EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 2, 2020, 323-345
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On the Category of Weakly $\mathcal{U}$-Complexes

Gustina Elfiyanti[1,2,*], Intan Muchtadi-Alamsyah[1], Fajar Yuliawan[1], Dellavitha Nasution[1]

[1] *Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung, Indonesia*
[2] *Mathematics Department, Faculty of Sciences and Technology, UIN Jakarta, Jakarta, Indonesia*

Abstract. Motivated by a study of Davvaz and Shabbani which introduced the concept of $\mathcal{U}$-complexes and proposed a generalization on some results in homological algebra, we study the category of $\mathcal{U}$-complexes and the homotopy category of $\mathcal{U}$-complexes. In [8] we said that the category of $\mathcal{U}$-complexes is an abelian category. Here, we show that the object that we claimed to be the kernel of a morphism of $\mathcal{U}$-complexes does not satisfy the universal property of the kernel, hence we can not conclude that the category of $\mathcal{U}$-complexes is an abelian category. The homotopy category of $\mathcal{U}$-complexes is an additive category. In this paper, we propose a weakly chain $\mathcal{U}$-complex by changing the second condition of the chain $\mathcal{U}$-complex. We prove that the homotopy category of weakly $\mathcal{U}$-complexes is a triangulated category.

2020 Mathematics Subject Classifications: 18E05,18G35, 18G80

Key Words and Phrases: $\mathcal{U}$-complexes, weakly $\mathcal{U}$-complexes, homotopy category of weakly $\mathcal{U}$-complexes, triangulated category.

1. Introduction

The notion of $\mathcal{U}$-complexes was introduced by Davvaz and Shabani-Solt in [6] as a generalization of chain complexes of R-modules. They established some results in homological algebra such as Lambek Lemma, Snake Lemma and Connecting homomorphism and Exact Triangle. Their study was motivated by results from Freni and Sureau in [12] and Davvaz and Parnian-Garamaleky in [5]. Freni and Sureau introduced a notion of exact sequences of hypergroups by defining the kernel of a hypergroup homomorphism as the inverse image of $\mathcal{U}$ where $\mathcal{U}$ is the intersection of all ultra-closed subhypergroups of its codomain (note that a hypergroup does not always has zero element). Inspired by this, Davvaz and Parnian-Garamaleky proposed a generalization of exact sequences

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i2.3673

Email addresses: gustina.elfiyanti@uinjkt.ac.id (G. Elfiyanti), {ntan, fajar.yuliawan, dellavitha}@math.itb.ac.id (I. Muchtadi-Alamsyah, F. Yuliawan, D. Nasution)

of R-modules, called $\mathcal{U}$-exact sequences, by replacing the kernel of any differential with the preimage of a submodule $\mathcal{U}$ of its codomain. Then, Anvariyeh and Davvaz studied application of $\mathcal{U}$-exactness and $\mathcal{U}$-split exact sequences [1]. Further results on $\mathcal{U}$-exactness given by Anvariyeh and Davvaz in [2] and Madanshekaf in [16].

Recently some authors continued working on $\mathcal{U}$-exactness. Mahatma and Muchtadi-Alamsyah defined $\mathcal{U}$-projective resolutions and $\mathcal{U}$-extension modules [17]. Baur et al. then computed the $\mathcal{U}$-projective resolution of modules over kQ where Q is quiver of type A_n and $\tilde{A}_n$ [3]. Fitriani, Surojo and Wijayanti introduced X-sub-exact sequence as a generalization of $\mathcal{U}$-exact sequence [9]. By using the concept of X-sub-exact sequence, they studied X-sub-linearly independent [10]. Furthermore, the authors generalized the $\mathcal{U}$-generator and M-subgenerator related to category $\sigma[M]$ [11]. In [7] and [8], we study the category of $\mathcal{U}$-complexes and its homotopy category of $\mathcal{U}$-complexes. We proved that the category of $\mathcal{U}$-complexes and its homotopy category are additive categories.

In this article we provide a corrigendum to the result in [8] which stated that the category of $\mathcal{U}$-complexes is an abelian category. Then, we introduce a generalization of chain $\mathcal{U}$-complexes, called weakly chain $\mathcal{U}$-complexes, by changing the second condition of in the definition of the chain $\mathcal{U}$-complexes. We show that the homotopy category of weakly $\mathcal{U}$-complexes is a triangulated category.

The paper is organized as follows. In section 2, we give the definition of additive category, triangulated category and we review the category of complexes. In section 3, we recall some results in [6], [7] and [8] that will be needed in the next section. Section 4 is the central section of our paper. In this section we introduce weakly $\mathcal{U}$-complexes and show that the homotopy category of weakly $\mathcal{U}$-complexes is a triangulated category.

Convention: Throughout this paper, unless otherwise specified, we use the following notations: R denotes a ring with identity. Chain complexes and its generalizations are over R-Mod, the category of R modules. $\mathcal{C}(R), \mathcal{U}\text{-}\mathcal{C}(R), \mathcal{C}_{\mathcal{U}}(R)$ denote the category of complexes, $\mathcal{U}$-complexes and weakly $\mathcal{U}$-complexes respectively. We denote 0 and 1 for the zero and identity morphisms respectively .

2. Preliminaries

In this section we recall some basic concepts that will be needed in the following sections. For more detail we refer to [4], [13], [14], [15], [18] and [19].

2.1. Additive and Triangulated Categories

In this section we review the definition of additive category and triangulated category.

Definition 1 ([14]). *A category $\mathcal{A}$ is called an additive category if the following conditions hold:*

A1 For every pair of objects X, Y the set of morphisms $Hom_{\mathcal{A}}(X, Y)$ is an abelian group and the composition of following morphisms is bilinear over the integers.

$$Hom_{\mathcal{A}}(Y,Z) \times Hom_{\mathcal{A}}(X,Y) \to Hom_{\mathcal{A}}(X,Z) \quad (1)$$

A2 $\mathcal{A}$ *contains a zero object* 0 *(i.e for every objects* X *in* $\mathcal{A}$ *each morphism set* $Hom_{\mathcal{A}}(X,0)$ *and* $Hom_{\mathcal{A}}(0,X)$ *has precisely one element).*

A3 For every pair of objects X, Y *in* $\mathcal{A}$ *there exists a coproduct* $X \oplus Y$*.*

A category satisfying (A1) and (A2) is called a preadditive category. If $\mathcal{A}$ is a preadditive category, then by using the following proposition we can replace the condition A3 above with the existence of a biproduct in $\mathcal{A}$.

Proposition 1 ([4])**.** *Given two objects* A, B *of a preadditive category* $\mathcal{C}$*, the following conditions are equivalent:*

(i) the product (P, p_A, p_B) *of* A, B *exists;*

(ii) the coproduct (P, s_A, s_B) *of* A, B *exists;*

(iii) the biproduct (P, p_A, p_B, s_A, s_B) *of* A, B *exists, i.e. there exists an object* P *and morphisms*

$$p_A : P \longrightarrow A,\ p_B : P \longrightarrow B,\ s_A : A \longrightarrow P,\ s_B : B \longrightarrow P \tag{2}$$

with the properties

$$p_A s_A = 1,\ p_B s_B = 1,\ p_A s_B = 0,\ p_B s_A = 0 \tag{3}$$

$$s_A p_A + s_A p_B = 1 \tag{4}$$

Moreover, under these conditions $s_A = \ker\ p_B,\ s_B = \ker\ p_A,\ p_A = co\ker s_B,\ p_B = co\ker s_A$*.*

Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \longrightarrow \mathcal{T}$ be an additive automorphism. A triangle in $\mathcal{T}$ is a sequence of objects and morphism in $\mathcal{T}$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \tag{5}$$

A morphism of triangles is a triple (f, g, h) of morphisms in $\mathcal{T}$ such that the following diagram is commutative in $\mathcal{T}$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd} \tag{6}$$

The triple (f, g, h) is called an isomorphism of triangles if the morphisms f, g and h are isomorphisms in $\mathcal{T}$.

Definition 2 ([14])**.** *A triangulated category is an additive category* $\mathcal{T}$ *together with an additive automorphism* Σ*, the translation or shift functor, and a colllection of distinguished triangles satisfying the following axioms:*

TR0 *Any triangle isomorphic to a distinguised triangle is again a distinguised triangle.*

TR1 *For every object* X *in* $\mathcal{T}$, *the triangle*

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X \tag{7}$$

is a distinguised triangle.

TR2 *For every morphism* $f : X \longrightarrow Y$ *in* $\mathcal{T}$ *there is a distinguised triangle of the form*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X \tag{8}$$

TR3 *If*

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(\alpha(f)) \xrightarrow{\beta(f)} \Sigma X \tag{9}$$

is a distinguised triangle then the following rotated triangle is also a distinguised triangle.

$$Y \xrightarrow{\alpha(f)} M(\alpha(f)) \xrightarrow{\beta(f)} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \tag{10}$$

TR4 *Given distinguished triangles* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ *and* $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$ *then each commutative diagram*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X' \end{tikzcd} \tag{11}$$

can be completed to a morphism of triangles (but not necessarily uniquely).

TR5 *(Octahedral axiom) Given the following distinguised triangles*

$$\begin{gathered} X \xrightarrow{u} Y \longrightarrow Z' \longrightarrow \Sigma X \\ Y \xrightarrow{v} Z \longrightarrow X' \longrightarrow \Sigma Y \\ X \xrightarrow{vu} Z \longrightarrow Y' \longrightarrow \Sigma X \end{gathered} \tag{12}$$

then there exists a distinguished triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow \Sigma Z'$ *making the following diagram commutative*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \longrightarrow & Z' & \longrightarrow & \Sigma X \\
\downarrow 1 & & \downarrow v & & \downarrow & & \downarrow 1 \\
X & \xrightarrow{vu} & Z & \longrightarrow & Y' & \longrightarrow & \Sigma X \\
\downarrow u & & \downarrow 1 & & \downarrow & & \downarrow \Sigma u \\
Y & \xrightarrow{v} & Z & \longrightarrow & X' & \longrightarrow & \Sigma Y \\
\downarrow & & \downarrow & & \downarrow 1 & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & \Sigma Z'
\end{array} \tag{13}$$

2.2. The Category of Complexes

2.2.1. Chain Complexes

A complexes (over R-Mod) is a family $X = \left(X_n, d_n^X\right)_{n\in\mathbb{Z}}$

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \longrightarrow \cdots \tag{14}$$

where X_n are R-modules, and $d_n^X : X_n \longrightarrow X_{n-1}$ are R-modules homomorphisms such that $d_{n+1}^X d_n^X = 0$ for all $n \in \mathbb{Z}$. The morphism d_n^X is called the differential of X on degree n. A morphism $f : X \to Y$ of complexes is a family $f = (f_n)_{n\in\mathbb{Z}}$ of morphisms $h_n : X_n \longrightarrow Y_n$ such that $f_n d_{n+1}^X = d_{n+1}^Y f_{n+1}$ for all $n \in \mathbb{Z}$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{n+1} & \xrightarrow{d_{n+1}^X} & X_n & \xrightarrow{d_n^X} & X_{n-1} & \longrightarrow & \cdots \\
 & & {\scriptstyle f_{n+1}}\downarrow & & {\scriptstyle f_n}\downarrow & & {\scriptstyle f_{n-1}}\downarrow & & \\
\cdots & \longrightarrow & Y_{n+1} & \xrightarrow{d_{n+1}^Y} & Y_n & \xrightarrow{d_n^Y} & X_{n-1} & \longrightarrow & \cdots
\end{array} \tag{15}$$

The chain complexes together with morphism of complexes form a category $\mathbf{C}(R)$, the category of complexes. This category is an abelian category.

2.2.2. The Homotopy Category of Complexes

Let X and Y be two objects in $\mathbf{C}(R)$. A morphsim $f \in \mathrm{Hom}_{\mathbf{C}(R)}(X, Y)$ is called homotopic to zero (or null homotopic) if there exists a family $h = (h_n)_{n\in\mathbb{Z}}$ of morphisms $h_n : X_n \to Y_{n+1}$

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+1} \arrow[r, "d_{n+1}^X"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n^X"] \arrow[d, "f_n"'] \arrow[dl, "h_n"'] & X_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & Y_{n+1} \arrow[r, "d_{n+1}^Y"] & Y_n \arrow[r, "d_n^Y"] & X_{n-1} \arrow[r] & \cdots
\end{tikzcd} \tag{16}$$

satisfying

$$f_n = d_{n+1}^Y h_n + h_{n-1} d_n^X \text{ for all } n \in \mathbb{Z} \tag{17}$$

The morphism h is called a (chain) homotopy map. Two morphisms $f, g \in \mathrm{Hom}_{\mathbf{C}(R)}(X, Y)$ are called homotopy equivalent, denoted by $f \sim g$, if and only if $f - g$ is homotopic to zero. A complex X is called homotopic to zero if the identity morphism on X is homotopic to zero.

The homotopy relation is an equivalence relation on the class of morphism in $\mathbf{C}(R)$. Moreover if $H_t(X, Y)$ is the set of morphisms from X to Y which are homotopic to zero, then the collection of all $H_t(X, Y)$ form and ideal in $\mathbf{C}(R)$. This implies the composition of two equivalence classes (modulo homotopy) can be defined as the equivalence classes of composition of two representative morphisms from each equivalence class. The quotient category of $\mathbf{C}(R)$ modulo this ideal is called homotopy category.

Definition 3 ([15])**.** *The homotopy category of complexes, denote by $\mathbf{K}(R)$, has the same object as the category $\mathbf{C}(R)$. The morphisms in $\mathbf{K}(R)$ are the equivalence classes of morphism in $\mathbf{C}(R)$ modulo homotopy, i.e.*

$$Hom_{\mathbf{K}(R)}(X, Y) = Hom_{\mathbf{C}(R)}(X, Y) / H_t(X, Y) \tag{18}$$

and the composition of two equivalence classes (modulo homotopy) is defined as the equivalence classes of composition of two representative morphisms from each equivalence class, i.e. $\bar{g} \circ \bar{f} = \overline{g \circ f}$ for all $\bar{f} \in Hom_{\mathbf{C}(R)}(X, Y) / H_t(X, Y)$ and $\bar{g} \in Hom_{\mathbf{C}(R)}(Y, Z) / H_t(X, Y)$.

Proposition 2 ([14])**.** *The homotopy category $\mathbf{K}(R)$ is an additive category.*

2.2.3. Triangulated Structure of the Homotopy Category of Complexes

In this section we recall a method to get a triangulated structure on $\mathbf{K}(R)$. At first we need an additive automorphism on $\mathbf{K}(R)$, then we find a suitable set distinguished triangles in $\mathbf{K}(R)$. The additive automorphism can be defined on the level of the cateogry $\mathbf{C}(R)$ as follow.

Definition 4 ([14])**.** *A translation functor or (left) shift Σ in $\mathbf{C}(R)$ is defined by shifting any complex one degree to the left. More precisely, for an object $X = \left(X_n, d_n^X\right)_{n \in \mathbb{Z}}$ in $\mathbf{C}(R)$, define $\Sigma X = \left((\Sigma X)_n, d_n^{\Sigma X}\right)_{n \in \mathbb{Z}}$ with $(\Sigma X)_n = X_{n-1}$ and $d_n^{\Sigma X} = -d_{n-1}^X$. For a morphism $f : X \longrightarrow Y$ in $\mathbf{C}(R)$, set $\Sigma f = ((\Sigma f)_n)_{n \in \mathbb{Z}}$ where $(\Sigma f)_n = f_{n-1}$. This functor is an additive functor, i.e. for every pair objects X, Y in $\mathbf{C}(R)$ the map $Hom(X, Y) \longrightarrow Hom(\Sigma X, \Sigma Y)$ is a morphism of abelian groups. Moreover it is an automorphism of the category $\mathbf{C}(R)$, where the inverse is given by (right) shift.*

To find the set of distinguished triangles in $\mathbf{K}(R)$ we need the following construction of the mapping cone.

Definition 5 ([14])**.** *Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}(R)$. The mapping cone of f is the object $M(f)$ in $\mathbf{C}(R)$ defined by*

$$M(f)_n = X_{n-1} \oplus Y_n \quad \textit{and } d_n^{M(f)} = \begin{pmatrix} -d_{n-1}^X & 0 \\ f_{n-1} & d_n^Y \end{pmatrix} \tag{19}$$

The following morphisms are canonical morphisms in $\mathbf{C}(R)$

$$\alpha(f) \;:\; Y \longrightarrow M(f), \text{ where } \alpha(f) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$

$$\beta(f) \;:\; M(f) \longrightarrow \Sigma X, \text{ where } \beta(f) = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{21}$$

The morphisms above are also well-defined in $\mathbf{K}(R)$. Hence a distinguished triangle in $\mathbf{K}(R)$ can be defined as follow.

Definition 6 ([14])**.** *A standard triangle in* $\mathbf{K}(R)$ *is a sequence*

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma X \tag{22}$$

A distinguished triangle in $\mathbf{K}(R)$ *is a triangle which is isomorphic (in* $\mathbf{K}(R)$*) to a standard triangle.*

With this class of distinghuished triangles we can prove the following proposition.

Proposition 3. *[14]The homotopy category* $\mathbf{K}(R)$ *of complexes is a triangulated category.*

3. A Generalization of the Category of Complexes

In this section we review some results in [6], [7] and [8]. In the first subsection we also provide a corrigendum to [8].

3.1. The Category of $\mathcal{U}$-Complexes

A chain $\mathcal{U}$-complex (over R-Mod) is a family $X = \left(X_n, U_n^X, d_n^X\right)_{n\in\mathbb{Z}}$

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \xrightarrow{d_{n-1}^X} X_{n-2} \longrightarrow \cdots \tag{23}$$

where X_n and U_n^X are R-modules, U_n^X is a submodule of X_n, and $d_n^X : X_n \longrightarrow X_{n-1}$ are R-modules homomorphisms such that for all $n \in \mathbb{Z}$:

(i) $d_n^X d_{n+1}^X (X_{n+1}) \subseteq U_{n-1}^X$, and

(ii) $Im\left(d_n^X\right) \supseteq U_{n-1}^X$

A morphism of $\mathcal{U}$-complexes $f : X \to Y$ is a family $f = (f_n : X_n \longrightarrow Y_n)_{n\in\mathbb{Z}}$ of R-modules homomorphisms such that every rectangle commutes and $f_n\left(U_n^X\right) \subseteq U_n^Y$ for all $n \in \mathbb{Z}$. The morphism f is called an isomorphism of $\mathcal{U}$-complexes if each f_n is an R-module isomorphism and the sequence of R-module morphisms $f^{-1} = \left(f_n^{-1} : Y_n \longrightarrow X_n\right)_{n\in\mathbb{Z}}$ is also a morphism of $\mathcal{U}$-complexes. The following are examples of chain $\mathcal{U}$-complexes and morphisms of $\mathcal{U}$-complexes.

Example 1. *(i) Every chain complex is a chain $\mathcal{U}$-complex with $U_n = 0$ for all $n \in \mathbb{Z}$.*

(ii) Suppose we have the following sequence of R-modules and R-modules homomorphsim

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \longrightarrow \cdots \tag{24}$$

Then the families $X = (X_n, d_{n+1}d_{n+2}(X_{n+2}), d_n)_{n\in\mathbb{Z}}$ and $Y = (X_n, d_n(X_n), d_n)_{n\in\mathbb{Z}}$ are chain $\mathcal{U}$-complexes. A morphsim $f : X \longrightarrow Y$ defined by $f_n = 1$ is a morphism of $\mathcal{U}$-complexes, but generally it is not an isomorphism of $\mathcal{U}$-complexes.

Suppose $f = (f_n : X_n \longrightarrow Y_n)_{n\in\mathbb{Z}}$ and $g = (g_n : Y_n \longrightarrow Z_n)_{n\in\mathbb{Z}}$ are morphisms of $\mathcal{U}$-complexes, then it is clear that $gf = (g_nf_n : X_n \longrightarrow Z_n)_{n\in\mathbb{Z}}$ is also a morphism of $\mathcal{U}$-complexes. We define the category of $\mathcal{U}$-complexes, denote by $\mathcal{U}$-$\mathbf{C}(R)$, as a category whose objects are chain $\mathcal{U}$-complexes and the morphisms are morphism of $\mathcal{U}$-complexes. This category is an additive category.

In [8], we also stated that $\mathcal{U}$-$\mathbf{C}(R)$ is an abelian category by claiming the kernel of a morphism $\mathcal{U}$-complexes $f : X \longrightarrow Y$ is $K = \left(K_n, U_n^K, d_n^K\right)_{n\in\mathbb{Z}}$ with

$$K_n = \ker f_n = \{x \in X_n \mid f_n(x) = 0\}, U_n^K = \left(d_{n+1}^K d_{n+2}^K\right)(K_{n+2}) \tag{25}$$

and d_n^K is the resitriction of d_n^X on K_n. But in the following example we can see that generally it does not satisfy the universal property of kernel. Hence we can not conclude that $\mathcal{U}$-$\mathbf{C}(R)$ is an abelian category by defining the kernel as in (25).

Example 2. *Suppose X be the chain $\mathcal{U}$-complex defined by $X_0 = X_{-1} = \mathbb{Z}$ and zero otherwise, $d_0^X = 1$ and zero otherwise, $U_{-1}^X = \mathbb{Z}$ and zero otherwise. Let Y be the chain $\mathcal{U}$-complex defined by shifting X one degree to the left. If $f : X \longrightarrow Y$ is defined by $f_0 = 1$ and zero otherwise, then we have $K = X$ as follow:*

$$\begin{tikzcd}
K: \arrow[d, "k"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \subseteq \mathbb{Z} \arrow[r] \arrow[d, "1"] & 0 \arrow[d] \\
X: \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \subset \mathbb{Z} \arrow[r, "1"] \arrow[d, "1"] & \mathbb{Z} \subseteq \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
Y: & 0 \arrow[r] & 0 \subset \mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \subseteq \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd} \tag{26}$$

Let $L = X$, then $l : L \longrightarrow X$ defined by $l_{-1} = 1$ and zero otherwise is a morphism of

$\mathcal{U}$-complexes, moreover $fl = 0$.

$$
\begin{tikzcd}
 & K: \arrow[dd, bend right, "k"'] & 0 \arrow[r] \arrow[dd, bend right] & 0 \arrow[r] \arrow[dd, bend right] & 0 \arrow[r] \arrow[dd, bend right] & 0 \subset \mathbb{Z} \arrow[r] \arrow[dd, bend right] & 0 \arrow[dd, bend right] \\
L: \arrow[ur, dashed, "g"] \arrow[dr, "l"'] & 0 \arrow[r] \arrow[ur, dashed] \arrow[dr] & 0 \arrow[r] \arrow[ur, dashed] \arrow[dr] & 0 \subset \mathbb{Z} \arrow[r, "1"] \arrow[ur, dashed] \arrow[dr] & \mathbb{Z} \subseteq \mathbb{Z} \arrow[r] \arrow[ur, dashed, "1"] \arrow[dr, "1"] & 0 \arrow[ur, dashed] \arrow[dr] & \\
 & X: \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \subset \mathbb{Z} \arrow[r, "1"] \arrow[d, "1"] & \mathbb{Z} \subseteq \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & Y: & 0 \arrow[r] & 0 \subset \mathbb{Z} \arrow[r, "1"] & \mathbb{Z} \subseteq \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd} \tag{27}
$$

The morphism $g : L \longrightarrow X$ defined by $g_{-1} = 1$ and zero otherwise is the only morphism such that $kg = l$, but g is not a morphism of $\mathcal{U}$-complexes since $g\left(U_{-1}^{L}\right) = \mathbb{Z} \nsubseteq U_{-1}^{K} = 0$. Hence K is not the kernel of f.

3.2. The Homotopy Category of $\mathcal{U}$-Complexes

A morphism $f : X \longrightarrow Y$ in $\mathcal{U}$-$\mathbf{C}(R)$ is called homotopic to zero (or null homotopic) if there exists a chain homotopy map $h = (h_n : X_n \longrightarrow Y_{n+1})_{n\in\mathbb{Z}}$ such that

$$f_n = d_{n+1}^{Y} h_n + h_{n-1} d_n^{X} \quad \text{and } h_n\left(U_n^X\right) \subseteq U_{n+1}^{Y} \tag{28}$$

We call two morphisms $f, g : X \longrightarrow Y$ in $\mathcal{U}$-$\mathbf{C}(R)$ homotopic (or homotopy equivalent), if $f - g$ is null homotopic. We write $f \sim g$ if they are homotopy equivalent. The homotopy relation $\sim$ is also an equivalence relation on the class of morphisms in $\mathcal{U}$-$\mathbf{C}(R)$. Furthemore the collections of homotopy equivalence classes of morphisms of $\mathcal{U}$-complexes form an ideal in $\mathcal{U}$-$\mathbf{C}(R)$.

Lemma 1. *Suppose X and Y are any objects in $\mathcal{U}$-$\mathbf{C}(R)$. Then the collections of all*

$$H_t(X,Y) = \left\{ f \in Hom_{\mathbf{C}_{\mathcal{U}}(R)}(X,Y) \mid f \sim 0 \right\} \tag{29}$$

forms an ideal in $\mathcal{U}$-$\mathbf{C}(R)$.

Proof. Let $f, g \in H_t(X,Y)$, $\alpha \in \mathrm{Hom}_{\mathcal{U}-\mathbf{C}(R)}(Y,Z)$ and $\beta \in \mathrm{Hom}_{\mathcal{U}-\mathbf{C}(R)}(W,X)$. Suppose $r = (r_n : X_n \to Y_{n+1})_{n\in\mathbb{Z}}$ and $s = (s_n : X_n \to Y_{n+1})_{n\in\mathbb{Z}}$ be homotopy maps such that $f_n = d_{n+1}^{Y} r_n + r_{n-1} d_n^{X}$ and $g_n = d_{n+1}^{Y} s_n + s_{n-1} d_n^{X}$. Then

$$
\begin{aligned}
\beta_n (f_n - g_n)\alpha_n &= \beta_n \left(d_{n+1}^{Y} r_n + r_{n-1} d_n^{X} - d_{n+1}^{Y} s_n - s_{n-1} d_n^{X} \right) \alpha_n \\
&= \beta_n d_{n+1}^{Y} (r_n - s_n)\alpha_n + \beta_n (r_{n-1} - s_{n-1}) d_n^{X} \alpha_n \\
&= d_{n+1}^{Z} \beta_{n+1} (r_n - s_n)\alpha_n + \beta_n (r_{n-1} - s_{n-1}) \alpha_{n-1} d_{n-1}^{W}
\end{aligned}
$$

Set $t = (t_n = \beta_{n+1}(r_n - s_n)\alpha_n : W_n \to Z_{n+1})_{n\in\mathbb{Z}}$, then t_n is a homotopy map. Hence $\beta_n (f_n - g_n)\alpha_n \sim 0$.

Therefore we can define the homotopy category of chain $\mathcal{U}$-complexes as the quotient of $\mathcal{U}$-$\mathbf{C}(R)$ modulo this ideal.

Definition 7. *The homotopy category of $\mathcal{U}$-complexes, denote by $\mathcal{U}$-$\mathbf{K}(R)$, has the same object as the category $\mathcal{U}$-$\mathbf{C}(R)$. The morphisms in $\mathcal{U}$-$\mathbf{K}(R)$ are the equivalence classes of morphism in $\mathcal{U}$-$\mathbf{C}(R)$ modulo homotopy, i.e.*

$$Hom_{\mathcal{U}-\mathbf{K}(R)}(X,Y) = Hom_{\mathcal{U}-\mathbf{C}(R)}(X,Y)/H_t(X,Y) \tag{30}$$

The homotopy category of $\mathcal{U}$-complexes is also an additive category [7]. To check whether the homotopy category of $\mathcal{U}$-complexes $\mathcal{U}$-$\mathbf{K}(R)$ carries a triangulated structure, we need to construct a mapping cone in $\mathcal{U}$-$\mathbf{C}(R)$.

Let $f : X \longrightarrow Y$ be a morphism in $\mathcal{U}$-$\mathbf{C}(R)$. Suppose

$$M(f)_n = X_{n-1} \oplus Y_n,\ U_n^{M(f)} = U_{n-1}^X \oplus U_n^Y \text{ and } d_n^{M(f)} = \begin{pmatrix} -d_{n-1}^X & 0 \\ f_{n-1} & d_n^Y \end{pmatrix} \tag{31}$$

For any $(x,y) \in X_{n-1} \oplus Y_n$, observe that

$$\begin{aligned} d_n^{M(f)} d_{n+1}^{M(f)}(x,y) &= \begin{pmatrix} d_n^X d_{n-1}^X & 0 \\ d_n^Y f_n - d_n^X f_{n-1} & d_n^Y d_{n+1}^Y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &\in \begin{pmatrix} U_{n-2}^X \\ U_{n-1}^Y \end{pmatrix} = U_{n-1}^{M(f)} \end{aligned} \tag{32}$$

and

$$d_n^{M(f)}(x,y) = \begin{pmatrix} -d_{n-1}^X(x) \\ f_{n-1}(x) + d_n^Y(y) \end{pmatrix} \tag{33}$$

In the following example we note that in general $\mathrm{Im}\left(d_n^{M(f)}\right)$ does not contain $U_{n-1}^{M(f)}$. Hence we can not define the mapping cone in $\mathcal{U}$-$\mathbf{C}(R)$ as in (31).

Example 3. *Suppose we have the following morphism of chain $\mathcal{U}$-complexes*

$$\begin{tikzcd} X: \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & R \arrow[r, "1"] \arrow[d, "f_0"] & R \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y: & 0 \arrow[r] & R \arrow[r, "d_1^Y"] & R\oplus R \arrow[r, "d_0^Y"] & 0 \arrow[r] & 0 \end{tikzcd} \tag{34}$$

where $d_0^X = 1$, $f_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $d_1^Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $d_0^Y = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $U_{-1}^X = R$, $U_0^Y = R\oplus 0$. Then $M(f)$ is

$$0 \longrightarrow R\oplus R \xrightarrow{\partial} R\oplus(R\oplus R) \longrightarrow 0 \longrightarrow 0 \tag{35}$$

For any $(x,y) \in R\oplus R$, observe that

$$\partial \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix} \tag{36}$$

Since $Im(\partial) = -R\oplus(R\oplus R) \not\supseteq R\oplus(R\oplus 0) = U_0^X \oplus U_1^Y$ we conclude that $M(f)$ is not an object in $\mathcal{U}$-$\mathbf{C}(R)$.

Observe that for $M(f)$ in the construction (31) we have $d_n^{M(f)}\left(U_n^{M(f)}\right) \subseteq U_{n-1}^{M(f)}$. Furthermore for any chain $\mathcal{U}$-complex X, it also satisfies $d_n^X\left(U_n^X\right) \subseteq U_{n-1}^X$. This motivate us to define a weakly chain $\mathcal{U}$-complex.

4. A Generalization of the Category of $\mathcal{U}$-Complexes

In this section we propose a generalization of chain $\mathcal{U}$-complex, called weakly chain $\mathcal{U}$-complex. Then, we prove that the homotopy category of weakly $\mathcal{U}$-complexes carries triangulated structure.

Let $X = \left(X_n, U_n^X, d_n^X\right)_{n\in\mathbb{Z}}$ be a family of R-modules and R-modules homomorphisms where U_n^X is a submodule of X_n. We define a weakly chain $\mathcal{U}$-complex (over R-Mod) by replacing the second condition of chain $\mathcal{U}$-complex i.e $d_n(X_n) \supseteq U_{n-1}$ with $d_n(U_n) \subseteq U_{n-1}$. It is easy to check that every chain complex and chain $\mathcal{U}$-complex are weakly chain $\mathcal{U}$-complexes.

We define a morphism of weakly chain $\mathcal{U}$-complexes analog to the definition of morphism of $\mathcal{U}$-complexes, i.e. $f : X \longrightarrow Y$ is a morphism of weakly chain $\mathcal{U}$-complexes if $f = (f_n : X_n \longrightarrow Y_n)_{n\in\mathbb{Z}}$ is a family of R-modules homomorphisms such that every rectangle commutes and $f_n\left(U_n^X\right) \subseteq U_n^Y$ for all $n \in \mathbb{Z}$. We denote the category of weakly chain $\mathcal{U}$-complexes as $\mathbf{C}_{\mathcal{U}}(R)$.

Proposition 4. *The category $\mathbf{C}_{\mathcal{U}}(R)$ of weakly chain $\mathcal{U}$-complexes is an additive category.*

Proof. The structure of an abelian group of $\mathrm{Hom}_{\mathbf{C}_{\mathcal{U}}(R)}(X,Y)$ and the billinearity of composition of morphisms are inherited from $\mathrm{Hom}_{\mathbf{C}(R)}(X,Y)$. The zero object in $\mathbf{C}(R)$ is also a zero object in $\mathbf{C}_{\mathcal{U}}(R)$. A biproduct of two objects X and Y is quintuple $(X \oplus Y, p_X, p_Y, s_X, s_Y)$ where $X \oplus Y, p_X, p_Y, s_X$ and s_Y are defined as follow:

$$X \oplus Y = \left(X \oplus Y, U^{X\oplus Y}, d^{X\oplus Y}\right) = \left(X_n \oplus Y_n, U_n^{X\oplus Y},\ d_n^{X\oplus Y}\right)_{n\in\mathbb{Z}} \tag{37}$$

where

$$U_n^{X\oplus Y} = \begin{pmatrix} U_n^X \\ U_n^Y \end{pmatrix} \text{ and } d_n^{X\oplus Y} = \begin{pmatrix} d_n^X & 0 \\ 0 & d_n^Y \end{pmatrix} \tag{38}$$

$$(s_X)_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (s_Y)_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{39}$$

$$(p_X)_n = \begin{pmatrix} 1 & 0 \end{pmatrix}, (p_Y)_n = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{40}$$

Analog to the definition of homotopy equivalent in the category of $\mathcal{U}$-complexes, we call two morphisms $f, g \in \mathrm{Hom}_{\mathbf{C}_{\mathcal{U}}(R)}(X,Y)$ are homotopy equivalent if $f - g$ is homotopic to zero (or null homotopic), i.e., there exists a chain map $s = (s_n : X_n \longrightarrow Y_{n+1})_{n\in\mathbb{Z}}$ such that

$$f_n - g_n = d_{n+1}^Y s_n + s_{n-1} d_n^X \text{ and } s_n\left(U_n^X\right) \subseteq U_{n+1}^Y \tag{41}$$

We called a weakly chain $\mathcal{U}$-complex X is homotopic to zero (or null homotopic) if the identity morphism on X is homotopic to zero.

It is clear that the homotopy relation is an equivalence relation on the class of morphisms in $\mathbf{C}_{\mathcal{U}}(R)$ and the collection of homotopy equivalence classes of morphisms in $\mathbf{C}_{\mathcal{U}}(R)$ form an ideal in $\mathbf{C}_{\mathcal{U}}(R)$. We define the homotopy category $\mathbf{K}_{\mathcal{U}}(R)$ of weakly chain $\mathcal{U}$-complexes as the quotient of $\mathbf{C}_{\mathcal{U}}(R)$ modulo this ideal. Since composition and addition are well defined on the homotopy classes, it follows that $\mathbf{K}_{\mathcal{U}}(R)$ inherits the bilinear composition from $\mathbf{C}_{\mathcal{U}}(R)$. Therefore we have the following result.

Proposition 5. *The homotopy category $\mathbf{K}_{\mathcal{U}}(R)$ of weakly chain $\mathcal{U}$-complexes is an additive category.*

Next, we will show that $\mathbf{K}_{\mathcal{U}}(R)$ is a triangulated category. We construct a translation functor Σ on $\mathbf{C}_{\mathcal{U}}(R)$ analog to translator functor on $\mathbf{K}(R)$.

Definition 8. *The translation functor shift Σ of X is an object $\Sigma X = \left(\Sigma X_n, U_n^{\Sigma X}, d_n^{\Sigma X}\right)_{n\in\mathbb{Z}}$ defined by*

$$\Sigma X_n = X_{n-1},\ U_n^{\Sigma X} = U_{n-1}^{X},\ \text{and } d_n^{\Sigma X} = -d_{n-1}^{X} \tag{42}$$

and for a morphism $f = (f_n)_{n\in\mathbb{Z}}$ in $\mathbf{K}_{\mathcal{U}}(R)$ we set

$$\Sigma f = (\Sigma f_n)_{n\in\mathbb{Z}}\ \text{ where } \Sigma f_n = f_{n-1}. \tag{43}$$

The functor Σ above is an additive automorphism in $\mathbf{C}_{\mathcal{U}}(R)$. Moreover it is compatible with homotopies, hence we have a well-defined induced functor Σ on $\mathbf{K}_{\mathcal{U}}(R)$. A triangle and morphism of triangles in $\mathbf{C}_{\mathcal{U}}(R)$ is defined analog to the definition of triangle and morphism of triangles in homotopy category $\mathbf{C}(R)$ of complexes.

Lemma 2. *Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}_{\mathcal{U}}(R)$ then*

$$M(f) = \left(M(f)_n, U_n^{M(f)}, d_n^{M(f)}\right)_{n\in\mathbb{Z}} \tag{44}$$

where

$$M(f)_n = X_{n-1} \oplus Y_n,\ U_n^{M(f)} = U_{n-1}^{X} \oplus U_n^{Y},\ \text{and } d_n^{M(f)} = \begin{pmatrix} -d_{n-1}^{X} & 0 \\ f_{n-1} & d_n^{Y} \end{pmatrix} \tag{45}$$

is an object in $\mathbf{K}_{\mathcal{U}}(R)$

Proof. From (31) we know that $d_n^{M(f)} d_{n+1}^{M(f)}\left(M(f)_{n+1}\right) \subseteq U_{n-1}^{M(f)}$. Now let $(a,b) \in U_{n-1}^{X} \oplus U_n^{Y}$, note that

$$\begin{aligned} d_n^{M(f)}(a,b) &= \begin{pmatrix} -d_{n-1}^{X} & 0 \\ f_{n-1} & d_n^{Y} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} -d_{n-1}^{X}(a) \\ f_{n-1}(a) + d_n^{Y}(b) \end{pmatrix} \in U_{n-1}^{M(f)} \end{aligned} \tag{46}$$

The object $M(f)$ above is called the mapping cone of f.

Lemma 3. *The mapping cone $M(1)$ of identity morphims on X is homotopic to zero.*

Proof. The mapping cone of identity morphism on X is

$$M(1) = \left(X_{n-1} \oplus X_n, U_{n-1}^X \oplus U_n^X, d_n^{M(1)}\right)_{n\in\mathbb{Z}} \tag{47}$$

where

$$d_n^{M(1)} = \begin{pmatrix} -d_{n-1}^X & 0 \\ 1 & d_n^X \end{pmatrix} : X_{n-1} \oplus X_n \longrightarrow X_{n-2} \oplus X_{n-1} \tag{48}$$

Look at the following diagram.

$$\begin{tikzcd} \cdots \arrow[r] & X_n \oplus X_{n+1} \arrow[r, "d_{n+1}^{M(1)}"] & X_{n-1} \oplus X_n \arrow[r, "d_n^{M(1)}"] \arrow[dl, "s_n"'] \arrow[d, "1"] & X_{n-2} \oplus X_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"'] & \cdots \\ \cdots \arrow[r] & X_n \oplus X_{n+1} \arrow[r, "d_{n+1}^{M(1)}"] & X_{n-1} \oplus X_n \arrow[r, "d_n^{M(1)}"] & X_{n-2} \oplus X_{n-1} \arrow[r] & \cdots \end{tikzcd} \tag{49}$$

Suppose $s_n : X_{n-1} \oplus X_n \to X_n \oplus X_{n+1}$ is defined by $s_n = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
It is clear that $s_n\left(U_n^{M(1)}\right) \subseteq U_{n+1}^{M(1)}$ and $d_n^{M(1)}\left(U_n^{M(1)}\right) \subseteq U_{n-1}^{M(1)}$. Observe that

$$\begin{aligned} d_{n+1}^{M(1)} s_n + s_{n-1} d_n^{M(1)} &= \begin{pmatrix} -d_n^X & 0 \\ 1 & d_{n+1}^X \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -d_{n-1}^X & 0 \\ 1 & d_n^X \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{50}$$

Hence, in the homotopy category $\mathbf{K}_{\mathcal{U}}(R)$ of weakly chain $\mathcal{U}$-complexes, the identity morphism on $M(1)$ is equal to the zero map. As a censequence, in the $\mathbf{K}_{\mathcal{U}}(R)$, the mapping cone $M(1)$ is isomorphic to zero complex.

Lemma 4. *If $f : X \longrightarrow Y$ is a morphism in $\mathbf{C}_{\mathcal{U}}(R)$, then the following canonical morphisms are also morphisms in $\mathbf{C}_{\mathcal{U}}(R)$:*

$$\alpha(f) : Y \longrightarrow M(f) \ \textit{ where } \alpha(f) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{51}$$

and

$$\beta(f) : M(f) \longrightarrow \Sigma X, \ \textit{ where } \beta(f) = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{52}$$

Furthermore,

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma X \tag{53}$$

is a short exact sequence of chain complexes.

Proof. We only need to show that $\alpha(f)$ dan $\beta(f)$ satisfy the second condition of morphsim of $\mathcal{U}$-complexes. Suppose $x \in U_n^Y$ and $(v,w) \in U_n^{M(f)}$, then

$$\alpha(f)_n(x) = \begin{pmatrix} 0 \\ x \end{pmatrix} \in \begin{pmatrix} U_{n-1}^X \\ U_n^Y \end{pmatrix} = U_n^{M(f)} \tag{54}$$

and

$$\beta(f)_n \begin{pmatrix} v \\ w \end{pmatrix} = v \in U_n^{\Sigma X} \tag{55}$$

The morphisms $\alpha(f)$ and $\beta(f)$ above are also well-defined in $\mathbf{K}_{\mathcal{U}}(R)$. This bring us to the following definition.

Definition 9. *A distinguished triangle in $\mathbf{K}_{\mathcal{U}}(R)$ is a triangle which is isomorphic (in $\mathbf{K}_{\mathcal{U}}(R)$ to the following standard triangle*

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma X \tag{56}$$

We use this class of distinguished triangles to prove that the homotopy category of weakly $\mathcal{U}$-complexes has a triangulated structure.

Theorem 1. *The homotopy category $\mathbf{K}_{\mathcal{U}}(R)$ of weakly $\mathcal{U}$-complexes is a triangulated category.*

Proof. By Definition 9 and Lemma 4 it is clear that axioms (TR0) and (TR2) are satisfied. Suppose X, Y are objects in $\mathbf{K}_{\mathcal{U}}(R)$.

TR1 Consider the triangle

$$X \xrightarrow{1} X \longrightarrow M(1) \xrightarrow{\beta(f)} \Sigma X \tag{57}$$

From Lemma 3 we know that the mapping cone $M(1)$ is isomorphic to a zero object in $\mathbf{K}_{\mathcal{U}}(R)$. Hence, the following is a distinguished triangle.

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow \Sigma X \tag{58}$$

TR3 Suppose $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma X$ be a distinguised triangle. We will show that the rotated triangle

$$Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma X \xrightarrow{f} \Sigma Y \tag{59}$$

is a distinguished triangle by proving that it is isomorphic in $\mathbf{K}_{\mathcal{U}}(R)$ to the following standard triangle

$$Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\alpha(\alpha(f))} M(\alpha(f)) \xrightarrow{\beta(\alpha(f))} \Sigma Y \tag{60}$$

To construct an isomorphism between (59) and (60), we take identity map for the first, second and fourth entries.

$$
\begin{tikzcd}
Y \arrow[r, "\alpha(f)"] \arrow[d, equal] & M(f) \arrow[r, "\beta(f)"] \arrow[d, equal] & \Sigma X \arrow[r, "-\Sigma f"] \arrow[d, shift right, "\phi"'] & \Sigma Y \arrow[d, equal] \\
Y \arrow[r, "\alpha(f)"] & M(f) \arrow[r, "\alpha(\alpha(f))"] & M(\alpha(f)) \arrow[r, "\beta(\alpha(f))"] \arrow[u, shift right, "\psi"'] & \Sigma Y
\end{tikzcd}
\tag{61}
$$

and define $\phi_n = \begin{pmatrix} -f_{n-1} \\ 1 \\ 0 \end{pmatrix}$ and $\psi_n = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. First we will show that ϕ and ψ are morphisms in $\mathbf{K}_{\mathcal{U}}(R)$. Look at the following diagram

$$
\begin{tikzcd}
X_n \arrow[r, "-d_n^X"] \arrow[d, shift right] & X_{n-1} \arrow[r, "-d_{n-1}^X"] \arrow[d, shift right, "\phi_n"'] & X_{n-2} \arrow[d, shift right] \\
Y_n \oplus X_n \oplus Y_{n+1} \arrow[r, "d_{n+1}^{M(\alpha(f))}"] \arrow[u, shift right] & Y_{n-1} \oplus X_{n-1} \oplus Y_n \arrow[r, "d_n^{M(\alpha(f))}"] \arrow[u, shift right, "\psi_n"'] & Y_{n-2} \oplus X_{n-2} \oplus Y_{n-1} \arrow[u, shift right]
\end{tikzcd}
\tag{62}
$$

where

$$
d_n^{M(\alpha(f))} = \begin{pmatrix} -d_{n-1}^Y & 0 & 0 \\ 0 & -d_{n-1}^X & 0 \\ 1 & f_n & d_n^Y \end{pmatrix} \tag{63}
$$

It is clear that $\phi_n\left(U_n^{\Sigma X}\right) \subseteq U_n^{M(\alpha(f))}$ and $\psi_n\left(U_n^{M(\alpha(f))}\right) \subseteq U_n^{\Sigma X}$. Moreover

$$
\phi_n\left(-d_n^X\right) \begin{pmatrix} f_n d_n^X \\ -d_n^X \\ 0 \end{pmatrix} = d_{n+1}^{M(\alpha(f))}\phi_{n+1} \tag{64}
$$

and

$$
-d_{n-1}^X\psi_n \;=\; \begin{pmatrix} 0 & -d_{n-1}^X & 0 \end{pmatrix} = \psi_{n-1} d_n^{M(\alpha(f))} \tag{65}
$$

Now we will show that ϕ and ψ give a morphism of triangle in $\mathbf{K}_{\mathcal{U}}(R)$. Note that

$$
\beta(\alpha(f))_n \phi_n = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -f_{n-1} \\ 1 \\ 0 \end{pmatrix} = -f_{n-1} \tag{66}
$$

Hence $\beta(\alpha(f))\phi = -\Sigma f$. Observe the following diagram

$$
\begin{tikzcd}
X_n \oplus Y_{n+1} \arrow[r, "d_{n+1}^{M(f)}"] \arrow[d] & X_{n-1} \oplus Y_n \arrow[r, "d_n^{M(f)}"] \arrow[dl, "h_n"'] \arrow[d, "\phi_n\beta(f)_n"', "\alpha(\alpha(f))_n"] & X_{n-2} \oplus Y_{n-1} \arrow[d] \arrow[dl, "h_{n-1}"'] \\
Y_n \oplus X_n \oplus Y_{n+1} \arrow[r, "d_{n+1}^{M(\alpha(f))}"] & Y_{n-1} \oplus X_{n-1} \oplus Y_n \arrow[r, "d_n^{M(\alpha(f))}"] & Y_{n-2} \oplus X_{n-2} \oplus Y_{n-1}
\end{tikzcd}
\tag{67}
$$

where

$$d_{n+1}^{M(\alpha(f))}\begin{pmatrix}-d_n^Y & 0 & 0\\ 0 & -d_n^X & 0\\ 1 & f_n & d_{n+1}^Y\end{pmatrix} \text{ and } d_n^{M(f)}=\begin{pmatrix}-d_{n-1}^X & 0\\ f_{n-1} & d_n^Y\end{pmatrix} \tag{68}$$

Let

$$h_n=\begin{pmatrix}0 & -1\\ 0 & 0\\ 0 & 0\end{pmatrix} : M\left(f\right)_n \longrightarrow M\left(\alpha\left(f\right)\right)_{n+1} \tag{69}$$

then it is clear that $h_n\left(U_n^{M(f)}\right)\subseteq U_{n+1}^{M(\alpha(f))}$ and

$$\phi_n\beta\left(f\right)_n-\alpha\left(\alpha\left(f\right)\right)_n=\begin{pmatrix}-f_{n-1} & 0\\ 0 & 0\\ 0 & -1\end{pmatrix}=d_{n+1}^{M(\alpha(f))}h_{n-1}+h_nd_n^{M(f)} \tag{70}$$

Thus $\phi\beta\left(f\right)\sim\alpha\left(\alpha\left(f\right)\right)$. We also have $\beta(f)=\psi\alpha\left(\alpha(f)\right)$ since

$$\beta\left(f\right)_n-\psi_n\alpha\left(\alpha\left(f\right)\right)_n=\begin{pmatrix}1 & 0\end{pmatrix}-\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\end{pmatrix} \tag{71}$$

Now we will show $-\Sigma f\psi\sim\beta\left(\alpha\left(f\right)\right)$. Consider the folowing diagram

$$\begin{tikzcd}
Y_n\oplus X_n\oplus Y_{n+1} \arrow[r, "d_{n+1}^{M(\alpha(f))}"] \arrow[d] & Y_{n-1}\oplus X_{n-1}\oplus Y_n \arrow[r, "d_n^{M(f)}"] \arrow[dl, "g_n"'] \arrow[d, shift right, "-\Sigma f_n\psi_n"'] \arrow[d, shift left, "\beta(\alpha(f))_n"] & Y_{n-2}\oplus X_{n-2}\oplus Y_{n-1} \arrow[d] \arrow[dl] \\
Y_n \arrow[r, "-d_n^Y"] & Y_{n-1} \arrow[r, "-d_{n-1}^Y"] & Y_{n-2}
\end{tikzcd} \tag{72}$$

Let $g_n=\begin{pmatrix}0 & 0 & -1\end{pmatrix}$ then it is clear that $g_n\left(U_n^{M(\alpha(f))}\right)\subseteq U_{n+1}^{\Sigma Y}$ and

$$\begin{aligned}-\Sigma f_n\psi_n-\beta\left(\alpha\left(f\right)\right)_n &= \begin{pmatrix}-1 & -f_{n-1} & 0\end{pmatrix}\\ &= -d_n^Yg_n+g_{n-1}d_n^{M(\alpha(f))}\end{aligned} \tag{73}$$

We get $-\Sigma f\psi\sim\beta\left(\alpha\left(f\right)\right)$. Next, we will show that $\psi\phi=1$ and $\phi\psi\sim 1$. Note that

$$\psi\phi=\begin{pmatrix}0 & 1 & 0\end{pmatrix}\begin{pmatrix}-\Sigma f\\ 1\\ 0\end{pmatrix}=1 \tag{74}$$

Let

$$p_n : M\left(\alpha\left(f\right)\right)_n \longrightarrow M\left(\alpha\left(f\right)\right)_{n+1}=\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{75}$$

Then it is clear that $p_n\left(U_n^{M(\alpha(f))}\right) \subseteq U_{n+1}^{M(\alpha(f))}$ and

$$\phi_n\psi_n - 1 = \begin{pmatrix} -1 & -f_{n-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = d_{n+1}^{M(\alpha(f))}p_{n-1} + p_n d_n^{M(f)} \tag{76}$$

Thus $\phi\psi \sim 1$. So the following is a distinguished triangle.

$$Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma \xrightarrow{-\Sigma f} \Sigma Y \tag{77}$$

TR4 Suppose we have a diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "\alpha(u)"] \arrow[d, "g"] & M(u) \arrow[r, "\beta(u)"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "\alpha(u')"] & M(u') \arrow[r, "\beta(u')"] & \Sigma X' \end{tikzcd} \tag{78}$$

where the left square commutes in $\mathbf{K}_{\mathcal{U}}(R)$, i.e. there exist homotopy map $s_n : X_n \longrightarrow Y'_{n+1}$ such that $g_n u_n - u'_n f_n = d_{n+1}^{Y'} s_n + s_{n-1}d_n^X$ and $s_n\left(U_n^X\right) \subseteq U^{Y'_{n+1}}$ for all $n \in \mathbb{Z}$. Define

$$h = (h_n) : M(u) \longrightarrow M(u') \text{ where } h_n = \begin{pmatrix} f_{n-1} & 0 \\ s_{n-1} & g_n \end{pmatrix} \tag{79}$$

Observe that

$$h_n\alpha(u)_n = \begin{pmatrix} 0 \\ g_n \end{pmatrix} = \alpha(u')\, g_n \tag{80}$$

and

$$\beta(u')_n h_n = \begin{pmatrix} f_{n-1} & 0 \end{pmatrix} = (\Sigma f)_n \beta(u)_n \tag{81}$$

and for any $(a,b) \in U_n^{M(u)} = U_{n-1}^X \oplus U_n^Y$ we have

$$h\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_{n-1}(a) \\ g_n(b) + s_{n-1}(a) \end{pmatrix} \in \begin{pmatrix} U_{n-1}^{X'} \\ U_n^{Y'} \end{pmatrix} = U_n^{M(u')} \tag{82}$$

Hence (f,g,h) is a morphism of triangle in $\mathbf{K}_{\mathcal{U}}(R)$.

TR5 Assume that we have the following diagram in $\mathbf{K}_{\mathcal{U}}(R)$.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "\alpha(u)"] \arrow[d, "v"] & M(u) \arrow[r, "\beta(u)"] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "vu"] \arrow[d, "u"'] & Z \arrow[r, "\alpha(vu)"] \arrow[d, equal] & M(vu) \arrow[r, "\beta(vu)"] & \Sigma X \arrow[d, "\Sigma u"] \\ Y \arrow[r, "v"] \arrow[d, "\alpha(u)"'] & Z \arrow[r, "\alpha(v)"] \arrow[d, "\alpha(vu)"] & M(v) \arrow[r, "\beta(v)"] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma\alpha(u)"] \\ M(u) & M(vu) & M(v) & \Sigma M(u) \end{tikzcd} \tag{83}$$

We define the missing morphisms as follows.

$$f \ : \ M(u) \longrightarrow M(vu) \ \text{ where } f_n = \begin{pmatrix} 1 & 0 \\ 0 & v_n \end{pmatrix} \tag{84}$$

$$g \ : \ M(vu) \longrightarrow M(v) \ \text{ where } g_n = \begin{pmatrix} u_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{85}$$

$$h \ : \ M(v) \longrightarrow \Sigma M(u) \ \text{ where } h_n = \Sigma\alpha(u)\beta(v) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{86}$$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "\alpha(u)"] \arrow[d, "v"] & M(u) \arrow[r, "\beta(u)"] \arrow[d, red, "f"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "vu"] \arrow[d, "u"'] & Z \arrow[r, "\alpha(vu)"] \arrow[d, equal] & M(vu) \arrow[r, "\beta(vu)"] \arrow[d, red, "g"] & \Sigma X \arrow[d, "\Sigma u"] \\
Y \arrow[r, "v"] \arrow[d, "\alpha(u)"'] & Z \arrow[r, "\alpha(v)"] \arrow[d, "\alpha(vu)"] & M(v) \arrow[r, "\beta(v)"] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma\alpha(u)"] \\
M(u) \arrow[r, red, "f"] & M(vu) \arrow[r, red, "g"] & M(v) \arrow[r, red, "h"] & \Sigma M(u)
\end{tikzcd} \tag{87}$$

It easy to check that $f_n\left(U_n^{M(u)}\right) \subseteq U_n^{M(vu)}$, $g\left(U_n^{M(vu)}\right) \subseteq U_n^{M(v)}$ and $h_n\left(U_n^{M(v)}\right) \subseteq U_n^{\Sigma M(v)}$. Moreover

$$f_n\alpha(u)_n - \alpha(vu)_n v_n = \begin{pmatrix} 1 & 0 \\ 0 & v_n \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{88}$$

$$\beta(vu)_n f_n - 1\beta(u)_n = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & v_n \end{pmatrix} - 1\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \tag{89}$$

$$g_n\alpha(vu)_n - \alpha(v)_n = \begin{pmatrix} u_{n-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{90}$$

$$\beta(v)_n g_n - u_{n-1}\beta(vu)_n = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} u_{n-1} & 0 \\ 0 & 1 \end{pmatrix} - u_{n-1}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \tag{91}$$

Hence (f, g, h) is a morphism of triangles in $\mathbf{K}_{\mathcal{U}}(R)$. Now we need to show that the bottom line

$$M(u) \xrightarrow{f} M(vu) \xrightarrow{g} M(v) \xrightarrow{h} \Sigma M(u) \tag{92}$$

is a distinguished triangle di $\mathbf{K}_{\mathcal{U}}(R)$. For this we construct an isomorphism to the standard triangle

$$M(u) \xrightarrow{f} M(vu) \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \Sigma M(u) \tag{93}$$

Since only the third entries in triangles are different, it suffices to find morphisms $\sigma : M(v) \longrightarrow M(f)$ and $\tau : M(f) \longrightarrow M(v)$ such that the diagrams commute in

$\mathbf{K}_{\mathcal{U}}(R)$, i.e. $\beta(f)\sigma = h,\ h\tau = \beta(f),\ \sigma g = \alpha(f)$ and $\tau\alpha(f) = g$, up to homotopy. Moreover, we have to show that they are isomorphisms in $\mathbf{K}_{\mathcal{U}}(R)$. Set

$$\sigma_n = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \tau_n = \begin{pmatrix} 0 & 1 & u_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{94}$$

Look at the following diagram

$$\begin{tikzcd} M(u) \arrow[r, "f"] \arrow[d, equal] & M(vu) \arrow[r, "g"] \arrow[d, equal] & M(u) \arrow[r, "h"] \arrow[d, shift right, "\sigma"'] & \Sigma M(u) \arrow[d, equal] \\ M(u) \arrow[r, "f"] & M(vu) \arrow[r, "\alpha(f)"] & M(f) \arrow[r, "\beta(f)"] \arrow[u, shift right, "\tau"'] & \Sigma M(u) \end{tikzcd} \tag{95}$$

By definition we can check that $\sigma_n\left(U_n^{M(v)}\right) \subseteq U_n^{M(f)}$ and $\tau_n\left(U_n^{M(f)}\right) \subseteq U_n^{M(v)}$. Moreover

$$\tau_n\alpha(f)_n - g_n = \begin{pmatrix} 0 & 1 & u_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} u_{n-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{96}$$

and

$$\beta(f)_n\sigma_n - h_n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus $\tau\alpha(f) = g$ and $\beta(f)\sigma = h$. Next we will show that $\alpha(f) \sim \sigma g$. Consider the following diagram.

$$\begin{tikzcd} & X_n \oplus Z_{n+1} \arrow[r, "d_{n+1}^{M(vu)}"] \arrow[d] & X_{n-1} \oplus Z_n \arrow[r, "d_n^{M(vu)}"] \arrow[dl, "s_n"'] \arrow[d, "\alpha(f))_n"', shift right] \arrow[d, "\sigma_n g_n", shift left] & X_{n-2} \oplus Z_{n-1} \arrow[d] \arrow[dl, "s_{n-1}"'] \\ & X_{n-1} \oplus Y_n \oplus X_n \oplus Z_{n+1} \arrow[r, "d_{n+1}^{M(f)}"] & X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n \arrow[r, "d_n^{M(f)}"] & X_{n-3} \oplus Y_{n-2} \oplus X_{n-2} \oplus Z_{n-1} \end{tikzcd} \tag{97}$$

Note that

$$d_n^{M(vu)} = \begin{pmatrix} -d_{n-1}^X & 0 \\ (vu)_{n-1} & d_n^Z \end{pmatrix} \tag{98}$$

$$d_n^{M(f)} = \begin{pmatrix} d_{n-2}^X & 0 & 0 & 0 \\ -u_{n-2} & -d_{n-1}^Y & 0 & 0 \\ 1 & 0 & -d_{n-1}^X & 0 \\ 0 & v_{n-1} & (vu)_{n-1} & d_n^Z \end{pmatrix} \tag{99}$$

Define $r_n : M(vu)_n \to M(f)_{n+1}$ by $r_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $r_n\left(U_n^{M(vu)}\right) \subseteq U_{n+1}^{M(f)}$ and

$$\begin{aligned}\alpha(f)_n - \sigma_n g_n &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -u_{n-1} & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} = d_{n+1}^{M(f)} r_n + r_{n-1} d_n^{M(vu)} \end{aligned} \tag{100}$$

Therefore we obtain $\alpha(f) \sim \sigma g$.

Now we will show that $\beta(f) \sim h\tau$. Look at the following diagram

$$\begin{tikzcd}
X_{n-1} \oplus Y_n \oplus X_n \oplus Z_{n+1} \arrow[r, "d_{n+1}^{M(f)}"] \arrow[d] & X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n \arrow[r, "d_n^{M(f)}"] \arrow[dl, "s_n"'] \arrow[d, "\beta f_n"', "h_n \tau_n"] & X_{n-3} \oplus Y_{n-2} \oplus X_{n-2} \oplus Z_{n-1} \arrow[dl, "s_{n-1}"'] \arrow[d] \\
X_{n-1} \oplus Y_n \arrow[r, "d_{n+1}^{\Sigma M(u)}"] & X_{n-2} \oplus Y_{n-1} \arrow[r, "d_n^{\Sigma M(u)}"] & X_{n-3} \oplus Y_{n-2}
\end{tikzcd} \tag{101}$$

with

$$d_n^{\Sigma M(u)} = \begin{pmatrix} -d_{n-2}^X & 0 \\ u_{n-2} & d_{n-1}^Y \end{pmatrix} \tag{102}$$

Let $s_n = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ then it is clear that $s_n\left(U_n^{M(f)}\right) \subseteq U_{n+1}^{\Sigma M(u)}$ and

$$\begin{aligned}\beta(f)_n - h_n\tau_n &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & u_{n-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -u_{n-1} & 0 \end{pmatrix} = d_{n+1}^{\Sigma M(u)} s_n + s_{n-1} d_n^{M(f)} \end{aligned} \tag{103}$$

Hence we have $\beta(f) \sim h\tau$.

Last we will show that τ and σ are isomorphisms in the homotopy category $\mathbf{K}_{\mathcal{U}}(R)$. Note that

$$\tau_n\sigma_n - 1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{104}$$

Let $t_n : M(f)_n \longrightarrow M(f)_{n+1}$ defined by

$$t_n = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{105}$$

then it is clear that $t_n\left(U_n^{M(f)}\right) \subseteq U_{n+1}^{M(f)}$ and

$$\sigma_n\tau_n - 1_n = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & u_{n-1} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = d_{n+1}^{M(f)}t_n + t_{n-1}d_n^{M(f)}$$

Thus $\tau\sigma = 1$ and $\sigma\tau \sim 1$ which mean that σ and τ are isomorphism of triangle in $\mathbf{K}_{\mathcal{U}}(R)$. Hence, $M(u) \xrightarrow{f} M(vu) \xrightarrow{g} M(v) \xrightarrow{h} \Sigma M(u)$ is a distinguished triangle in $\mathbf{K}_{\mathcal{U}}(R)$ and we have proved the octahedral axiom for $\mathbf{K}_{\mathcal{U}}(R)$.

5. Conclusion

Category of $\mathcal{U}$-complexes is a generalization of category of complexes defined by replacing the objects with chain $\mathcal{U}$-complexes and the morphisms with morphisms of $\mathcal{U}$-complexes. It is an additive category. The homotopy category of $\mathcal{U}$-complexes is also an additive category.

Let $X = \left(X_n, U_n^X, d_n^X\right)_{n\in\mathbb{Z}}$ be a chain $\mathcal{U}$-complex, then $d_n^X\left(U_n^X\right) \subseteq U_{n-1}^X$. We introduce a weakly chain $\mathcal{U}$-complex by replacing the second condition of chain $\mathcal{U}$-complex with $d_n^X\left(U_n^X\right) \subseteq U_{n-1}^X$. The category of weakly $\mathcal{U}$-complexes is again an additive category and its homotopy category is a triangulated category.

Every chain complex is a chain $\mathcal{U}$-complex with $U_n = 0 forall n \in mathbbZ$. From the first and the second condition of chain $\mathcal{U}$-complex we know that chain $\mathcal{U}$-complexes is also a weakly $\mathcal{U}$-complexes.

Acknowledgements

The first author wish to thank Alexander Zimmermann for many discussions during her visit to Université de Picardie Jules Verne, France, in 2015 and 2017. She also thanks LPDP for funding her study in ITB. The authors thank ITB for supported this research. We are also grateful to the referees and the editor for valuable remarks which contributed to the improvement of the paper.

References

[1] SM. Anvariyeh and B. Davvaz. U-split-exact sequences. *Far East Journal of Mathematical Sciences*, 4(2):209–220, 2002.

[2] SM. Anvariyeh and B. Davvaz. On quasi-exact sequences. *Bulletin of the Korean Mathematical Society*, 42(1):149–155, 2005.

[3] K. Baur, Y. Mahatma, and I. Muchtadi-Alamsyah. The U-projective resolution of modules over path algebras of types A_n and $\tilde{A}_n$. *Communications of the Korean Mathematical Society*, 34(3):701–718, 2019.

[4] F. Borceux. *Handbook of Categorical Algebra: Volume 2, Categories and Structures*, volume 50. Cambridge University Press, 1994.

[5] B. Davvaz and Y.A. Parnian-Garamaleky. A note on exact sequences. *Bulletin of the Malaysian Mathematical Sciences Society*, 22(1), 1999.

[6] B. Davvaz and H. Shabani-Solt. A generalization of homological algebra. *Journal of the Korean Mathematical Society*, 39(6):881–898, 2002.

[7] G. Elfiyanti. Sifat aditif kategori homotopi kompleks-U. *LOGIK@*, 6(1):62–70, 2016.

[8] G. Elfiyanti, I. Muchtadi-Alamsyah, D. Nasution, and U. Amartiwi. Abelian property of the category of U-complexes. *International Journal of Mathematical Analysis*, 10(17):849–853, 2016.

[9] Fitriani, B. Surodjo, and I.E. Wijayanti. On sub-exact sequences. *Far East Journal of Mathematical Sciences*, 100:1055–65, 2016.

[10] Fitriani, B. Surodjo, and I.E Wijayanti. On X-sub-linearly independent modules. In *Journal of Physics: Conference Series*, volume 893, page 012008. IOP Publishing, 2017.

[11] Fitriani, I.E Wijayanti, and B. Surodjo. Generalization of U-generator and M-subgenerator related to category $\sigma[M]$. *Journal of Mathematics Research*, 10(4):1055–65, 2016.

[12] D. Freni and Y. Sureau. Hypergroupes de type-u et homologie de complexes. *Algebra Universalis*, 35(1):34–62, 1996.

[13] S.I. Gelfand and Y.I. Manin. *Methods of Homological Algebra*. Springer Science & Business Media, 2013.

[14] T. Holm, P. Jørgensen, and R. Rouquier. *Triangulated Categories*, volume 375. Cambridge University Press, 2010.

[15] S. König and A. Zimmermann. *Derived Equivalences for Group Rings*, volume 1685. Springer, 2006.

[16] A. Madanshekaf. Quasi-exact sequence and finitely presented modules. *Iranian Journal of Mathematical Sciences and Informatics*, 3(2):49–53, 2008.

[17] Y. Mahatma and I. Muchtadi-Alamsyah. Construction of U-extension module. In *AIP Conference Proceedings*, volume 1867, page 020025. AIP Publishing LLC, 2017.

[18] C.A. Weibel. *An Introduction to Homological Algebra.* Number 38. Cambridge University Press, 1995.

[19] A. Zimmermann. Representation theory. *Springer, Amiens*, 5(9):11, 2014.