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# On Intuitionistic Fuzzy Hyper GR-ideals in Hyper GR-algebras

Amila P. Macodi-Ringia[1,*], Gaudencio C. Petalcorin, Jr.[2]

[1] *Mathematics Department, College of Natural Sciences and Mathematics, MSU-General Santos, General Santos City, Philippines*
[2] *Department of Mathematics and Statistics, College of Science and Mathematics, MSU-Iligan Institute of Technology, Iligan City, Philippines*

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**Abstract.** In this paper, fuzzy set and intuitionistic fuzzy set are applied to hyper GR-algebra. Particularly, the fuzzy hyper GR-ideal of type 1 and the intuitionistic fuzzy hyper GR-ideal are introduced, and a relationship between them are obtained. Moreover, some of their characterizations are established by the use of their level subsets.

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## 1. Introduction

In 1934, hyperstructure theory was introduced in 1934 by F. Marty [13] during the 8*th* Congress of Scandinavian Mathematicians. Around the 40's, several authors worked on hypergroups, especially, in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing hyperstructures has been seen. Hyperstructures have many application to several sectors of both pure and applied sciences. Davvaz et al. [5] applied this concept to elementary particles in physical theory. While, Xin [11] applied this concept to BCI-algebras and proved that hyper BCI-algebras are one of the generalizations of BCI-algebras. After the introduction on the concept of hyper BCI-algebras, several researches were conducted. One of these studies is the hyper GR-algebras. In 2016, Indangan et al. [6] introduced hyper GR-algebras and the faithful hyper GR-algebra's hyper operation properties were established including some properties of hyper GR-ideals. In 2017, some hyper homomorphic properties on hyper

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*Corresponding author.
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*Email addresses:* amila.macodi-ringia@g.msuiit.edu.ph (A. Macodi-Ringia),
gaudencio.petalcorin@g.msuiit.edu.ph (G. Petalcorin, Jr.)

 

GR-algebra together with the construction of the quotient hyper GR-algebra via regular congruence relation were presented by Indangan et al. [7].

Uncertainty is an attribute of information and uncertain data are presented in various domains. The most appropriate theory for dealing with uncertainties is the theory of fuzzy sets developed by Zadeh [16] in 1965. It has been well developed in the context of hyperstructure theory. Several studies were fuzzy theory is applied to hyperstructure are fuzzy hyper BCK-ideals of hyper BCK-algebras [8], fuzzy ideals in hyper BCI-algebras [14], fuzzy implicative hyper BCK-ideals of hyper BCK-algebras [9], and some results on fuzzy implicative hyper GR-ideals [12]. These studies obtained some characterizations where level subsets of fuzzy set are being used. On the contrary, fuzzy set theory has no means to incorporate the hesitation or uncertainty in the membership degrees. Atanassov [1, 2] introduced the concept of intuitionistic fuzzy sets in a non-empty set $X$ which give both a membership degree and a non-membership degree. Since then, the notion of intuitionistic fuzzy set has been explored by researchers and a number of theoritical and practical results have appeared. The relations between intuitionistic fuzzy sets and algebraic hyperstructures have been already considered by many mathematicians. Some of these studies are intuitionistic fuzzy hyper BCK-ideals of hyper BCK-algebras [3] and intuitionistic fuzzy ideals in hyper BCI-algebras [15]. Both of these researches used level subsets of intuitionistic fuzzy set to establish some characterizations of intuitionistic fuzzy hyper BCK-ideals and intuitionistic fuzzy hyper BCI-ideals.

In this paper, we intoduce fuzzy hyper GR-ideals of type 1 and intuitionistic fuzzy hyper GR-ideals including some of thier properties and characterizations by following the works of Borzooei et al. [3], Jun et al. [8, 9], Nisar et al. [14], and Palaniappan et al. [15].

## 2. Preliminaries

Let $H$ be a nonempty set with a hyperoperation "$\circledast$". For any two subsets $A$ and $B$ of $H$ and $x \in H$, we define

$$A \circledast B = \bigcup_{a \in A, b \in B} a \circledast b, A \circledast x = A \circledast \{x\}, \text{ and } x \circledast B = \{x\} \circledast B.$$

Moreover, $x \ll y$ is defined by $0 \in x \circledast y$ and $A \ll B$ is defined by for all $a \in A$, there exist $b \in B$ such that $a \ll b$. The symbol "$\ll$"is called a hyperorder on $H$.

**Definition 2.1.** [10] Let $H$ be a nonempty set endowed with a hyperoperation $\circledast$ and a constant $0$. Then $(H, \circledast, 0)$ is called a hyper BCK-algebra if it satisfies the following axioms, for all $x, y, z \in H$:

**(i)** $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

**(ii)** $(x \circledast y) \circledast z = (x \circledast z) \circledast y$;

**(iii)** $x \circledast H \ll \{x\}$; and

**(iv)** $x \ll y$ and $y \ll x$ imply $x = y$.

**Definition 2.2.** [11] Let $H$ be a nonempty set and $\circledast$ be a hyperoperation on $H$. Then $(H, \circledast, 0)$ is called a hyper BCI-algebra if it contains a constant $0 \in H$ and satisfies the following axioms, for all $x, y, z \in H$:

**(i)** $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$;

**(ii)** $(x \circledast y) \circledast z = (x \circledast z) \circledast y$;

**(iii)** $x \ll x$;

**(iv)** $x \ll y$ and $y \ll x$ imply $x = y$; and

**(v)** $0 \circledast (0 \circledast x) \ll x, x \neq 0$;

**Definition 2.3.** [6] Let $H$ be a nonempty set and $\circledast$ be a hyperoperation on $H$. If $H$ contains a constant 0 and the following axioms

**(HGR1)** $(x \circledast z) \circledast (y \circledast z) \ll x \circledast y$,

**(HGR2)** $(x \circledast y) \circledast z = (x \circledast z) \circledast y$,

**(HGR3)** $x \ll x$,

**(HGR4)** $0 \circledast (0 \circledast x) \ll x, x \neq 0$, and

**(HGR5)** $(x \circledast y) \circledast z \ll y \circledast z$

are satisfied for all $x, y, z \in H$, then $(H, \circledast, 0)$ is said to be a hyper GR-algebra.
For the sake of simplicity, we say $H$ is a hyper GR-algebra.

**Definition 2.4.** [6] A subset $I$ of a hyper GR-algebra $H$ is called a *hyper GR-ideal* of $H$ if it contains 0 and for all $x, y \in H$, $x \circledast y \subseteq I$ and $y \in I$ imply that $x \in I$.

**Definition 2.5.** [16] A *fuzzy set* $\mu$ of a nonempty set $M$ is a function $\mu : M \to [0, 1]$.

**Definition 2.6.** [4] Let $\mu$ be a fuzzy set of $M$. For a fixed $t \in [0, 1]$, the set $\mu_t = \{x \in M | \mu(x) \geq t\}$ is called a *level subset* of $\mu$.

**Definition 2.7.** [1] An *intuitionistic fuzzy set* $A$ in a nonempty set $H$ is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in H\}$ where the function $\mu_A : H \to [0, 1]$ and $\gamma_A : H \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and for all $x \in H$,

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1.$$

Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of $x$ in $A$. $\pi_A(x)$ is the degree of indeterminancy of $x \in H$ to intuitionistic fuzzy set $A$ and $\pi_A(x) \in [0, 1]$. $\pi_A(x)$ expresses the lack of knowledge of whether $x$ belongs to intuitionistic fuzzy set $A$ or not.

We shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in H\}$.

**Definition 2.8.** [15] For an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in $H$ and $s, t \in [0, 1]$, the set $A_{\langle t,s\rangle} = \{x \in H | \mu_A(x) \geq t, \gamma_A(x) \leq s\}$ is called a *level subset* of $A$.

## 3. Fuzzy Hyper GR-ideals of Type 1

**Definition 3.1.** A fuzzy set $\mu$ in a hyper GR-algebra $H$ is a *fuzzy hyper GR-ideal of type* 1 if for all $x, y \in H$,

$$\text{(F1) } \mu(0) \geq \mu(x) \geq \min\left\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\right\}.$$

**Example 3.2.** Let $H = [0, 1]$ such that for any $a, b \in [0, 1]$,

$$a \circledast b = \begin{cases} [0, 0.3], & \text{if } b \neq 0 \text{ or } a = 0 = b; \\ \{a\}, & \text{if } a \neq 0 \text{ and } b = 0. \end{cases}$$

It can be seen that $H$ is a hyper GR-algebra. Define a fuzzy set $\mu$ in $H$ by

$$\mu(a) = \begin{cases} l, & \text{if } a = 0 \\ k, & \text{if } a \neq 0. \end{cases}$$

where $k, l \in [0, 1]$ and $k < l$. By routine calculations, we see that $\mu$ is a fuzzy hyper GR-ideal of type 1 in $H$.

**Example 3.3.** Consider the hyper GR-algebra $H$ in Example 3.2. Define a fuzzy set $\mu$ in $H$ by

$$\mu(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0.3 + x, & \text{if } x \in (0, 0.3], \\ 0.7, & \text{if } x \in (0.3, 1]. \end{cases}$$

It can be shown that $\mu$ is a fuzzy hyper GR-ideal of type 1 in $H$.

**Proposition 3.4.** *Let $H$ be a hyper GR-algebra. If $\mu$ is a fuzzy hyper GR-ideal of type* 1 *in $H$ such that $\inf_{u \in x \circledast y} \mu(u) = \mu(0) = \inf_{u \in y \circledast x} \mu(u)$ for $x \neq y$, then $\mu(x) = \mu(y)$.*

*Proof.* Let $x, y \in H$ such that $x \neq y$ and $\inf_{u \in x \circledast y} \mu(u) = \mu(0) = \inf_{u \in y \circledast x} \mu(u)$. By F1,

$$\mu(x) \geq \min\left\{\inf_{u \in x \circledast y} \mu(u), \mu(y)\right\} = \min\{\mu(0), \mu(y)\} = \mu(y) \text{ and}$$
$$\mu(y) \geq \min\left\{\inf_{u \in y \circledast x} \mu(u), \mu(x)\right\} = \min\{\mu(0), \mu(x)\} = \mu(x).$$

Thus, $\mu(x) = \mu(y)$. □

**Theorem 3.5.** *A fuzzy set $\mu$ in a hyper GR-algebra $H$ is a fuzzy hyper GR-ideal of type 1 if and only if $\mu_t$ is a hyper GR-ideal of $H$ whenever $\mu_t \neq \emptyset$ and $t \in [0,1]$.*

*Proof.* Suppose $\mu$ is a fuzzy hyper GR-ideal of type 1. Let $t \in [0,1]$ such that $\mu_t \neq \emptyset$. Then there exists $a \in \mu_t$. By F1, $\mu(0) \geq \mu(a) \geq t$. Then, $0 \in \mu_t$. Let $x, y \in H$ such that $x \circledast y \subseteq \mu_t$ and $y \in \mu_t$. Then $\mu(y) \geq t$ and $\mu(u) \geq t$ for any $u \in x \circledast y$. This implies that $t$ is a lowerbound of $\{\mu(u) : u \in x \circledast y\}$. Thus, $\inf_{u \in x\circledast y} \mu(u) \geq t$. By F1, $\mu(x) \geq \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\} \geq \min\{t, t\} = t$. Hence, $x \in \mu_t$ and so $\mu_t$ is a hyper GR-ideal of $H$.

Conversely, let $\mu_t$ be a hyper GR-ideal of $H$ for any $t \in [0,1]$. Let $x \in H$ and let $k \in [0,1]$ such that $k = \mu(x)$. Since $0 \in \mu_k$, $\mu(0) \geq k = \mu(x)$. Moreover, let $x, y, z \in H$ and let $l \in [0,1]$ such that $l = \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\}$. Since $\mu(y) \geq \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\} = l$, $y \in \mu_l$. Let $w \in x \circledast y$. Then $\mu(w) \geq \inf_{u\in x\circledast y} \mu(u) \geq \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\} = l$. It follows that $w \in \mu_l$ and so $x \circledast y \subseteq \mu_l$. Since $\mu_l$ is a hyper GR-ideal of $H$, $x \in \mu_l$. It implies that $\mu(x) \geq l = \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\}$. Thus, $\mu$ is a fuzzy hyper GR-ideal of type 1. □

**Corollary 3.6.** *For any nonempty subset $A$ of $H$, let $\mu_A$ be a fuzzy set in hyper GR-algebra $H$ defined by*

$$\mu_A(x) = \begin{cases} n, & \text{if } x \in A, \\ m, & \text{otherwise,} \end{cases}$$

*for all $x \in H$ where $n, m \in [0,1]$ with $n > m$. Then $A$ is a hyper GR-ideal of $H$ if and only if $\mu_A$ is a fuzzy hyper GR-ideal of type 1 in $H$.*

*Proof.* Note that

$$(\mu_A)_t = \begin{cases} \emptyset, & \text{if } n < t \leq 1, \\ A, & \text{if } m < t \leq n, \\ H, & \text{if } 0 \leq t \leq m \end{cases} \tag{1}$$

are all possible level subsets of $\mu_A$ where $t \in [0,1]$. Then, $(\mu_A)_t = A$ is a nonempty level subsets of $\mu_A$. Thus, by Theorem 3.5, $A$ is a hyper GR-ideal of $H$ if and only if $\mu_A$ is a fuzzy hyper GR-ideal of type 1 in $H$. □

**Theorem 3.7.** *If $\mu$ is a fuzzy hyper GR-ideal of type 1 of a hyper GR-algebra $H$, then the set $A = \{x \in H | \mu(x) = \mu(0)\}$ is a hyper GR-ideal of $H$.*

*Proof.* Suppose $\mu$ is a fuzzy hyper GR-ideal of type 1 of a hyper GR-algebra $H$. Let $x, y \in H$ such that $x \circledast y \subseteq A$ and $y \in A$. Then $\mu(y) = \mu(0)$ and $\mu(u) = \mu(0)$ for all $u \in x \circledast y$. By the hypothesis, $\mu(0) \geq \mu(x) \geq \min\left\{\inf_{u\in x\circledast y} \mu(u), \mu(y)\right\} = \mu(0)$. Hence, $\mu(x) = \mu(0)$ and so $x \in A$. Thus, $A$ is a hyper GR-ideal of $H$. □

## 4. Intuitionistic Fuzzy Hyper GR-ideals

**Definition 4.1.** An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a hyper GR-algebra $H$ is an *intuitionistic fuzzy hyper GR-ideal* if for all $x, y \in H$ the following hold:

**(IFGR1)** $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$;

**(IFGR2)** $\mu_A(x) \geq \min\left\{\inf\limits_{u \in x \circledast y} \mu_A(u), \mu_A(y)\right\}$; and

**(IFGR3)** $\gamma_A(x) \leq \max\left\{\sup\limits_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\}$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in H\}$.

**Example 4.2.** Consider the hyper GR-algebra $H$ in Example 3.3 and its fuzzy set $\mu$. Let $A = (\mu_A, \gamma_A)$ in $H$ be an intuitionistic fuzzy set where $\mu_A = \mu$ and

$$\gamma_A(x) = \begin{cases} 0, & \text{if } x = 0, \\ 0.7 - x, & \text{if } x \in (0, 0.3], \\ 0.1, & \text{if } x \in (0.3, 1]. \end{cases}$$

By routine calculations, $A$ is an intuitionistic fuzzy hyper GR-ideal in $H$.

**Theorem 4.3.** *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in a hyper GR-algebra $H$. $A_{\langle t,s \rangle}$ is a hyper GR-ideal of $H$ if and only if $A$ is an intuitionistic fuzzy hyper GR-ideal of $H$ whenever $A_{\langle t,s \rangle} \neq \emptyset$ and $t, s \in [0, 1]$.*

*Proof.* Suppose $A_{\langle t,s \rangle}$ is a hyper GR-ideal of $H$ for any $t, s \in [0, 1]$. Let $x \in H$ and let $k, l \in [0, 1]$ such that $k = \mu_A(x)$ and $l = \gamma_A(x)$. Since $A_{\langle k,l \rangle}$ is a hyper GR-ideal of $H$, $0 \in A_{\langle k,l \rangle}$. Then $\mu_A(0) \geq k = \mu_A(x)$ and $\gamma_A(0) \leq l = \gamma_A(x)$. Moreover, let $x, y \in H$ and let $\tilde{t}, \tilde{s} \in [0, 1]$ such that $\tilde{t} = \min\left\{\inf\limits_{u \in x \circledast y} \mu_A(u), \mu_A(y)\right\}$ and $\tilde{s} = \max\left\{\sup\limits_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\}$. Suppose $w \in x \circledast y$. Then,

$$\mu_A(w) \geq \inf_{u \in x \circledast y} \mu_A(u) \geq \min\left\{\inf_{u \in x \circledast y} \mu_A(u), \mu_A(y)\right\} = \tilde{t} \text{ and}$$

$$\gamma_A(w) \leq \sup_{v \in x \circledast y} \gamma_A(v) \leq \max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\} = \tilde{s}.$$

These imply that $w \in A_{\langle \tilde{t}, \tilde{s} \rangle}$ and so $x \circledast y \subseteq A_{\langle \tilde{t}, \tilde{s} \rangle}$. Note that

$$\mu_A(y) \geq \min\left\{\inf_{u \in x \circledast y} \mu_A(u), \mu_A(y)\right\} = \tilde{t} \text{ and}$$

$$\gamma_A(y) \leq \max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\} = \tilde{s}.$$

Then, $y \in A_{\langle \tilde{t}, \tilde{s} \rangle}$. Since $A_{\langle \tilde{t}, \tilde{s} \rangle}$ is a hyper GR-ideal of $H$, $x \in A_{\langle \tilde{t}, \tilde{s} \rangle}$. It follows that

$$\mu_A(x) \geq \tilde{t} = \min\left\{\inf_{u\in x\circledast y} \mu_A(u), \mu_A(y)\right\} \text{ and}$$
$$\gamma_A(x) \leq \tilde{s} = \max\left\{\sup_{v\in x\circledast y} \gamma_A(v), \gamma_A(y)\right\}.$$

By Definition 4.1, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper GR-ideal in $H$.

Conversely, suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper GR-ideal in $H$. Then, $\mu_A(0) \geq \mu_A(x)$ for all $x \in H$. Let $t, s \in [0, 1]$. Since $A_{\langle t,s\rangle} \neq \emptyset$, there exists $z \in A_{\langle t,s\rangle}$ such that $\mu_A(0) \geq \mu_A(z) \geq t$ and $\gamma_A(0) \leq \gamma_A(z) \leq s$. Thus, $0 \in A_{\langle t,s\rangle}$. Let $x, y \in H$ such that $x \circledast y \subseteq A_{\langle t,s\rangle}$ and $y \in A_{\langle t,s\rangle}$. Then, $\mu_A(y) \geq t$, $\gamma_A(y) \leq s$, $\mu_A(u) \geq t$ and $\gamma_A(v) \leq s$ for any $u, v \in x \circledast y$. It follows that $t$ is a lowerbound for $\{\mu_A(u) : u \in x \circledast y\}$ and $s$ is an upperbound for $\{\gamma_A(v) : v \in x \circledast y\}$. Then, $\inf\limits_{u\in x\circledast y} \mu_A(u) \geq t$ and $\sup\limits_{v\in x\circledast y} \gamma_A(v) \leq s$. By IFGR2 and IFGR3,

$$\mu_A(x) \geq \min\left\{\inf_{u\in x\circledast y} \mu_A(u), \mu_A(y)\right\} \geq \min\{t, t\} = t \text{ and}$$
$$\gamma_A(x) \leq \max\left\{\sup_{v\in x\circledast y} \gamma_A(v), \gamma_A(y)\right\} \leq \max\{s, s\} = s.$$

Hence, $x \in A_{\langle t,s\rangle}$ and so $A_{\langle t,s\rangle}$ is a hyper GR-ideal of $H$. □

**Lemma 4.4.** *Let $\mu : H \to [0, 1]$ be a fuzzy set and $S \subseteq H$. Then*

(a) $1 - \sup\limits_{x\in S} \mu(x) = \inf\limits_{x\in S}(1 - \mu(x))$, *and*

(b) $1 - \inf\limits_{x\in S} \mu(x) = \sup\limits_{x\in S}(1 - \mu(x))$.

*Proof.* Let $x \in S$.

(a) Since $\mu(x) \leq \sup\limits_{x\in S} \mu(x)$, $1 - \mu(x) \geq 1 - \sup\limits_{x\in S} \mu(x)$. Then, $1 - \sup\limits_{x\in S} \mu(x)$ is a lowerbound for $\{1-\mu(x)|x \in S\}$. It implies that $1-\sup\limits_{x\in S} \mu(x) \leq \inf\limits_{x\in S}(1 - \mu(x))$. Since $\inf\limits_{x\in S}(1-\mu(x)) \leq 1-\mu(x)$, $\mu(x) \leq 1 - \inf\limits_{x\in S}(1 - \mu(x))$. Thus, $1 - \inf\limits_{x\in S}(1 - \mu(x))$ is an upperbound for $\{\mu(x) : x \in S\}$. Then $\sup\limits_{x\in S} \mu(x) \leq 1 - \inf\limits_{x\in S}(1 - \mu(x))$ and so $\inf\limits_{x\in S}(1 - \mu(x)) \leq 1 - \sup\limits_{x\in S} \mu(x)$. Therefore, $1 - \sup\limits_{x\in S} \mu(x) = \inf\limits_{x\in S}(1 - \mu(x))$.

(b) Note that $\mu(x) \geq \inf\limits_{x\in S} \mu(x)$. Then, $1 - \mu(x) \leq 1 - \inf\limits_{x\in S} \mu(x)$. This implies that $1 - \inf\limits_{x\in S} \mu(x)$ is an upperbound for $\{1 - \mu(x)|x \in S\}$. It follows that $\sup\limits_{x\in S}(1 - \mu(x)) \leq 1 - \inf\limits_{x\in S} \mu(x)$. Since $1 - \mu(x) \leq \sup\limits_{x\in S}(1 - \mu(x))$, $1 - \sup\limits_{x\in S}(1 - \mu(x)) \leq \mu(x)$. Then, $1 - \sup\limits_{x\in S}(1 - \mu(x))$ is a lowerbound for $\{\mu(x)|x \in S\}$. This implies that $1 - \sup\limits_{x\in S}(1 - \mu(x)) \leq \inf\limits_{x\in S} \mu(x)$ and so $1 - \inf\limits_{x\in S} \mu(x) \leq \sup\limits_{x\in S}(1 - \mu(x))$. Hence, $1 - \inf\limits_{x\in S} \mu(x) = \sup\limits_{x\in S}(1 - \mu(x))$. □

The following corollary follows from Lemma 4.4.

**Corollary 4.5.** *Let $\mu : H \to [0,1]$ be a fuzzy set and $S \subseteq H$. Then*

(a) $1 - \max_{x \in S} \mu(x) = \min_{x \in S}(1 - \mu(x))$,

(b) $1 - \min_{x \in S} \mu(x) = \max_{x \in S}(1 - \mu(x))$.

**Lemma 4.6.** *An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper GR-ideal in a hyper GR-algebra $H$ if and only if the fuzzy sets $\mu_A$ and $\bar{\gamma}_A$ are fuzzy hyper GR-ideals of type 1 in $H$.*

*Proof.* Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper GR-ideal in $H$. Clearly, $\mu_A$ is a fuzzy hyper GR-ideal of type 1 in $H$ and $\gamma_A(x) \geq \gamma_A(0)$ for all $x \in H$. Then, $\bar{\gamma}_A(x) = 1 - \gamma_A(x) \leq 1 - \gamma_A(0) = \bar{\gamma}_A(0)$. Let $x, y \in H$. Then,

$$\bar{\gamma}_A(x) = 1 - \gamma_A(x) \geq 1 - \max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\}. \tag{2}$$

Case 1. Suppose $\max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\} = \sup_{v \in x \circledast y} \gamma_A(v)$.

Then, $1 - \max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\} = 1 - \sup_{v \in x \circledast y} \gamma_A(v)$. By Corollary 4.5,

$$1 - \sup_{v \in x \circledast y} \gamma_A(v) = \inf_{v \in x \circledast y}(1 - \gamma_A(v)) = \inf_{v \in x \circledast y} \bar{\gamma}_A(v) \geq \min\left\{\inf_{v \in x \circledast y} \bar{\gamma}_A(v), \bar{\gamma}_A(y)\right\}.$$

By (2), we have $\bar{\gamma}_A(x) \geq \min\left\{\inf_{v \in x \circledast y} \bar{\gamma}_A(v), \bar{\gamma}_A(y)\right\}$.

Case 2. Suppose $\max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\} = \gamma_A(y)$.
Then,

$$1 - \max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\} = 1 - \gamma_A(y) = \bar{\gamma}_A(y) \geq \min\left\{\inf_{v \in x \circledast y} \bar{\gamma}_A(v), \bar{\gamma}_A(y)\right\}.$$

It follows from (2) that $\bar{\gamma}_A(x) \geq \min\left\{\inf_{v \in x \circledast y} \bar{\gamma}_A(v), \bar{\gamma}_A(y)\right\}$. Therefore, $\bar{\gamma}_A$ is a fuzzy hyper GR-ideal of type 1 in $H$.

Conversely, suppose $\mu_A$ and $\bar{\gamma}_A$ are fuzzy hyper GR-ideals of type 1. Let $x \in H$. Clearly, $\bar{\gamma}_A(x) \leq \bar{\gamma}_A(0)$. Then, $1 - \gamma_A(x) \leq 1 - \gamma_A(0)$ and so $\gamma_A(x) \geq \gamma_A(0)$. Let $x, y \in H$. By IFGR3 and Lemma 4.4,

$$1 - \gamma_A(x) = \bar{\gamma}_A(x) \geq \min\left\{\inf_{v \in x \circledast y} \bar{\gamma}_A(v), \bar{\gamma}_A(y)\right\}$$

$$
\begin{aligned}
&= \min\left\{\inf_{v\in x\circledast y}(1-\gamma_A(v)), 1-\gamma_A(y)\right\}\\
&= \min\left\{1-\sup_{v\in x\circledast y}\gamma_A(v), 1-\gamma_A(y)\right\}\\
&=1-\max\left\{\sup_{v\in x\circledast y}\gamma_A(v), \gamma_A(y)\right\}.
\end{aligned}
$$

It follows that $-\gamma_A(x) \geq -\max\left\{\sup_{v\in x\circledast y}\gamma_A(v), \gamma_A(y)\right\}$ and so

$$\gamma_A(x) \leq \max\left\{\sup_{v\in x\circledast y}\gamma_A(v), \gamma_A(y)\right\}.$$

Therefore, $A$ is an intuitionistic fuzzy hyper GR-ideal in $H$. □

**Theorem 4.7.** *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in a hyper GR-algebra $H$. Then, $A$ is an intuitionistic fuzzy hyper GR-ideal in $H$ if and only if $\hat{A} = (\mu_A, \bar{\mu}_A)$ and $\tilde{A} = (\bar{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy hyper GR-ideals of $H$.*

*Proof.* Suppose $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper GR-ideal in $H$. By Lemma 4.6, $\mu_A$ and $\bar{\gamma}_A$ are fuzzy hyper GR-ideals of type 1 in $H$. Let $x, y \in H$. Then, $\bar{\mu}_A(x) = 1-\mu_A(x) \geq 1-\mu_A(0) = \bar{\mu}_A(0)$. By Lemma 4.4,

$$
\begin{aligned}
\bar{\mu}_A(x) &= 1-\mu_A(x)\\
&\leq 1-\min\left\{\inf_{u\in x\circledast y}\mu_A(u), \mu_A(y)\right\}\\
&= \max\left\{1-\inf_{u\in x\circledast y}\mu_A(u), 1-\mu_A(y)\right\}\\
&= \max\left\{\sup_{u\in x\circledast y}(1-\mu_A(u)), \bar{\mu}_A(y)\right\}\\
&= \max\left\{\sup_{u\in x\circledast y}\bar{\mu}_A(u), \bar{\mu}_A(y)\right\}.
\end{aligned}
$$

Hence by Definition 4.1, $\hat{A} = (\mu_A, \bar{\mu}_A)$ and $\tilde{A} = (\bar{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy hyper GR-ideals in $H$.

Conversely, let $\hat{A} = (\mu_A, \bar{\mu}_A)$ and $\tilde{A} = (\bar{\gamma}_A, \gamma_A)$ be intuitionistic fuzzy hyper GR-ideals in $H$. Then by Lemma 4.6, $\mu_A$ and $\bar{\gamma}_A$ are fuzzy hyper GR-ideals of type 1 in $H$. Thus by Lemma 4.6, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper GR-ideal in $H$. □

**Theorem 4.8.** *For any subset $I$ of a hyper GR-algebra $H$, let $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ be an intuitionistic fuzzy set in $H$ defined by the following, respectively:*

$$(\mu_{A(I)})(x) = \begin{cases} k_1, & \text{if } x \in I \\ k_2, & \text{otherwise} \end{cases}$$

$$(\gamma_{A(I)})(x) = \begin{cases} m_1, & \text{if } x \in I \\ m_2, & \text{otherwise} \end{cases}$$

*for all $x \in H$, where $k_1, k_2, m_1, m_2 \in [0,1]$ with $k_1 > k_2$, $m_1 < m_2$, $k_i + m_i \leq 1$ for $i = 1, 2$. Then, $I$ is a hyper GR-ideal of $H$ if and only if $A(I) = (\mu_{A(I)}, \gamma_{A(I)})$ is an intuitionistic fuzzy hyper GR-ideal in $H$.*

*Proof.* Note that the level subsets of $\mu_{A(I)}$ are

$$\left(\mu_{A(I)}\right)_{t_1} = \begin{cases} \varnothing, \text{ if } k_1 < t_1 \leq 1 \\ I, \text{ if } k_2 < t_1 \leq k_1 \\ H, \text{ if } 0 \leq t_1 \leq k_2. \end{cases} \tag{3}$$

Since

$$(\bar{\gamma}_{A(I)})(x) = \begin{cases} 1 - m_1, & \text{if } x \in I \\ 1 - m_2, & \text{otherwise} \end{cases}$$

and $1 - m_1 > 1 - m_2$,

$$\left(\bar{\gamma}_{A(I)}\right)_{t_2} = \begin{cases} \varnothing, & \text{if } 1 - m_1 < t_2 \leq 1 \\ I, & \text{if } 1 - m_2 < t_2 \leq 1 - m_1 \\ H, & \text{if } 0 \leq t_2 \leq 1 - m_2. \end{cases}$$

Suppose $I$ is a hyper GR-ideal of $H$. Then the nonempty level subsets $\left(\mu_{A(I)}\right)_{t_1}$ and $\left(\bar{\gamma}_{A(I)}\right)_{t_2}$ are hyper GR-ideals of $H$. By Theorem 3.5, $\mu_{A(I)}$ and $\bar{\gamma}_{A(I)}$ are fuzzy hyper GR-ideals of type 1. By Lemma 4.6, $A(I) = \left(\mu_{A(I)}, \gamma_{A(I)}\right)$ is an intuitionistic fuzzy hyper GR-ideal in $H$.

Conversely, suppose $A(I) = \left(\mu_{A(I)}, \gamma_{A(I)}\right)$ is an intuitionistic fuzzy hyper GR-ideal in $H$. By Lemma 4.6, $\mu_{A(I)}$ and $\bar{\gamma}_{A(I)}$ are fuzzy hyper GR-ideals in $H$. It follows from Theorem 3.5 that $I = \left(\mu_{A(I)}\right)_{t_1}$ is a hyper GR-ideal of $H$. □

**Theorem 4.9.** *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy hyper GR-ideal of a hyper GR-algebra $H$, then the set $I = \{x \in H | \mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(0)\}$ is a hyper GR-ideal of $H$.*

*Proof.* Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy hyper GR-ideal in $H$. Clearly, $0 \in I$. Let $x, y \in H$ such that $x \circledast y \subseteq I$ and $y \in I$. Then, $\mu_A(y) = \mu_A(0)$, $\gamma_A(y) = \gamma_A(0)$, $\mu_A(u) = \mu_A(0)$ and $\gamma_A(v) = \gamma_A(0)$ for any $u, v \in x \circledast y$. By IFGR1 and IFGR2,

$$\mu_A(0) \geq \mu_A(x) \geq \min\left\{\inf_{u \in x \circledast y} \mu_A(u), \mu_A(y)\right\} = \mu_A(0) \text{ and}$$

$$\gamma_A(0) \leq \gamma_A(x) \leq \max\left\{\sup_{v \in x \circledast y} \gamma_A(v), \gamma_A(y)\right\} = \gamma_A(0).$$

This implies that $\mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(0)$ and so $x \in I$. Hence, $I$ is a hyper GR-ideal of $H$. □

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