq c),\ E(f > c),\ \text{ or }\ E(f \leq c)$$

then χ_A is McShane integrable on $[a, b]$.

Hence, by Lemma 3 and Theorem 5, we have the following result:

Corollary 1. *Let c be any real number. If $f : [a, b] \to \mathbb{R}$ is McShane integrable on $[a, b]$, then the sets $E(f < c)$, $E(f \geq c)$, $E(f > c)$ and $E(f \leq c)$ are integrable sets.*

Below is a version of the Monotone Convergence Theorem (MCT) for McShane integrals.

Theorem 6 (Monotone Convergence Theorem)**.** [1, 5] *Let $\{f_n\}_{n=1}^{\infty}$ be an increasing sequence of McShane integrable functions on $[a, b]$ such that*

$$\lim_{n\to\infty} f_n(x) = f(x)\ \text{, for each } x \in [a, b].$$

If $\sup\left\{ \int_a^b f_n : n \in \mathbb{N}\right\} < \infty$, *then* f *is McShane integrable on* $[a,b]$ *and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Recall that if $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$, then f is McShane integrable on every sub-interval $[c,d]$ of $[a,b]$. However, McShane integrability on $[a,b]$ does not imply McShane integrability on any $E \subseteq [a,b]$. One necessary condition is that a subset $E \subseteq [a,b]$ must be an integrable set.

Theorem 7. [3] *If* $f : [a,b] \to \mathbb{R}$ *is McShane integrable on* $[a,b]$, *then* f *is McShane integrable on every integrable subset* E *of* $[a,b]$.

3. Results

In what follows, we denote the family of all sub-intervals of $[a,b]$ by $\mathcal{I}([a,b])$ and δ is a gauge on $[a,b]$.

Definition 5. [8] Let $F : \mathcal{I}([a,b]) \to \mathbb{R}$. For any subset X of $[a,b]$, the *McShane* δ*-variation* of F on X is given by

$$V(F, X, \delta) = \sup_{P\in\mathcal{P}([a,b])} (P)\sum |F(u,v)|,$$

where $\mathcal{P}([a,b])$ is the collection of all McShane δ-fine partial division $P = \{([u,v],\xi)\}$ of $[a,b]$ with $\xi \in X$. The *McShane variational measure* of F on X is given by

$$V_M F(X) = \inf\left\{V(F, X, \delta) : \delta \text{ is a gauge on } X\right\}.$$

If $f : [a,b] \to \mathbb{R}$ is McShane integrable on $[a,b]$ with primitive F, then we write

$$F(u,v) = F(v) - F(u),$$

for any $u \leq v$ in $[a,b]$.

Lemma 4. *Let* $f : [a,b] \to \mathbb{R}$ *be McShane integrable on* $[a,b]$ *with primitive* F *and* $X \subseteq [a,b]$. *If* f *is McShane integrable on* X, *then* $V_M F(X) < \infty$ *and*

$$\int_a^b |f \cdot \chi_X| = V_M F(X).$$

Proof. Let $\varepsilon > 0$. By Henstock Lemma, there exists a gauge δ_1 on $[a,b]$ such that

$$(P)\sum |F(u,v) - f(\xi)(v-u)| < \frac{\varepsilon}{3}$$

whenever $P = \{([u,v],\xi)\}$ is a McShane δ_1-fine partial division of $[a,b]$.

Note that if f is McShane integrable on $X \subseteq [a,b]$, then $f \cdot \chi_X$ is McShane integrable on $[a,b]$. By Theorem 1, $|f \cdot \chi_X|$ is McShane integrable on $[a,b]$. Again by Henstock Lemma, there exists a gauge δ_2 on $[a,b]$ such that

$$(P)\sum \left| |(f \cdot \chi_X)(\xi)|(v-u) - \int_u^v |f \cdot \chi_X| \right| < \frac{\varepsilon}{3}$$

whenever $P = \{([u,v],\xi)\}$ is a McShane δ_2-fine partial division of $[a,b]$.

Let $\delta_3 = \min\{\delta_1, \delta_2\}$ and $P = \{([u,v],\xi)\}$ be a McShane δ_3-fine partial division of $[a,b]$ such that $\xi \in X$. Then

$$\begin{aligned}
(P)\sum |F(u,v)| &\leq (P)\sum |F(u,v) - f(\xi)(v-u)| + (P)\sum |f(\xi)(v-u)| \\
&< \frac{\varepsilon}{3} + (P)\sum |f(\xi)(v-u)| \\
&\leq \frac{\varepsilon}{3} + (P)\sum \left| |f(\xi) \cdot \chi_X(\xi)|(v-u) - \int_u^v |f \cdot \chi_X| \right| \\
&\qquad + (P)\sum \int_u^v |f \cdot \chi_X| \\
&< \frac{2\varepsilon}{3} + (P)\sum \int_u^v |f \cdot \chi_X| \\
&\leq \frac{2\varepsilon}{3} + \int_a^b |f \cdot \chi_X|.
\end{aligned}$$

This implies that

$$V_M F(X) \leq V(F, X, \delta_3) \leq \frac{2\varepsilon}{3} + \int_a^b |f \cdot \chi_X|.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$V_M F(X) \leq \int_a^b |f \cdot \chi_X|. \tag{1}$$

By definition of $V_M F(X)$, there exists $\delta_4(\xi) > 0$ such that

$$V(F, X, \delta_4) \leq V_M F(X) + \frac{\varepsilon}{3}.$$

Let $\delta_0(\xi) = \min\{\delta_3(\xi), \delta_4(\xi)\}$, for all $\xi \in [a,b]$. Let $P = \{([u,v],\xi)\}$ be any McShane δ_0-fine partial division of $[a,b]$ with $\xi \in X$. Suppose $D = P \cup P'$ is a McShane δ_0-fine division of $[a,b]$, where $P' = \{([u,v],\xi)\}$ such that $\xi \notin X$. Then

$$(D)\sum |f(\xi) \cdot \chi_X(\xi)|(v-u) = (P)\sum |f(\xi) \cdot \chi_X(\xi)|(v-u).$$

Note that D is also a McShane δ_0-fine partial division of $[a,b]$. Thus,

$$\int_a^b |f \cdot \chi_X| \leq (D)\sum \left| \int_u^v |f \cdot \chi_X| - |f(\xi) \cdot \chi_X(\xi)|(v-u) \right| + (D)\sum |f(\xi) \cdot \chi_X(\xi)|(v-u)$$

$$
\begin{aligned}
&\leq (D)\sum \left| \int_u^v |f\cdot\chi_X| - |f(\xi)\cdot\chi_X(\xi)|(v-u)\right| + (P)\sum |f(\xi)\cdot\chi_X(\xi)|(v-u)\\
&< \frac{\varepsilon}{3} + (P)\sum |f(\xi)\cdot\chi_X(\xi)|(v-u)\\
&= \frac{\varepsilon}{3} + (P)\sum |f(\xi)|(v-u)\\
&\leq \frac{\varepsilon}{3} + (P)\sum \big|f(\xi)(v-u) - F(u,v)\big| + (P)\sum \big|F(u,v)\big|\\
&< \frac{2\varepsilon}{3} + (P)\sum \big|F(u,v)\big|\\
&\leq \frac{2\varepsilon}{3} + V(F,X,\delta_0).
\end{aligned}
$$

Hence,

$$\int_a^b |f\cdot\chi_X| \leq \varepsilon + V_M F(X).$$

Again, since $\varepsilon > 0$ is arbitrary, we have

$$\int_a^b |f\cdot\chi_X| \leq V_M F(X). \tag{2}$$

Combining (1) and (2), we have

$$\int_a^b |f\cdot\chi_X| = V_M F(X).$$ □

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be McShane integrable on $[a,b]$ with primitive F and $X \subseteq [a,b]$. Then f is McShane integrable on X if and only if $V_M F(X) < \infty$. In this case,*

$$\int_a^b |f\cdot\chi_X| = V_M F(X).$$

Proof. Necessity follows from Lemma 4. Conversely, assume that

$$V_M F(X) < \infty.$$

For each $n \in \mathbb{N}$, we let

$$X_n = \{x \in X : |f(x)| \leq n\}.$$

Then each X_n is an integrable set, $X_n \subseteq X_{n+1}$ for all n and

$$X = \bigcup_{n=1}^{\infty} X_n.$$

By Theorem 7, f is McShane integrable on X_n for each n. By Lemma 4, $V_M F(X_n) < \infty$ and

$$\int_a^b |f\cdot\chi_{X_n}| = V_M F(X_n).$$

Consider the sequence $\{f_n\}_{n=1}^{\infty}$, defined by $f_n = |f \cdot \chi_{X_n}|$ for all $n \in \mathbb{N}$. Since $X_n \subseteq X_{n+1}$ for each n, $\{f_n\}_{n=1}^{\infty}$ is an increasing sequence of McShane integrable functions on $[a,b]$ and $f_n \to |f \cdot \chi_X|$ pointwisely on $[a,b]$. Note that

$$\sup\left\{\int_a^b |f \cdot \chi_{X_n}| : n \in \mathbb{N}\right\} = \sup\Big\{V_M F(X_n) : n \in \mathbb{N}\Big\} \leq V_M F(X) < \infty.$$

By Theorem 6, $|f \cdot \chi_X|$ is McShane integrable on $[a,b]$ and

$$\int_a^b |f \cdot \chi_X| = \lim_{n\to\infty} \int_a^b |f \cdot \chi_{X_n}| = \lim_{n\to\infty} V_M F(X_n) = V_M F(X). \qquad \square$$

4. Conclusion

Let $f : [a,b] \to \mathbb{R}$ to be McShane (Lebesgue) integrable on $[a,b]$ with primitive F. It is shown in this paper that for f to be McShane integrable on $X \subseteq [a,b]$, a necessary and sufficient condition is that the primitive must be of finite McShane variational measure on X, that is, $V_M F(X) < \infty$. The authors recommend that interested readers may investigate this study for non-absolute integrals (since McShane integrals are absolute).

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