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Mixed Type Symmetric and Self-Duality for Multiobjective Variational Problems

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Abstract. In this paper, a new formulation of multiobjective symmetric dual pair, called mixed type multiobjective symmetric dual pair, for multiobjective variational problems is presented. This mixed formulation unifies two existing Wolfe and Mond-Weir type symmetric dual pairs of multiobjective variational problems. For this pair of mixed type multiobjective variational problems, various duality theorems are established under invexity-incavity and pseudoinvexity-pseudoincavity of kernel functions appearing in the problems. Under additional hypotheses, a self duality theorem is validated. It is also pointed that our duality theorems can be viewed as dynamic generalization of the corresponding (static) symmetric and self duality of multiobjective nonlinear programming already existing in the literature.

2000 Mathematics Subject Classifications: Primary 90C30, Secondary 90C11, 90C20, 90C26.

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1. Introduction

Following Dorn [7], symmetric duality results in mathematical programming have been derived by a number of authors, notably, Dantzig et al [8], Mond [12], Bazaraa and Goode [1]. In these researches, the authors have studied symmetric duality under the hypothesis of convexity-concavity of the kernel function involved. Mond and Cottle [13] presented self duality for the problems of [8] by assuming skew symmetric of the kernel function. Later Mond-Weir [14] formulated a different pair of symmetric dual nonlinear program with a view to generalize convexity-concavity of the kernel function to pseudoconvexity-pseudoconcavity.

Symmetric duality for variational problems was first introduced by Mond and Hanson [15] under the convexity-concavity conditions of a scalar functions like $\psi(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ with $x(t) \in R^n$ and $y(t) \in R^m$. Bector, Chandra and Husain [3] presented a different pair of symmetric dual variational problems in order to relax the requirement of convexity-concavity to that of pseudoconvexity-pseudoconcavity while in [6] Chandra and Husain gave a fractional analogue.

Bector and Husain [4] probably were the first to study duality for multiobjective variational problems under appropriate convexity assumptions. Subsequently, Gulati, Husain and Ahmed [9] presented two distinct pairs of symmetric dual multiobjective variational problems and established various duality results under appropriate invexity requirements. In this reference, self duality theorem is also given under skew symmetric of the integrand of the objective functional. Husain and Jabeen [10] formulated a pair of mixed type symmetric dual variational problem in order to unify the Wolfe and Mond-Weir symmetric dual pairs of variational problems studied in [9].

The purpose of this research is to unify formulations of Wolfe and Mond-Weir type symmetric dual pairs of multiobjective variational problems incorporated by Gulati, Husain and Ahmed [9] and also present multiobjective version of the formulation of a pair of mixed type symmetric dual of Husain and Jabeen [10] and hence study symmetric and self duality for a pair of mixed multiobjective variational problem. This research is motivated by the work of Xu [18]. The problems, treated in this research are quite hard to solve. So to expect any immediate application of these problems would be far from reality. Unfortunately, there has not always been sufficient flow between the researchers in the multiple criteria decision making and the researchers applying it to their problems. Of course, one can find optimal control applications in galore which reflect the utility of our model. Special cases are deduced and it is also pointed out that our results can be considered as dynamic generalizations of corresponding (static) symmetric duality results of multiobjective nonlinear nonlinear treated by Bector et al. [3].

2. Notations and Preliminaries

The following notation will be used for vectors in R^n.

$$
\begin{aligned}
x < y \quad &\Leftrightarrow \quad x_i < y_i, \quad i = 1, 2, \ldots, n. \\
x \leqq y \quad &\Leftrightarrow \quad x_i \leqq y_i, \quad i = 1, 2, \ldots, n. \\
x \leq y \quad &\Leftrightarrow \quad x_i \leq y_i, \quad i = 1, 2, \ldots, n, \text{ but } x \neq y \\
&x \nleq y, \text{ is the negation of } x \leq y
\end{aligned}
$$

Let $I = [a, b]$ be the real interval, and $\phi^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ be a scalar function and twice differentiable function for $i = 1, 2, \ldots, p$ where $x : I \to R^n$ and $y : I \to R^n$ with derivatives $\dot{x}$ and $\dot{y}$. In order to consider $\phi^i(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ denote the first partial derivatives of ϕ^i with respect to t, $x(t)$, $\dot{x}(t)$, $y(t)$, $\dot{y}(t)$

respectively, by ϕ^i_t, ϕ^i_x, $\phi^i_{\dot{x}}$, ϕ^i_y, $\phi^i_{\dot{y}}$, that is,

$$\phi^i_t = \frac{\partial \phi^i}{\partial t}$$
$$\phi^i_x = \left[\frac{\partial \phi^i}{\partial x_1}, \frac{\partial \phi^i}{\partial x_2}, \ldots, \frac{\partial \phi^i}{\partial x_n}\right], \quad \phi^i_{\dot{x}} = \left[\frac{\partial \phi^i}{\partial \dot{x}_1}, \frac{\partial \phi^i}{\partial \dot{x}_2}, \ldots, \frac{\partial \phi^i}{\partial \dot{x}_n}\right]$$
$$\phi^i_y = \left[\frac{\partial \phi^i}{\partial y_1}, \frac{\partial \phi^i}{\partial y_2}, \ldots, \frac{\partial \phi^i}{\partial y_n}\right], \quad \phi^i_{\dot{y}} = \left[\frac{\partial \phi^i_i}{\partial \dot{y}_1}, \frac{\partial \phi^i}{\partial \dot{y}_2}, \ldots, \frac{\partial \phi^i}{\partial \dot{y}_n}\right].$$

The twice partial derivatives of ϕ^i with respect to $t, x(t), \dot{x}(t), y(t)$ and $\dot{y}(t)$, respectively are the matrices

$$\phi^i_{xx} = \left(\frac{\partial^2 \phi^i}{\partial x_k x_s}\right)_{n\times n}, \quad \phi^i_{x\dot{x}} = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{x}_s}\right)_{n\times n}, \quad \phi^i_{xy} = \left(\frac{\partial^2 \phi^i}{\partial x_k y_s}\right)_{n\times n}$$
$$\phi^i_{x\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{y}_s}\right)_{n\times n}, \quad \phi^i_{\dot{x}y} = \left(\frac{\partial^2 \phi^i}{\partial \dot{x}_k y_s}\right)_{n\times n}, \quad \phi^i_{x\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial x_k \dot{y}_s}\right)_{n\times n}$$
$$\phi^i_{yy} = \left(\frac{\partial^2 \phi^i}{\partial y_k y_s}\right)_{n\times n}, \quad \phi^i_{y\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial y_k \dot{y}_s}\right)_{n\times n}, \quad \phi^i_{\dot{y}\dot{y}} = \left(\frac{\partial^2 \phi^i}{\partial \dot{y}_k \dot{y}_s}\right)_{n\times n}$$

for $i = 1, 2, \ldots, p$.

Noting that

$$\frac{d}{dt}\phi^i_{\dot{y}} = \phi^i_{\dot{y}t} + \phi^i_{\dot{y}y}\dot{y} + \phi^i_{\dot{y}\dot{y}}\ddot{y} + \phi^i_{\dot{y}x}\dot{x} + \phi^i_{\dot{y}\dot{x}}\ddot{x}$$

and hence

$$\frac{\partial}{\partial y}\frac{d}{dt}\phi^i_{\dot{y}} = \frac{d}{dt}\phi^i_{y\dot{y}}, \quad \frac{\partial}{\partial \dot{y}}\frac{d}{dt}\phi^i_{\dot{y}} = \frac{d}{dt}\phi^i_{\dot{y}\dot{y}} + \phi^i_{\dot{y}y}, \quad \frac{\partial}{\partial \ddot{y}}\frac{d}{dt}\phi^i_{\dot{y}} = \phi^i_{\dot{y}y}$$
$$\frac{\partial}{\partial x}\frac{d}{dt}\phi^i_{\dot{y}} = \frac{d}{dt}\phi^i_{\dot{y}x}, \quad \frac{\partial}{\partial \dot{x}}\frac{d}{dt}\phi^i_{\dot{y}} = \frac{d}{dt}\phi^i_{\dot{y}\dot{x}} + \phi^i_{\dot{y}x}, \quad \frac{\partial}{\partial \ddot{x}}\frac{d}{dt}\phi^i_{\dot{y}} = \phi^i_{\dot{y}\dot{x}}$$

In order to establish our main results, the following are needed.

Definition 1 (Partially Invex). *If there exists a vector function $\eta(t, x(t), y(t), u(t), v(t)) \in \mathbf{R}^n_+$ with $\eta = 0$ at $x(t) = u(t)$ or $y(t) = v(t)$, such that for the scalar function $h(t, x(t), \dot{x}(t), y(t), \dot{y}(t))$ the functional*

$$H(x, \dot{x}, y, \dot{y}) = \int_I h(t, x(t), \dot{x}(t), y(t), \dot{y}(t))dt$$

satisfies

$$H(x,\dot{x},y,\dot{y})-H(u,\dot{u},v,\dot{v}) \geq \int_I [\eta^T h_x(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) + (D\eta)^T h_{\dot{x}}(t,x(t),\dot{x}(t),y(t),\dot{y}(t))]dt$$

then $H(x,\dot{x},y,\dot{y})$ *is said to be partially invex in* x *and* $\dot{x}$ *on* I *with respect to* η*, for fixed* y*. if* H *satisfies*

$$H(x,\dot{x},y,\dot{y})-H(x,\dot{x},v,\dot{v}) \geq \int_I [\eta^T h_y(t,x(t),\dot{x}(t),v(t),\dot{v}(t)) +(D\eta)^T h_{\dot{y}}(t,x(t),\dot{x}(t),v(t),\dot{v}(t))]dt,$$

then $H(x,\dot{x},y,\dot{y})$ *is said to be partially invex in* y *and* $\dot{y}$ *on* I *with respect to* η*, for fixed* x*. If* $-H$ *is partially invex in* x *and* $\dot{x}$ *(or in* y *and* $\dot{y}$*) on* I *with respect to* η*, for fixed* y*(or for fixed* x*), then* H *is said to be partially incave in* x *and* $\dot{x}$ *(or in* y *and* $\dot{y}$*) on* I *with respect to* η*, for fixed* y *(or for fixed* x*).*

Definition 2 (Partially Pseudoinvex)**.** *The functional* H *is said to be partially pseudoinvex in* x *and* $\dot{x}$ *with respect to* η*, for fixed* y *if* H *satisfies*

$$\int_I [\eta^T h_x(t,x,\dot{x},y,\dot{y})+(D\eta)^T h_{\dot{x}}(t,x,\dot{x},y,\dot{y})]dt \geqq 0$$

implies

$$H(x,\dot{x},u,\dot{u}) \geqq H(x,\dot{x},y,\dot{y})$$

and H *is said to be partially pseudoinvex in* y *and* $\dot{y}$ *with respect to* η*, for fixed* x *If* H *satisfies*

$$\int_I [\eta^T h_y(t,x,\dot{x},y,\dot{y})+(D\eta)^T h_{\dot{y}}(t,x,\dot{x},y,\dot{y})]dt \geqq 0$$

implies

$$H(x,\dot{x},v,\dot{v}) \geqq H(x,\dot{x},y,\dot{y}).$$

Definition 3 (Partially Quasi-invex). *The functional H is said to be partially quasi-invex in x and $\dot{x}$ with respect to η, for fixed y if H satisfies*

$$H(x,\dot{x},u,\dot{u}) \leqq H(x,\dot{x},y,\dot{y})$$

implies

$$\int_I [\eta^T h_x(t,x,\dot{x},y,\dot{y}) + (D\eta)^T h_{\dot{x}}(t,x,\dot{x},y,\dot{y})]dt \leqq 0$$

and H is said to be partially quasi-invex in y and $\dot{y}$ with respect to η, for fixed x if H satisfies

$$H(x,\dot{x},v,\dot{v}) \leqq H(x,\dot{x},y,\dot{y})$$

implies

$$\int_I [\eta^T h_y(t,x,\dot{x},y,\dot{y}) + (D\eta)^T h_{\dot{y}}(t,x,\dot{x},y,\dot{y})]dt \leqq 0.$$

If h is independent of t, then the above definitions become the usual definitions of invexity and generalized invexity, discussed by several authors, notably Ben-Israel and Mond [5], Martin [11], and Rueda and Hanson [16].

Definition 4 (Skew Symmetry). *The function $h : I \times R^n \times R^n \times R^n \times R^n \to R$ is said to be skew symmetric if for all x and y in the domain of h if*

$$h(t,x(t),\dot{x}(t),y(t),\dot{y}(t)) = -h(t,y(t),\dot{y}(t),x(t),\dot{x}(t)), \quad t \in I$$

where x and y are piecewise smooth on I.

Now consider the following multiobjective variational problem considered in [4]:

$$(\text{VP}_0) \qquad \text{Minimize}\left(\int_I \phi^1(t,x,\dot{x})dt, \int_I \phi^2(t,x,\dot{x})dt, \ldots, \int_I \phi^p(t,x,\dot{x})dt,\right)$$

Subject to

$$x(a) = \alpha, \qquad x(b) = \beta$$

$$h(t, x, \dot{x}) \leqq 0, \qquad t \in I,$$

where $\phi^i : I \times R^n \times R^n \times R^n \to R$ $(i = 1, 2, \ldots, p)$ and $h : I \times R^n \times R^n \times R^n \to R^m$. Let the set of feasible solution of (VP_0) be represented by K.

Definition 5 (Efficiency). *A point $\bar{x} \in K$ is an efficient (Pareto optimal) solution of (VP_0) if for all $x \in K$,*

$$\int_I \phi^i(t, x, \dot{x})dt \nleq \int_I \phi^i(t, \bar{x}, \dot{\bar{x}})dt, \quad (i = 1, 2, \ldots, p)$$

3. Statement of the Problems

For $N = \{1, 2, \ldots, n\}$ and $M = \{1, 2, \ldots, m\}$, let $J_1 \subset N, K_1 \subset M, J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in the subset J_1. The other symbol $|J_2|, |K_1|$ and $|K_2|$ are similarly defined. Let $x^1 : I \to R^{|J_1|}$ and $x^2 : I \to R^{|J_2|}$, then any $x : I \to R^n$ can be written as $x = (x^1, x^2)$. Similarly for $y^1 : I \to R^{|K_1|}$ and $y^2 : I \to R^{|K_2|}$ can be written as $y = (y^1, y^2)$ where $x : I \to R^n$, $y : I \to R^m$. Let $f : I \times R^{|J_1|} \times R^{|K_1|} \to R^p$ and $g : I \times R^{|J_2|} \times R^{|K_2|} \to R^p$ be twice continuously differentiable functions.

We state the following pair of mixed type multiobjective symmetric dual variational problems involving vector functions f and g.

(Mix SP) Minimize
$$F(x^1, x^2, y^1, y^2) = \int_I \{f(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g(t, x^2, \dot{x}^2, y^2, \dot{y}^2) - y^1(t)^T(\lambda^T f_{y^1}(t, x^1, \dot{x}^1, y^1, \dot{y}^1) - D\lambda^T f_{\dot{y}^1}(t, x^1, \dot{x}^1, y^1, \dot{y}^1))e\}dt$$

Subject to

$$x^1(a) = 0 = x^1(b), \qquad y^1(a) = 0 = y^1(b), \tag{1}$$

$$x^2(a) = 0 = x^2(b), \qquad y^2(a) = 0 = y^2(b), \tag{2}$$

$$\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) \leqq 0, t \in I, \tag{3}$$

$$\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2) - D\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2) \leqq 0, t \in I, \tag{4}$$

$$\int_I y^2(t)^T(\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2) - D\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2)) \geqq 0, \tag{5}$$

$$\lambda \in \Lambda^+. \tag{6}$$

(Mix SD) Maximize $G(u^1,u^2,v^1,v^2) = \int_I \{f(t,u^1,\dot{u}^1,v^1,\dot{v}^1) + g(t,u^2,\dot{u}^2,v^2,\dot{v}^2) - u^1(t)^T(\lambda^T f_{y^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - D\lambda^T f_{\dot{y}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))e\}dt$

Subject to

$$u^1(a) = 0 = u^1(b), \qquad v^1(a) = 0 = v^1(b), \tag{7}$$

$$u^2(a) = 0 = u^2(b), \qquad v^2(a) = 0 = v^2(b), \tag{8}$$

$$\lambda^T f_{u^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - D\lambda^T f_{\dot{u}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) \geqq 0, \quad t \in I, \tag{9}$$

$$\lambda^T g_{u^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2) - D\lambda^T g_{\dot{u}^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2) \geqq 0, \quad t \in I, \tag{10}$$

$$\int_I u^2(t)^T(\lambda^T g_{u^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2) - D\lambda^T g_{\dot{u}^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)) \geqq 0, \tag{11}$$

$$\lambda \in \Lambda^+. \tag{12}$$

where $\Lambda^+ = \{\lambda \in R^p | \lambda > 0, \lambda^T e = 1, e = (1,1,\ldots,1)^T \in R^p\}$.

4. Mixed Type Multiobjective Symmetric Duality

In this section, we present various duality results and the appropriate invexity and generalized invexity assumptions.

Theorem 1 (Weak Duality). *Let* $(x^1,x^2,y^1,y^2,\lambda)$ *be feasible for* (Mix SP) *and* $(u^1,u^2,v^1,v^2,\lambda)$ *be feasible for* (Mix SD).

Let

H$_1$ $\int_I f(t,.,.,y^1(t),\dot{y}^1(t))dt$ *be partially invex in* $x^1, \dot{x}^1$ *on* I *for fixed* $y^1, \dot{y}^1$ *with respect to* $\eta_1(t,x^1,u^1) \in R^{|J_1|}$.

$\int_I f(t,x^1(t),\dot{x}^1(t),.,.)dt$ *be partially incave in* $y^1, \dot{y}^1$ *on* I *for fixed* $x^1, \dot{x}^1$ *with respect to* $\eta_2(t,y^1,v^1) \in R^{|K_1|}$.

H$_2$ $\int_I \lambda^T g(t,.,.,y^2(t),\dot{y}^2(t))dt$ *be partially pseudoinvex in* $x^2, \dot{x}^2$ *on* I *for fixed* $y^2, \dot{y}^2$ *with respect* $\eta_3(t,x^2,u^2) \in R^{|J_2|}$ *and* $\int_I \lambda^T g(t,x^2,\dot{x}^2,\ldots)dt$ *be partially pseudoincave in* $y^2, \dot{y}^2$ *on* I *for fixed* $x^2, \dot{x}^2$ *with respect to* $\eta_4(t,y^2,v^2) \in R^{|K_2|}$.

H$_3$

$$\eta_1(t,x^1,u^1) + u^1(t) \geqq 0, \quad t \in I, \tag{13}$$

$$\eta_2(t,v^1,y^1) + y^1(t) \geqq 0, \quad t \in I, \tag{14}$$

$$\eta_3(t,x^2,u^2) + u^2(t) \geqq 0, \quad t \in I, \tag{15}$$

$$\eta_4(t,v^2,y^2) + y^2(t) \geqq 0, \quad t \in I, \tag{16}$$

then

$$F(x^1,x^2,y^1,y^2) \nleq G(u^1,u^2,v^1,v^2).$$

Proof. Because of the partial invexity-incavity of the function f, we have for each $i = \{1,2,\ldots,p\}$.

$$\int_I f^i(t,x^1,\dot{x}^1,v^1,\dot{v}^1)dt - \int_I f^i(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt$$
$$\geqq \int_I \{\eta_1^T f^i_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) + (D\eta_1)^T f^i_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)\}dt \tag{17}$$

$$\int_I f^i(t,x^1,\dot{x}^1,v^1,\dot{v}^1)dt - \int_I f^i(t,x^1,\dot{x}^1,y^1,\dot{y}^1)dt$$
$$\leqq \int_I \{\eta_2^T f^i_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) + (D\eta_2)^T f^i_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)\}dt \tag{18}$$

Multiplying (17) by $\lambda^i > 0$ and summing over i.

$$\int_I \lambda^T f(t,x^1,\dot{x}^1,v^1,\dot{v}^1)dt - \int_I \lambda^T f(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt$$
$$\geqq \int_I \{\eta_1^T(\lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) + (D\eta_1)^T \lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))\}dt$$

Integrating by parts, the above inequality becomes

$$\int_I \lambda^T f(t,x^1,\dot{x}^1,v^1,\dot{v}^1) - \int_I \lambda^T f(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt$$
$$\geqq \int_I \eta_1^T \lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt + \eta_1^T \lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)\Big|_{t=a}^{t=b}$$
$$- \int_I \eta_1^T D\lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt$$

Using the boundary conditions which at $t = a, t = b$ gives $\eta_1 = 0$, we have

$$\int_I \lambda^T f(t,x^1,\dot{x}^1,v^1,\dot{v}^1) - \int_I \lambda^T f(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt$$
$$\geqq \int_I \eta_1^T[\lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt - D(\lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))]dt \tag{19}$$

Multiplying (18) by $\lambda^i, i \in \{1,2,\ldots,p\}$ and summing over i, we get,

$$\int_I \lambda^T f(t,x^1,\dot{x}^1,v^1,\dot{v}^1) - \int_I \lambda^T f(t,x^1,\dot{x}^1,y^1,\dot{y}^1)dt$$
$$\leqq \int_I \{\eta_2^T(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)) + (D\eta_2)^T \lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)\}dt$$

On integrating by parts the R.H.S of the above inequality and using the boundary conditions which at $t = a, t = b$ gives $\eta_2 = 0$, we have

$$\int_I \lambda^T f(t,x^1,\dot{x}^1,v^1,\dot{v}^1) - \int_I \lambda^T f(t,x^1,\dot{x}^1,y^1,\dot{y}^1)dt$$
$$\leqq \int_I \eta_2^T[(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)) - D(\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1))]dt \tag{20}$$

Multiplying (20) by (-1) and adding to (19), we have

$$\int_I \lambda^T f(t,x^1,\dot{x}^1,y^1,\dot{y}^1)dt - \int_I \lambda^T f(t,u^1,\dot{u}^1,v^1,\dot{v}^1)dt$$

$$\geqq \int_I \eta_1^T[(\lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)) - D(\lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))]dt$$
$$-\int_I \eta_2^T[(\lambda^T f_{y^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)) - D(\lambda^T f_{\dot{y}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))]dt \quad (21)$$

Now from the inequality (9) along with (13), it follows

$$\int_I \eta_1^T(\lambda^T f_{u^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - D\lambda^T f_{\dot{u}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))dt$$
$$\geqq -\int_I u^1(t)^T[\lambda^T f_{u^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - D\lambda^T f_{\dot{u}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)]dt \quad (22)$$

Also from the inequality (3) together with (14) implies

$$-\int_I \eta_2^T(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1))dt$$
$$\geqq \int_I y^1(t)^T[\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)]dt \quad (23)$$

Using (22) and (23), in (21), we have

$$\int_I \lambda^T f(t,x^1,\dot{x}^1,y^1,\dot{y}^1)dt - y^1(t)^T\int_I \lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)dt$$
$$\geqq -\int_I u^1(t)^T[(\lambda^T f_{u^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)) - D(\lambda^T f_{\dot{u}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))]dt$$
$$+\int_I y^1(t)^T[(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1))$$
$$-D(\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1))]dt, \quad (24)$$

which implies

$$\int_I \{\lambda^T f(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - y^1(t)^T(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)$$
$$-D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1))\}dt$$
$$\geqq \int_I \{\lambda^T f_{u^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - u^1(t)^T(\lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)$$
$$-D\lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))\}dt \quad (25)$$

Now from the inequality (10) along with (15), we have

$$\int_I (\eta_3^T(t,x^2,\dot{x}^2,u^2,\dot{u}^2)+u^2(t))(\lambda^T g_{u^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)$$
$$-D\lambda^T g_{\dot{u}^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)) \geqq 0.$$

This implies

$$\int_I \eta_3^T(\lambda^T g_{u^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)-D(\lambda^T g_{\dot{u}^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)))dt$$
$$\geqq -\int_I u^2(t)^T[\lambda^T g_{u^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)-D(\lambda^T g_{\dot{u}^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2))]dt$$

Integrating by parts and using the boundary conditions which at $t=a, t=b$ gives $\eta_3=0$, we have

$$\int_I \{\eta_3^T(\lambda^T g_{u^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)+(D\eta_3)^T(\lambda^T g_{\dot{u}^2}(t,u^2,\dot{u}^2,v^2,\dot{v}^2)))\}dt \geqq 0$$

Because of the partial pseudo-invexity of $\int_I \lambda^T g_{u^2} dt$, this gives

$$\int_I \lambda^T g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)dt \geqq \int_I \lambda^T g(t,u^2,\dot{u}^2,v^2,\dot{v}^2)dt \tag{26}$$

Also from (4) together with (16), we have

$$\int_I (\eta_4^T(t,v^2,\dot{v}^2,y^2,\dot{y}^2)+y^2(t))(\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2)$$
$$-D\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2))dt \geqq 0$$

This implies,

$$\int_I \eta_4^T(\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2))-D(\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2))dt$$
$$\leqq -\int_I y^2(t)^T[\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2)-D(\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2))]dt$$

This in view of (5) yields,

$$\int_I \eta_4^T\{\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2)-D(\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2))\}dt \leqq 0$$

On integrating by parts and using the boundary conditions which at $t=a, t=b$ gives $\eta_4 = 0$, we have,

$$\int_I \eta_4^T\{\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2) + (D\eta_4)^T(\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2))\}dt \leqq 0$$

Because of partial pseudo-incavity of $\int_I \lambda^T g_{y^2} dt$, we have

$$\int_I (\lambda^T g(t,x^2,\dot{x}^2,v^2,\dot{v}^2))dt \leqq \int_I (\lambda^T g(t,x^2,\dot{x}^2,y^2,\dot{y}^2))dt \tag{27}$$

From (26) and (27), we get,

$$\int_I (\lambda^T g(t,x^2,\dot{x}^2,y^2,\dot{y}^2))dt \geqq \int_I (\lambda^T g(t,u^2,\dot{u}^2,v^2,\dot{v}^2))dt \tag{28}$$

Combining (25) and (28), we get

$$\begin{aligned}
&\int_I \{\lambda^T f(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - y^1(t)^T(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1)) \\
&-D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) + \lambda^T g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)\}dt \\
&\quad \geqq \int_I \{\lambda^T f(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - u^1(t)^T(\lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)) \\
&\qquad -D\lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) + \lambda^T g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)\}dt
\end{aligned}$$

This implies,

$$\begin{aligned}
&\lambda^T \int_I \{f(t,x^1,\dot{x}^1,y^1,\dot{y}^1) + g(t,x^2,\dot{x}^2,y^2,\dot{y}^2) \\
&-y^1(t)^T(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1))e\}dt \\
&\quad \geqq \lambda^T \int_I \{f(t,u^1,\dot{u}^1,v^1,\dot{v}^1) + g(t,u^2,\dot{u}^2,v^2,\dot{v}^2) \\
&\qquad -u^1(t)^T(\lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - D\lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))e\}dt
\end{aligned}$$

This implies,

$$\int_I \{f(t,x^1,\dot{x}^1,y^1,\dot{y}^1) + g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)$$

$$-y^1(t)^T(\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1))e\}dt$$
$$\not\leq \int_I \{f(t,u^1,\dot{u}^1,v^1,\dot{v}^1) + g(t,u^2,\dot{u}^2,v^2,\dot{v}^2)$$
$$-u^1(t)^T(\lambda^T f_{x^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1) - D\lambda^T f_{\dot{x}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1))e\}dt$$

This was to be proved.

Theorem 2 (Strong Duality). *Let* $(\bar{x}^1,\bar{x}^2,\bar{y}^1,\bar{y}^2,\bar{\lambda})$ *be an efficient solution of* (Mix SP). *Let* $\lambda=\bar{\lambda}$ *be fixed in* (Mix SD) *and*

$$(\mathrm{C}_1)\qquad \int_I [\{(\phi^1(t))^T(\lambda^T f_{y^1y^1} - D\lambda^T f_{y^1\dot{y}^1}) - D\phi^1(t)^T(-D\lambda^T f_{\dot{y}^1\dot{y}^1})$$
$$+D^2\phi^1(t)^T(-\lambda^T f_{\dot{y}^1\dot{y}^1})\}\phi^1(t)]dt > 0,$$

and

$$\int_I [\{(\phi^2(t))^T(\lambda^T g_{y^2y^2} - D\lambda^T g_{y^2\dot{y}^2}) - D\phi^2(t)^T(-D\lambda^T g_{\dot{y}^2\dot{y}^2})$$
$$+D^2\phi^2(t)^T(-\lambda^T g_{\dot{y}^2\dot{y}^2})\}\phi^2(t)]dt > 0,$$

$$(\mathrm{C}_2)\qquad \int_I [\{(\phi^1(t))^T(\lambda^T f_{y^1y^1} - D\lambda^T f_{y^1\dot{y}^1}) - D\phi^1(t)^T(-D\lambda^T f_{\dot{y}^1\dot{y}^1})$$
$$+D^2\phi^1(t)^T(-\lambda^T f_{\dot{y}^1\dot{y}^1})\}\phi^1(t)]dt = 0, t\in I \Rightarrow \phi^1(t)=0, t\in I,$$

and

$$\int_I [\{(\phi^2(t))^T(\lambda^T g_{y^2y^2} - D\lambda^T g_{y^2\dot{y}^2}) - D\phi^2(t)^T(-D\lambda^T g_{\dot{y}^2\dot{y}^2})$$
$$+D^2\phi^2(t)^T(-\lambda^T g_{\dot{y}^2\dot{y}^2})\}\phi^2(t)]dt = 0, t\in I \Rightarrow \phi^1(t)=0, t\in I.$$

and

$$(\mathrm{C}_3)\quad g^i_{y^2} - Dg^i_{\dot{y}^2} = 0, i=1,2,\dots,p \text{ are linearly independent.}$$

Let $\int_I f\,dt$ *and* $\int_I \lambda^T g\,dt$ *satisfy the invexity and generalized invexity as stated in Theorem 1, then* $(\bar{x}^1,\bar{x}^2,\bar{y},\bar{y}^2,\bar{\lambda})$ *and* $(\bar{u}^1,\bar{u}^2,\bar{v},\bar{v}^2,\bar{\lambda})$ *are efficient solution of (Mix SP) and (Mix SD) respectively.*

Proof. Since $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ is efficient, it is weak minimum. Hence there exists $\tau \in R^p, \eta \in R^p$, $\gamma \in R$ and piecewise smooth functions $\theta^1(t) : I \to R^{|K_1|}, \theta^2(t) : I \to R^{|k_2|}$ and $\mu : I \to R^m$ such that the following Fritz-John optimality conditions, in view of the analysis on [13, 9, 14], are satisfied

$$\begin{aligned} H &= \tau(f+g) + (\theta^1(t) - (\tau^T e) y^1(t)^T)(\lambda^T f_{y^1} - D\lambda^T f_{\dot{y}^1}) \\ &\quad + (\theta^2(t) - \gamma y^2(t)^T)(\lambda^T g_{y^2} - D\lambda^T g_{\dot{y}^2}) + \eta^T \lambda \end{aligned}$$

Satisfying

$$H_{x^1} - DH_{\dot{x}^1} + D^2 H_{\ddot{x}^1} = 0, \quad t \in I \tag{29}$$

$$H_{x^2} - DH_{\dot{x}^2} + D^2 H_{\ddot{x}^2} = 0, \quad t \in I \tag{30}$$

$$H_{y^1} - DH_{\dot{y}^1} + D^2 H_{\ddot{y}^1} = 0, \quad t \in I \tag{31}$$

$$H_{y^2} - DH_{\dot{y}^2} + D^2 H_{\ddot{y}^2} = 0, \quad t \in I \tag{32}$$

$$(\theta^1(t) - (\tau^T e) y^1(t))^T (f_{y^1} - Df_{\dot{y}^1}) + (\theta^2(t) - \gamma y^2(t))^T (g_{y^2} - Dg_{\dot{y}^2}) - \eta = 0, \quad t \in I \tag{33}$$

$$\theta^1(t)(\lambda^T f_{y^1} - D\lambda^T f_{\dot{y}^1}) = 0, \quad t \in I \tag{34}$$

$$\theta^2(t)(\lambda^T g_{y^2} - D\lambda^T g_{\dot{y}^2}) = 0, \quad t \in I \tag{35}$$

$$\gamma \int_I y^2(t)^T (\lambda^T g_{y^2} - D\lambda^T g_{\dot{y}^2}) = 0 \tag{36}$$

$$\eta^T \bar{\lambda} = 0 \tag{37}$$

$$(\tau, \theta^1(t), \theta^2(t), \eta, \gamma) \geqq 0, \quad t \in I \tag{38}$$

$$(\tau, \theta^1(t), \theta^2(t), \eta, \gamma) \neq 0, \quad t \in I \tag{39}$$

hold throughout I (except at the corners of $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t))$ where (29)-(32) are valid for unique right and left hand limits). Here θ^1 and θ^2 are continuous except possibly at corner of $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}^1(t), \bar{y}^2(t))$.

The relations (29)-(32) are all deducible from the classical Euler-Lagrange and Clebsch necessary optimality conditions. Particularly, the equations (29)-(32) are the famous Euler-Lagrange differential equation when second order derivatives appear in H. Using the analogies of the observation of $D\phi\ddot{y}$ from the notational section, the equations (29)-(32) become,

$$\begin{aligned}\tau(f_{x^1}-Df_{\dot{x}^1})-(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(\lambda^Tf_{y^1x^1}-D\lambda^Tf_{\dot{y}^1x^1})\\ -D(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(\lambda^Tf_{y^1\dot{x}^1}-D\lambda^Tf_{\dot{y}^1\dot{x}^1}-\lambda^Tf_{\dot{y}^1x^1})\\ +D^2((\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(-\lambda^Tf_{\dot{y}^1\dot{x}^1}))=0\end{aligned}\tag{40}$$

$$\begin{aligned}\tau(g_{x^2}-Dg_{\dot{x}^2})+(\theta^2(t)-\gamma\bar{y}^2(t))^T(\lambda^Tg_{y^2x^2}-D\lambda^Tg_{\dot{y}^2x^2})\\ -D(\theta^2(t)-\gamma\bar{y}^2(t))^T(\lambda^Tg_{y^2\dot{x}^2}-D\lambda^Tg_{\dot{y}^2\dot{x}^2}-\lambda^Tg_{\dot{y}^2x^2})\\ +D^2((\theta^2(t)-\gamma\bar{y}^2(t))^T(-\lambda^Tg_{\dot{y}^2\dot{x}^2}))=0\end{aligned}\tag{41}$$

$$\begin{aligned}(\tau-(\tau^Te)\lambda)^T(f_{y^1}-Df_{\dot{y}^1})+(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T\\ \times(\lambda^Tf_{y^1y^1}-D\lambda^Tf_{\dot{y}^1y^1})\\ -D(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(-D\lambda^Tf_{\dot{y}^1\dot{y}^1})\\ +D^2((\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(-\lambda^Tf_{\dot{y}^1\dot{y}^1}))=0\end{aligned}\tag{42}$$

$$\begin{aligned}(\tau-\gamma\lambda)^T(g_{y^2}-Dg_{\dot{y}^2})+(\theta^2(t)-\gamma\bar{y}^2(t))^T(\lambda^Tg_{y^2y^2}-D\lambda^Tg_{\dot{y}^2y^2})\\ -D(\theta^2(t)-\gamma\bar{y}^2(t))^T(-D\lambda^Tg_{\dot{y}^2\dot{y}^2})\\ +D^2(\theta^2(t)-\gamma\bar{y}^2(t))^T(-\lambda^Tg_{\dot{y}^2\dot{y}^2})=0\end{aligned}\tag{43}$$

Since $\lambda>0$, (37) implies $\eta=0$. Consequently, (33) reduces to

$$(\theta^1(t)-(\tau^Te)y^1(t))^T(f_{y^1}-Df_{\dot{y}^1})+(\theta^2(t)-\gamma y^2(t))^T(g_{y^2}-Dg_{\dot{y}^2})=0,t\in I\tag{44}$$

Postmultiplying (42) by $(\theta^1(t)-(\tau^Te)y^1(t))$, (43) by $(\theta^2(t)-\gamma y^2(t))$ and then adding, we have

$$\{(\tau-(\tau^Te)\lambda)^T(f_{y^1}-Df_{\dot{y}^1})+(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(\lambda^Tf_{y^1y^1}-D\lambda^Tf_{\dot{y}^1y^1})$$

$$-D[(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(-D\lambda^Tf_{\dot{y}^1\dot{y}^1})]$$
$$+D^2[(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(-\lambda^Tf_{\dot{y}^1\dot{y}^1})]\}(\theta^1(t)-(\tau^Te)\bar{y}^1(t))$$
$$+\{(\tau-\gamma\lambda)^T(g_{y^2}-Dg_{\dot{y}^2})+(\theta^2(t)-\gamma\bar{y}^2(t))^T(\lambda^Tg_{y^2y^2}-D\lambda^Tg_{\dot{y}^2y^2})$$
$$-D[(\theta^2(t)-\gamma\bar{y}^2(t))^T(-D\lambda^Tg_{\dot{y}^2\dot{y}^2})]$$
$$+D^2[(\theta^2(t)-\gamma\bar{y}^2(t))^T(-\lambda^Tg_{\dot{y}^2\dot{y}^2})]\}(\theta^2(t)-\gamma\bar{y}^2(t))=0 \tag{45}$$

Now multiplying (44) by $\bar{\lambda}$ and then using (35) and (36) we have

$$\int_I(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(\lambda^Tf_{y^1}-D\lambda^Tf_{\dot{y}^1})dt=0$$

that is

$$\int_I(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(\lambda^Tf_{y^1}-D\lambda^Tf_{\dot{y}^1})(\tau^Te)dt=0 \tag{46}$$

Multiplying (44) by τ, we have

$$\int_I[(\theta^1(t)-(\tau^Te)y^1(t))^T(\tau f_{y^1}-D\tau f_{\dot{y}^1})$$
$$+(\theta^2(t)-\gamma y^2(t))^T(\tau g_{y^2}-D\tau g_{\dot{y}^2})]dt=0 \tag{47}$$

Subtracting (46) and (47) and using (35) and (36), we have

$$\int_I[(\theta^1(t)-(\tau^Te)y^1(t))^T(f_{y^1}-Df_{\dot{y}^1})(\tau-(\tau^Te)\bar{\lambda})$$
$$+(\theta^2(t)-\gamma y^2(t))^T(\tau g_{y^2}-D\tau g_{\dot{y}^2})(\tau-\gamma\bar{\lambda})]dt=0 \tag{48}$$

From (45) and (48), we obtain

$$\int_I[\{(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(\lambda^Tf_{y^1y^1}-D\lambda^Tf_{y^1\dot{y}^1})$$
$$-D[(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(-D\lambda^Tf_{\dot{y}^1\dot{y}^1})]$$
$$+D^2[(\theta^1(t)-(\tau^Te)\bar{y}^1(t))^T(-\lambda^Tf_{\dot{y}^1\dot{y}^1})]\}.(\theta^1(t)-(\tau^Te)\bar{y}^1(t))]dt$$
$$+\int_I[\{(\theta^2(t)-\gamma\bar{y}^2(t))^T(\lambda^Tg_{y^2y^2}-D\lambda^Tg_{y^2\dot{y}^2})$$

$$-D[(\theta^2(t)-\gamma\bar{y}^2(t))^T(-D\lambda^T g_{\dot{y}^2\dot{y}^2})]$$
$$+D^2[(\theta^2(t)-\gamma\bar{y}^2(t))^T(-\lambda^T g_{\dot{y}^2\dot{y}^2})]\}(\theta^2(t)-\gamma\bar{y}^2(t))]dt=0$$

In view of the hypothesis (C_1), we have

$$\int_I [\{(\theta^1(t)-(\tau^T e)\bar{y}^1(t))^T(\lambda^T f_{y^1y^1}-D\lambda^T f_{y^1\dot{y}^1})$$
$$-D[(\theta^1(t)-(\tau^T e)\bar{y}^1(t))^T(-D\lambda^T f_{\dot{y}^1\dot{y}^1})]$$
$$+D^2[(\theta^1(t)-(\tau^T e)\bar{y}^1(t))^T(-\lambda^T f_{\dot{y}^1\dot{y}^1})]\}.(\theta^1(t)-(\tau^T e)\bar{y}^1(t))]dt=0$$

and

$$\int_I [\{(\theta^2(t)-\gamma\bar{y}^2(t))^T(\lambda^T g_{y^2y^2}-D\lambda^T g_{y^2\dot{y}^2})$$
$$-D[(\theta^2(t)-\gamma\bar{y}^2(t))^T(-D\lambda^T g_{\dot{y}^2\dot{y}^2})]$$
$$+D^2[(\theta^2(t)-\gamma\bar{y}^2(t))^T(-\lambda^T g_{\dot{y}^2\dot{y}^2})]\}(\theta^2(t)-\gamma\bar{y}^2(t))]dt=0$$

This in view of the hypothesis (C_2) yields,

$$\phi^1(t)=\theta^1(t)-(\tau^T e)\bar{y}^1(t)=0, \quad t\in I \tag{49}$$

$$\phi^2(t)=\theta^2(t)-\gamma\bar{y}^2(t)=0, \quad t\in I \tag{50}$$

From (50) and (43), we have

$$(\tau-\gamma\lambda)^T(g_{y^2}-Dg_{\dot{y}^2})=0$$

that is

$$\sum_{i=1}^{p}(\tau^i-\gamma\lambda^i)^T(g_{y^2}-Dg_{\dot{y}^2})=0$$

This in view of the (C_3) yields

$$\tau^i=\gamma\lambda^i, i=1,2,\ldots,p \tag{51}$$

Let if possible, $\gamma = 0$. Then from (51), we have $\tau = 0$ and therefore, from (49) and (50), we have

$$\phi^1(t) = 0, \theta^2(t) = 0, t \in I$$

Hence $(\tau, \theta^1(t), \theta^2(t), \eta, \gamma) = 0$, contradicting Fritz-John conditions (39). Hence $\gamma > 0$ and consequently $\tau > 0$.

From (40) and (4.29) along with (51), we obtain

$$(\bar{\lambda}^T f_{x^1} - D\bar{\lambda}^T f_{\dot{x}^1}) = 0, \quad t \in I \tag{52}$$

$$(\bar{\lambda}^T g_{x^2} - D\bar{\lambda}^T g_{\dot{x}^2}) = 0, \quad t \in I \tag{53}$$

which implies

$$\int_I x^2(t)^T(\bar{\lambda}^T g_{x^2} - D\bar{\lambda}^T g_{\dot{x}^2})dt = 0 \tag{54}$$

From (52)-(54) together with (49), we have

$$y^1(t)^T(\bar{\lambda}^T f_{y^1} - D\bar{\lambda}^T f_{\dot{y}^1}) = 0, t \in I \tag{55}$$

From the primal objective with (55)

$$\begin{aligned}
&\int_I \{f(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \\
&- \quad y^1(t)^T(\lambda^T f_{y^1} - D\lambda^T f_{\dot{y}^1})\}dt \\
&= \quad \int_I \{f(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g(t, x^2, \dot{x}^2, y^2, \dot{y}^2)\}dt
\end{aligned} \tag{56}$$

From the dual objective in view of (52), we have

$$\begin{aligned}
&\int_I \{f(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g(t, x^2, \dot{x}^2, y^2, \dot{y}^2) \\
&- \quad x^1(t)^T(\lambda^T f_{x^1} - D\lambda^T f_{\dot{x}^1})\}dt \\
&= \quad \int_I \{f(t, x^1, \dot{x}^1, y^1, \dot{y}^1) + g(t, x^2, \dot{x}^2, y^2, \dot{y}^2)\}dt
\end{aligned} \tag{57}$$

From (55) and (57), the equality of objective values is evident. Consequently, in view of the hypothesis of Theorem 1, the efficiency of $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ follows.

We now state converse duality whose proof follows by symmetry.

Theorem 3 (Converse Duality). *Let* $(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{\lambda})$ *be an efficient solution of* (Mix SP). *Let* $\lambda = \bar{\lambda}$ *be fixed in* (Mix SD) *and*

$$(\mathrm{A}_1) \qquad \int_I [\{\psi^1(t)^T(\lambda^T f_{x^1x^1} - D\lambda^T f_{x^1\dot{x}^1}) - D\psi^1(t)^T(-D\lambda^T f_{\dot{x}^1\dot{x}^1}) + D^2\psi^1(t)^T(-\lambda^T f_{\dot{x}^1\dot{x}^1})\}\psi^1(t)]dt > 0,$$

and

$$\int_I [\{\psi^2(t)^T(\lambda^T g_{x^2x^2} - D\lambda^T g_{x^2\dot{x}^2}) - D\psi^2(t)^T(-D\lambda^T g_{\dot{x}^2\dot{x}^2}) + D^2\psi^2(t)^T(-\lambda^T g_{\dot{x}^2\dot{x}^2})\}\psi^2(t)]dt > 0,$$

$$(\mathrm{A}_2) \qquad \int_I [\{\psi^1(t)^T(\lambda^T f_{x^1x^1} - D\lambda^T f_{x^1\dot{x}^1}) - D\psi^1(t)^T(-D\lambda^T f_{\dot{x}^1\dot{x}^1}) + D^2\psi^1(t)^T(-\lambda^T f_{\dot{x}^1\dot{x}^1})\}\psi^1(t)]dt = 0, t \in I \Rightarrow \psi^1(t) = 0, t \in I,$$

and

$$\int_I [\{\psi^2(t)^T(\lambda^T g_{x^2x^2} - D\lambda^T g_{x^2\dot{x}^2}) - D\psi^2(t)^T(-D\lambda^T g_{\dot{x}^2\dot{x}^2}) + D^2\psi^2(t)^T(-\lambda^T g_{\dot{x}^2\dot{x}^2})\}\psi^2(t)]dt = 0, t \in I \Rightarrow \psi^2(t) = 0, t \in I$$

and

$$(\mathrm{A}_3) \qquad g^i_{x^2} - Dg^i_{\dot{x}^2} = 0, i = 1, 2, \ldots, p \textit{ are linearly independent.}$$

Let $\int_I f\,dt$ and $\int_I \lambda^T g\,dt$ satisfy the invexity and generalized invexity as stated in Theorem 1, then $(\bar{x}^1, \bar{x}^2, \bar{y}, \bar{y}^2, \bar{\lambda})$ and $(\bar{u}^1, \bar{u}^2, \bar{v}, \bar{v}^2, \bar{\lambda})$ are efficient solution of (Mix SP) and (Mix SD) respectively.

5. Self Duality

A problem is said to be self-dual if it is formally identical with its dual, in general, the problems (Mix SP) and (Mix SD) are not formally in the absence of an additional restrictions of the function f and g. Hence skew symmetric of f and g is assumed in order to validate the following self-duality theorem.

Theorem 4 (Self Duality). *Let f^i and $g^i, i = 1, 2, \ldots, p$, be skew symmetric. Then the problem* (Mix SP) *is self dual. If the problems* (Mix SP) *and* (Mix SD) *are dual problems and $(\bar{x}^1(t), \bar{x}^2(t), \bar{y}(t), \bar{y}^2(t), \bar{\lambda})$ is a joint optimal solution of* (Mix SP) *and* (Mix SD), *then so is $(\bar{y}(t), \bar{y}^2(t), \bar{x}^1(t), \bar{x}^2(t), \bar{\lambda})$, and the common functional value is zero, i.e.*

$$\text{Minimum(Mix SP)} = \int_I \{f(x^1, \dot{x}^1, y^1, \dot{y}^1) + g(x^2, \dot{x}^2, y^2, \dot{y}^2)\}dt = 0$$

Proof. By skew symmetric of f^i and g^i, we have

$$f^i_{x^1}(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)) = -f^i_{y^1}(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t))$$
$$g^i_{x^2}(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)) = -g^i_{\dot{y}^2}(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t))$$
$$f^i_{y^1}(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)) = -f^i_{x^1}(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t))$$
$$g^i_{y^2}(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)) = -g^i_{\dot{x}^2}(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t))$$
$$f^i_{\dot{x}^1}(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)) = -f^i_{\dot{y}^1}(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t))$$
$$g^i_{\dot{x}^2}(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)) = -g^i_{\dot{y}^2}(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t))$$
$$f^i_{y^1}(t, x^1(t), \dot{x}^1(t), y^1(t), \dot{y}^1(t)) = -f^i_{\dot{x}^1}(t, y^1(t), \dot{y}^1(t), x^1(t), \dot{x}^1(t))$$
$$g^i_{\dot{y}^2}(t, x^2(t), \dot{x}^2(t), y(t), \dot{y}^2(t)) = -g^i_{\dot{x}^2}(t, y^2(t), \dot{y}^2(t), x^2(t), \dot{x}^2(t))$$

Recasting the dual problem (Mix SD) as a minimization problem and using the above relations, we have

$$\text{(Mix SD1)} \qquad \text{Minimize} - \int_I \{f(t, y^1, \dot{y}^1, x^1, \dot{x}^1) + g(t, y^2, \dot{y}^2, x^2, \dot{x}^2)$$

$$x^1(t)^T(\lambda^T f_{x^1}(t,y^1,\dot{y}^1,x^1,\dot{x}^1)$$
$$-D\lambda^T f_{\dot{x}^1}(t,y^1,\dot{y}^1,x^1,\dot{x}^1))e\}dt$$

Subject to

$$x^1(a)=0=x^1(b), y^1(a)=0=y^1(b)$$
$$x^2(a)=0=x^2(b), y^2(a)=0=y^2(b)$$
$$\lambda^T f_{x^1}(t,y^1,\dot{y}^1,x^1,\dot{x}^1)-D\lambda^T f_{\dot{x}^1}(t,y^1,\dot{y}^1,x^1,\dot{x}^1)\leqq 0, t\in I$$
$$\lambda^T g_{x^2}(t,y^2,\dot{y}^2,x^2,\dot{x}^2)-D\lambda^T g_{\dot{x}^2}(t,y^2,\dot{y}^2,x^2,\dot{x}^2)\leqq 0, t\in I$$
$$\int_I x^2(t)^T(\lambda^T g_{x^2}(t,y^2,\dot{y}^2,x^2,\dot{x}^2)$$
$$-D\lambda^T g_{\dot{x}^1}(t,y^2,\dot{y}^2,x^2,\dot{x}^2))dt\geqq 0$$
$$\lambda\in\Lambda^+$$

This shows that the problem (Mix SD_1) is just the primal problem (Mix SP). Therefore, $(\bar{x}^1(t),\bar{x}^2(t),\bar{y}^1(t),\bar{y}^2(t),\bar{\lambda})$ is an optimal solution of (Mix SD) implies that $(\bar{y}^1(t),\bar{y}^2(t),\bar{x}^1(t),\bar{x}^2(t),\bar{\lambda})$ is an optimal solution for (Mix SP), and by symmetric duality also for (Mix SD).

Now from (55)

$$\text{Minimum (Mix SP)}=\int_I\{f(t,x^1,\dot{x}^1,y^1,\dot{y}^1)+g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)\}dt$$

Correspondingly with the solution $(\bar{y}(t),\bar{y}^2(t),\bar{x}^1(t),\bar{x}^2(t),\bar{\lambda})$, we have

$$\text{Minimum (Mix SP)}=\int_I\{f(t,y^1,\dot{y}^1,x^1,\dot{x}^1)+g(t,y^2,\dot{y}^2,x^2,\dot{x}^2)\}dt$$

By the skew symmetric of f^i and g^i, we have

$$\begin{aligned}\text{Minimum (Mix SP)} &= \int_I\{f(t,x^1,\dot{x}^1,y^1,\dot{y}^1)+g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)\}dt\\ &= \int_I\{f(t,y^1,\dot{y}^1,x^1,\dot{x}^1)+g(t,y^2,\dot{y}^2,x^2,\dot{x}^2)\}dt\\ &= -\int_I\{f(t,x^1,\dot{x}^1,y^1,\dot{y}^1)+g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)\}dt\end{aligned}$$

this yields,

$$\text{Minimum (Mix SP)} = \int_I \{f(t,x^1,\dot{x}^1,y^1,\dot{y}^1) + g(t,x^2,\dot{x}^2,y^2,\dot{y}^2)\}dt = 0$$

This accomplishes the proof of the theorem.

6. Natural Boundary Conditions

The pair of mixed symmetric multiobjective variational problem with natural boundary values rather than fixed points may be formulated as,

Primal problem (Mix SP_0)

$$\begin{aligned}\text{Minimize} \int_I \{ & f(t,x^1,\dot{x}^1,y^1,\dot{y}^1) + g(t,x^2,\dot{x}^2,y^2,\dot{y}^2). \\ & -y^1(t)^T(\lambda^T f_{y^1}(x^1,\dot{x}^1,y^1,\dot{y}^1) \\ & -D\lambda^T f_{\dot{y}^1}(x^1,\dot{x}^1,y^1,\dot{y}^1))e\}dt\end{aligned}$$

Subject to

$$\lambda^T f_{y^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) - D\lambda^T f_{\dot{y}^1}(t,x^1,\dot{x}^1,y^1,\dot{y}^1) \leqq 0$$

$$\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2) - D\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2) \leqq 0$$

$$\int_I y^2(t)^T(\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2) \\ -D\lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2)) \geqq 0$$

$$\lambda^T f_{y^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)|_{t=a} = 0,\ \lambda^T f_{\dot{y}^1}(t,u^1,\dot{u}^1,v^1,\dot{v}^1)|_{t=b} = 0$$

$$\lambda^T g_{y^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2)|_{t=a} = 0,\ \lambda^T g_{\dot{y}^2}(t,x^2,\dot{x}^2,y^2,\dot{y}^2)|_{t=b} = 0$$

$$\lambda \in \Lambda^+$$

Dual problem (Mix SD_0)

$$\begin{aligned}\text{Maximize} \int_I \{ & f(u^1,\dot{u}^1,v^1,\dot{v}^1) + g(u^2,\dot{u}^2,v^2,\dot{v}^2) \\ & -u^1(t)^T(\lambda^T f_{u^1}(u^1,\dot{u}^1,v^1,\dot{v}^1)\end{aligned}$$

$$-D\lambda^T f_{\dot{u}^1}(u^1, \dot{u}^1, v^1, \dot{v}^1))e\}dt$$

Subject to

$$\lambda^T f_{u^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1) - D\lambda^T f_{\dot{u}^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1) \geqq 0$$

$$\lambda^T g_{u^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2) - D\lambda^T g_{\dot{u}^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2) \geqq 0$$

$$\int_I u^2(t)^T(\lambda^T g_{u^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2)$$

$$-D\lambda^T g_{\dot{u}^2}(t, u^2, \dot{u}^2, v^2, \dot{v}^2))dt \leqq 0,$$

$$\lambda^T f_{\dot{x}^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1)|_{t=a} = 0,\ \lambda^T f_{\dot{x}^1}(t, u^1, \dot{u}^1, v^1, \dot{v}^1)|_{t=b} = 0$$

$$\lambda^T g_{\dot{x}^2}(t, x^2, \dot{x}^2, y^2, \dot{y}^2)|_{t=a} = 0,\ \lambda^T g_{\dot{x}^2}(t, x^2, \dot{x}^2, y^2, \dot{y}^2)|_{t=b} = 0$$

$$\lambda \in \Lambda^+$$

For these problems, Theorem 1-3 will remain true except that some slight modifications in the arguments for these theorems are to be indicated.

7. Mathematical Programming

If the time dependency of (Mix SP) and (Mix SD) is removed and $b - a = 1$, we obtain following pair of static mixed type multiobjective dual problems studied by Bector, Chandra and Abha [2].

Primal (Mix SP$_1$) Minimize $f(x^1, y^1) + g(x^2, y^2) - (y^1)^T(\lambda^T f_{y^1}(x^1, y^1)$

Subject to

$$\lambda^T f_{y^1}(x^1, y^1) \leqq 0,$$

$$\lambda^T g_{y^2}(x^2, y^2) \leqq 0,$$

$$y^2(t)^T(\lambda^T g_{y^2}(x^2, y^2)) \geqq 0,$$

$$\lambda \in \Lambda^+$$

Dual (Mix SD1) Maximize $f(u^1, v^1) + g(u^2, v^2) - u^1(t)^T(\lambda^T f_{u^1}(u^1, v^1))$

Subject to

$$\lambda^T f_{u^1}(u^1, v^1) \geqq 0,$$

$$\lambda^T g_{u^2}(u^2, v^2) \geqq 0,$$

$$(u^2)^T(\lambda^T g_{u^2}(u^2, v^2). \leqq 0,$$

$$\lambda \in \Lambda^+.$$

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