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On Multi Poly-Genocchi Polynomials with Parameters a, b and c

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Abstract. Most identities of Genocchi numbers and polynomials are related to the well-known Benoulli and Euler polynomials. In this paper, multi poly-Genocchi polynomials with parameters a, b and c are defined by means of polylogarithm in multiple paramaters. Several properties of these polynomials are established including some recurrence relations and explicit formulas.

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1. Introduction

The Genocchi numbers and polynomials can be traced back to Angelo Genocchi (1817-1889). Genocchi numbers have been extensively studied in many different contexts in mathematics. For instance, Genocchi numbers have been studied by several authors in the context of Apostol-type polynomials, Hermite-type polynomials, polylogarithm, and their q-analogues [3, 6–8, 13, 20, 21, 23, 27, 29, 30]. Many studies and literature provide relations of Genocchi numbers to Bernoulli and Euler numbers, especially Euler numbers. Bernoulli, Euler and Genocchi numbers defined by exponential generating function (see [1, 19, 22])

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}, \quad |t| < 2\pi \tag{1}$$

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{2}{e^t + 1}, \quad |t| < \pi \tag{2}$$

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$$\sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = \frac{2t}{e^t+1}, \quad |t| < \pi. \tag{3}$$

The Bernoulli, Euler and Genocchi polynomials are defined via generating functions to be, respectively,

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t-1} e^{xt}, \quad |t| < 2\pi \tag{4}$$

$$\sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{xt}, \quad |t| < \pi \tag{5}$$

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t+1} e^{xt}, \quad |t| < \pi, \tag{6}$$

where, when $x = 0$, $B_n(0) = B_n$, $E_n(0) = E_n$ and $G_n(0) = G_n$. (see [5, 19, 22, 24])

Araci [19] and Kim et al. [26] did some researches on the so-called Genocchi polynomials of higher order arising from Genocchi basis, which were defined by

$$\left(\frac{2t}{e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!}. \tag{7}$$

The main objective of their studies is to derive interesting identities on (7) using a new method constructed by Kim et al. [11].

Moreover, Araci and He [7, 19, 30] introduced the Apostol-Genocchi polynomials as an extension of the Genocchi polynomials, which were defined by

$$\frac{2t}{\lambda e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x,\lambda) \frac{t^n}{n!}.$$

Based on this, Araci [23] introduced Apostol-Genocchi polynomials of higher order which is also called the generalized Apostol-Genocchi polynomials of order $k \in C$,

$$\left(\frac{2t}{\lambda e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda) \frac{t^n}{n!}, |t| < \pi \text{ when } \lambda = 1 \text{ and } \tag{8}$$
$$|t| < |\log(-\lambda)| \text{ when } \lambda \neq 1; \lambda \in \mathbb{C}.$$

In [15], Lim defined the degenerated Genocchi polynomials $G_n^{(k)}(x,\lambda)$ of order k to be

$$\left(\frac{2t}{(1+\lambda t)^{1/\lambda}}\right)^k (1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda) \frac{t^n}{n!}.$$

Besides these generalizations, Araci [20], Duran et al. [29] and Agyuz et al. [6] also introduced the q-analogue of the Genocchi polynomials as follows,

$$\sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!} = t \int_{z_p} q^{-y} e^{t[y+x]_q} d\mu_{-q}(y),$$

where

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q}.$$

This definition is constructed by p-adic fermionic q-integral on $\mathbb{Z}_p$ with respect to μ_{-q}. It can also be defined by

$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} = [2]_q t \sum_{m=0}^{\infty} (-1)^m e^{t[m+x]_q}.$$

In which when we take $x = 0$, it becomes $G_{n,q}(0) := G_{n,q}$, which we call it the nth q-Genocchi number.

When it comes to Genocchi numbers, the most common thing that comes to our mind is to determine the relationship between Genocchi numbers, Bernoulli numbers and Euler numbers [22]. Indeed, most researches on Genocchi numbers concern the relations between these three kinds of numbers [3, 4, 12, 22]. In other words, there are many literatures that provide identities on these three kinds of numbers. Similarly, when it comes to Genocchi polynomials, the most common thing is to establish relationship between Genocchi polynomials, Bernoulli polynomials and Euler polynomials [2–4, 8, 12, 22, 30].

Another form of generalization of Bernoulli polynomials was introduced by Kaneko [10]. This generalization was defined in terms of the following *kth* polylogarithm $\mathrm{Li}_k(z)$:

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}. \tag{9}$$

where $k \in \mathbb{Z}$ and $z \in \mathbb{C}$ with $|z| < 1$ which can be extended to $z \geq 1$ by the process of analytic continuation. When $z = 1$, the *kth* polylogarithm gives the Riemann zeta function. That is,

$$\mathrm{Li}_k(1) = \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

Also, when $k = 1$, the $1st$ polylogarithm yields the natural logarithmic function as follows:

$$\mathrm{Li}_1(z) = -\ln(1-z).$$

This special case of the polylogarithm motivates the construction of poly-Bernoulli numbers in the sense that

$$\mathrm{Li}_1(1-e^{-x}) = x.$$

The poly-Bernoulli numbers $B_n^{(k)}$ were defined by Kaneko [10] as

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t - 1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}. \tag{10}$$

Parallel to this, Kim et al. [27] defined poly-Genocchi polynomials as follows

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}. \tag{11}$$

Note that, when $x = 0$, (11) reduces to

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^t+1} = \sum_{n=0}^{\infty} G_n^{(k)}\frac{t^n}{n!}. \tag{12}$$

where $G_n^{(k)}$ are called the poly-Genocchi numbers. Moreover, they defined a modified poly-Genocchi polynomials, denoted by $G_{n,2}^{(k)}(x)$, as follows

$$\frac{\mathrm{Li}_k(1-e^{-2t})}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,2}^{(k)}(x)\frac{t^n}{n!}. \tag{13}$$

Note that

$$G_n^{(1)} := G_{n,2}^{(1)} := G_n(x).$$

Kim et al. [27] obtained several properties of these polynomials.

On the other hand, Kurt [13] defined two forms of generalized poly-Genocchi polynomials with parameters a, b, and c, as follows

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x;a,b,c)\frac{t^n}{n!} \tag{14}$$

$$\frac{2\mathrm{Li}_k(1-(ab)^{-2t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_{n,2}^{(k)}(x;a,b,c)\frac{t^n}{n!}, \tag{15}$$

which are motivated by the definitions in (11) and (13), respectively. Kurt [13] also derived several properties parallel to those of poly-Genocchi polynomials by Kim et al. [27].

The followings are some relations between poly-Bernoulli and poly-Genocchi numbers and polynomials; poly-Genocchi numbers, Euler number and Stirling numbers of the second kind; and modified poly-Bernoulli and poly-Genocchi polynomials:

$$nB_{n-1}^{(k)} = \frac{1}{2}nG_{n-1}^{(k)} + \sum_{m=0}^{n}\binom{n}{m}B_m G_{n-m}^{(k)} \tag{16}$$

$$B_{n,2}^{(k)} - 2^{n+1}B_n^{(k)} = \frac{1}{2}G_{n,2}^{(k)} \tag{17}$$

$$2nG_{n-1}^{(k)} - 2\sum_{m=0}^{n}\binom{n}{m}G_m G_{n-m}^{(k)} = \sum_{m=0}^{n}\binom{n}{m}G_m(1)G_{n-m}^{(k)}, \tag{18}$$

$$\sum_{m=0}^{n}\binom{n}{m}B_m^{(k)}(x)B_{n-m}^{(k)}(y)=\sum_{p=0}^{n}\binom{n}{p}B_p^{(k)}B_{n-p}^{(k)}(x+y), \tag{19}$$

and

$$G_{n,2}^{(k)}=2^{n+1}\left(B_n^{(k)}\left(x+\frac{1}{2}\right)-B_n^{(k)}(x)\right). \tag{20}$$

Moreover, using the generating function of the poly-Genocchi numbers and Stirling numbers of the second kind, we have

$$\begin{aligned}\sum_{n=0}^{\infty}G_n^{(k)}\frac{t^n}{n!}&=\frac{2}{e^t+1}\sum_{m=1}^{\infty}\frac{(-1)^m(e^{-t}-1)^m}{m^k}\\&=\sum_{m=1}^{\infty}\frac{(-1)^m}{m^k}\frac{2}{e^t+1}m!\sum_{l=0}^{\infty}S_2(l,m)(-1)^l\frac{t^l}{l!}\\&=\sum_{m=1}^{\infty}\frac{(-1)^m}{m^k}\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}m!\sum_{l=0}^{\infty}S_2(l,m)(-1)^l\frac{t^l}{l!}\\&=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^m}{m^k}E_nm!S_2(l,m)(-1)^l\frac{t^{n+l}}{n!l!}.\end{aligned}$$

Replacing $n+l$ with l, we get

$$\begin{aligned}\sum_{n=0}^{\infty}G_n^{(k)}\frac{t^n}{n!}&=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{l=n}^{\infty}\frac{(-1)^m}{m^k}E_nm!S_2(l-n,m)(-1)^{l-n}\frac{t^l}{n!(l-n)!}\frac{l!}{l!}\\&=\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{l=n}^{\infty}\binom{l}{n}\frac{(-1)^{m+l-n}}{m^k}E_nm!S_2(l-n,m)\frac{t^l}{l!}\\&=\sum_{l=0}^{\infty}\left\{\sum_{n=0}^{l}\sum_{m=1}^{\infty}\binom{l}{n}\frac{(-1)^{m+l-n}}{m^k}E_nm!S_2(l-n,m)\right\}\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n}\sum_{m=1}^{\infty}\binom{n}{r}\frac{(-1)^{m+n-r}}{m^k}E_rm!S_2(n-r,m)\right\}\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficient of $\frac{t^n}{n!}$, we obtain

$$G_n^{(k)}=\sum_{r=0}^{n}\binom{n}{r}\left\{\sum_{m=1}^{\infty}\frac{(-1)^{m+n-r}}{m^k}E_rm!S_2(n-r,m)\right\}. \tag{21}$$

The multi poly-Bernoulli numbers was first introduced by Imatomi et al. [9] using the concept of multiple polylogarithm also known as multiple zeta values, which is given by

$$\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(z)=\sum_{0<m_1<m_2<\ldots<m_r}\frac{z^{m_r}}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}. \tag{22}$$

When $r = 1$, (22) yields

$$\mathrm{Li}_{k_1}(z) = \sum_{m_1>0} \frac{z^{m_1}}{m_1^{k_1}},$$

which is exactly (9). The multi poly-Bernoulli numbers defined by Imatomi et al. [9] is given by

$$\frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!}.$$

These numbers possess the following recurrence relation and explicit formula

$$B_n^{(k_1,k_2,\ldots,k_r)} = \frac{1}{n+1}\left(B_n^{(k_1-1,k_2,\ldots,k_r)} - \sum_{m=1}^{n-1}\binom{n}{m-1} B_m^{(k_1,k_2,\ldots,k_r)}\right)$$

$$B_n^{(k_1,k_2,\ldots,k_r)} = (-1)^n \sum_{n+1\geq m_1>m_2>\ldots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!S(n,m_1-1)}{m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r}}.$$

Parallel to the above generalization is the generalized multi poly-Bernoulli polynomials which are denoted by $B_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$. These polynomials have been introduced in [18] by means of the above multiple poly-logarithm. More precisely, we have

$$\frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}-b^t)^r} c^{rxt} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}. \tag{23}$$

When $r = 1$, (23) boils down to the generalized poly-Bernoulli polynomials with three parameters a, b, c. Moreover, when $c = e$, (23) reduces to the multi poly-Bernoulli polynomials with two parameters a, b. These special cases have been discussed intensively in [18]. On the other hand, the generalized multi poly-Euler polynomials were also defined in [17] by means of multiple poly-logarithm.

This paper intends to investigate multi poly-Genocchi polynomials with parameters a, b and c.

2. Multi Poly-Genocchi Polynomials with Parameters a, b and c

In this section, using the concept of multiple polylogarithm, we introduce the multi poly-Genocchi polynomials with parameters a, b and c. Some properties of these polynomials are established parallel to those of the poly-Genocchi polynomials with parameters a, b and c.

Definition 2.1. The multi poly-Genocchi polynomials with parameters a, b and c are defined by

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!} = \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-2t})}{(a^{-t}+b^t)^r} c^{rxt} \tag{24}$$

When $c = e$, equation (24) reduces to

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,e)\frac{t^n}{n!} = \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-2t})}{(a^{-t}+b^t)^r}e^{rxt}.$$

For convenience, we use $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ to denote $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,e)$. That is,

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)\frac{t^n}{n!} = \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-2t})}{(a^{-t}+b^t)^r}e^{rxt}. \tag{25}$$

Furthermore, if we put $a = 1, b = e$ in (25), then this will reduce to

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;1,e)\frac{t^n}{n!} = \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-e^{-2t})}{(1+e^t)^r}e^{rxt}.$$

We use $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x)$ to denote $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;1,e)$. That is,

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x)\frac{t^n}{n!} = \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-e^{-2t})}{(1+e^t)^r}e^{rxt}. \tag{26}$$

When $x = 0$, equation (25) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(0;a,b)\frac{t^n}{n!} = \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-2t})}{(a^{-t}+b^t)^r}.$$

We use $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(a,b)$ to denote $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(0;a,b)$.

The following theorem is given without proof since it follows from [16, Theorems 2.1-2.3].

Theorem 2.2. *The generalized poly-Genocchi polynomials satisfy the relations*

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = (r(\ln a+\ln b))^n \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}\left(\frac{x\ln c+\ln a}{\ln ab}\right) \tag{27}$$

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{i=0}^{\infty}\binom{n}{i}(r\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(a,b)x^{n-i} \tag{28}$$

$$\frac{d}{dx}\mathcal{G}_{n+1}^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = (n+1)(r\ln c)\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c). \tag{29}$$

Equation (29) contains a differential identity that can be used to classify generalized poly-Genocchi polynomials as Appell polynomials [14, 25, 28]. When $c = e^{1/r}$, equation (29) reduces to

$$\frac{d}{dx}\mathcal{G}_{n+1}^{(k_1,k_2,\ldots,k_r)}(x;a,b,e^{1/r}) = (n+1)\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,e^{1/r}), \tag{30}$$

which is one of the property for the polynomial to be classified as Appell polynomial. Hence, the generalized poly-Genocchi polynomials $\mathcal{G}_n^{(k)}(x;a,b)$ must possess the following properties

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,e^{1/r}) = \sum_{i=0}^{n}\binom{n}{i}c_i x^{n-i}$$
$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,e^{1/r}) = \left(\sum_{i=0}^{\infty}\frac{c_i}{i!}D^i\right)x^n,$$

for some scalar $c_i \neq 0$. It is then necessary to find the sequence $\{c_n\}$. However, using equation (28), $c_i = \mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(a,b)$, which implies the following corollary.

Corollary 2.3. *The generalized poly-Genocchi polynomials satisfy the following formula*

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,e^{1/r}) = \left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(a,b)}{i!}D^i\right)x^n.$$

For example, when $n=3$, we have

$$\mathcal{G}_3^{(k_1,k_2,\ldots,k_r)}(x;a,b,e^{1/r}) = \left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(a,b)}{i!}D^i\right)x^3$$

$$\begin{aligned}&= \frac{\mathcal{G}_0^{(k_1,k_2,\ldots,k_r)}(a,b)}{0!}x^3 + \frac{\mathcal{G}_1^{(k_1,k_2,\ldots,k_r)}(a,b)}{1!}D^1x^3 + \frac{\mathcal{G}_2^{(k_1,k_2,\ldots,k_r)}(a,b)}{2!}D^2x^3\\ &\quad + \frac{\mathcal{G}_3^{(k_1,k_2,\ldots,k_r)}(a,b)}{3!}D^3x^3\\ &= \mathcal{G}_0^{(k)}(a,b)x^3 + 3\mathcal{G}_1^{(k_1,k_2,\ldots,k_r)}(a,b)x^2 + 3\mathcal{G}_2^{(k_1,k_2,\ldots,k_r)}(a,b)x + \mathcal{G}_3^{(k_1,k_2,\ldots,k_r)}(a,b).\end{aligned}$$

The following theorem contains the addition formula for $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$.

Theorem 2.4. *The generalized poly-Genocchi polynomials satisfy the following addition formula*

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+y;a,b,c) = \sum_{i=0}^{\infty}\binom{n}{i}(r\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)y^{n-i}$$

.

Proof. Using Definition 2.1,

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+y;a,b,c)\frac{t^n}{n!} = \frac{L_{i_k}(1-(ab)^{-2t})^r}{(a^{-t}+b^t)}c^{(x+y)rt}$$

$$= \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-2t})^r}{(a^{-t}+b^t)} c^{xrt} c^{yrt}$$

$$= \left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(yr\ln c)^n\frac{t^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty}\binom{n}{i}(yr\ln c)^{n-i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\right)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired result.

When $y=1$, Theorem 2.4 yields the following recurrence relation

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+1;a,b,c) = \sum_{m=0}^{n}\binom{n}{m}(r\ln c)^m \mathcal{G}_{n-m}^{(k_1,k_2,\ldots,k_r)}(x;a,b,c). \tag{31}$$

The following corollary immediately follows from equation (36) and the characterization of Appell polynomials [14, 25, 28].

Corollary 2.5. *The generalized poly-Genocchi polynomials satisfy the following addition formula*

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x+y;a,b,e^{1/r}) = \sum_{i=0}^{\infty}\binom{n}{i}\mathcal{G}_i^{(k_1,k_2,\ldots,k_r)}(x;a,b,e^{1/r})y^{n-i}. \tag{32}$$

Taking $x=0$ in formula (32) and using the fact $\mathcal{G}_n^{(k)}(0;a,b,c)=\mathcal{G}_n^{(k)}(a,b)$, Theorem 2.4 gives formula (28).

The next theorem contains an expression of generalized poly-Genocchi polynomials in terms of multiple parameters poly-Bernoulli polynomials.

Theorem 2.6. *The of generalized poly-Genocchi polynomials satisfy the relation*

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{m=0}^{r}\binom{r}{m}(-1)^m B_n^{(k_1,k_2,\ldots,k_r)}\left(\frac{rx\ln c+(r+m)\ln a+m\ln b}{2(\ln a+\ln b)}\right)(2\ln ab)^n. \tag{33}$$

Proof. We can rewrite equation (24) as

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!} = \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-2t})}{(1-(ab)^{2t})^r}(e^{-t\ln a}-e^{t\ln b})^r e^{rxt\ln c}e^{2rt\ln a}$$

$$= \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-e^{-2t(\ln ab)})}{(-1)^r(e^{2t\ln(ab)}-1)^r}\sum_{m=0}^{r}\binom{r}{m}(-1)^m e^{(r-m)t(-\ln a+x\ln c+2\ln a)}e^{mt(\ln b+x\ln c+2\ln a)}$$

$$= \sum_{m=0}^{r} \binom{r}{m} (-1)^m \frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-e^{-2t(\ln ab)})}{(-1)^r(e^{2t\ln(ab)}-1)^r} e^{((rx\ln c+(r+m)\ln a+m\ln b)/2\ln ab)(2t\ln ab)}.$$

Using the definition of poly-Bernoulli polynomials, we have

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{r} \binom{r}{m} (-1)^m B_n^{(k_1,k_2,\ldots,k_r)} \left(\frac{rx\ln c + (r+m)\ln a + m\ln b}{2(\ln a + \ln b)} \right) 2^n (\ln a + \ln b)^n \right\} \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (33).

The next two theorems are given without proof since it follows from [16, Theorem 3.4 and Theorem 2.6].

Theorem 2.7. *The generalized multi-poly-Genocchi polynomials have the following explicit formula*

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$$
$$= \sum_{i=0}^{n} \sum_{\substack{0\le m_1\le m_2\le\cdots\le m_r \\ c_1+c_2+\cdots=r}} \sum_{j=0}^{m_r} \frac{(rx\ln c - 2j\ln ab)^{n-i} r!(-1)^{j+s}(s\ln ab + r\ln a)^i \binom{m_r}{j}\binom{n}{j}}{(c_1!c_2!\cdots)(m_1^{k_1}m_2^{k_2}\cdots m_k^{k_r})} \tag{34}$$

where $s = c_1 + 2c_2 + \cdots$

Theorem 2.8. *The generalized poly-Genocchi polynomials* $\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b)$ *satisfy the following explicit formulas*

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(-m\ln c;a,b)(x)^{(m)} \tag{35}$$

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k_1,k_2,\ldots,k_r)}(a,b)(x)_m \tag{36}$$

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{l=0}^{n}\sum_{m=0}^{n-l} \binom{n}{l} \left\{ {l+s \atop s} \right\} \frac{\binom{n-l}{m}}{\binom{l+s}{s}} \mathcal{G}_{n-l-m}^{(k_1,k_2,\ldots,k_r)}(a,b) B_m^{(s)}(x\ln c) \tag{37}$$

$$\mathcal{G}_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{m=0}^{\infty} \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^{s} \binom{s}{j} (-\lambda)^{s-j} \mathcal{G}_{n-m}^{(k_1,k_2,\ldots,k_r)}(j;a,b) H_m^{(s)}(x;\lambda), \tag{38}$$

where $(x)^{(n)} = x(x+1)\cdots(x+n-1), (x)_n = x(x-1)\cdots(x-n+1),$

$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x)\frac{t^n}{n!} \quad \textit{and} \quad \left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x;\lambda)\frac{t^n}{n!}.$$

3. Symmetrized Generalization

In this section, we will consider the symmetrized generalization of multi poly-Genocchi polynomials with parameters a, b and c.

Definition 3.1. *For $m, n \geq 0$, we define the symmetrized generalization of multi poly-Genocchi polynomials with parameters a, b and c as follows*

$$\mathcal{S}_n^{(m)}(x,y;a,b,c) = \sum_{k_1+k_2+\ldots+k_r=m} \binom{m}{k_1,k_2,\ldots k_r} \frac{\mathcal{G}_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n} \left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}. \tag{39}$$

The following theorem contains the double generating function for $\mathcal{S}_n^{(m)}(x,y;a,b,c)$.

Theorem 3.2. *For $n, m \geq 0$, we have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{S}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!} = \frac{e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t} e^{\binom{r}{2}u+2(r-1)t}(1-e^{-2t})^{r-1}}{(1+e^t)^{r-1}\prod_{i=1}^{r-1}(e^{2t}+e^{iu}-e^{2t+iu})}. \tag{40}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathcal{S}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k_1+k_2+\ldots+k_r=m} \frac{\mathcal{G}_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}\frac{t^n}{n!}\times \\
&\quad \times\frac{u^m}{k_1!k_2!\ldots k_r!} \\
&= \sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_r\geq 0} \frac{\mathcal{G}_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}\frac{t^n}{n!}\times \\
&\quad \times\frac{u^{k_1+k_2+\ldots+k_r}}{k_1!k_2!\ldots k_r!}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0} \frac{\mathcal{G}_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\sum_{k_r\geq 0}\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}\frac{u^{k_r}}{k_r!}\times \\
&\quad \times\frac{t^n}{n!}\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!} \\
&= e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}\sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0} \frac{\mathcal{G}_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\frac{t^n}{n!}\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!}
\end{aligned}$$

Using identity (26), we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$\begin{aligned}
&= e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0}\sum_{n=0}^{\infty}\mathcal{G}_n^{(-k_1,-k_2,\ldots -k_{r-1})}\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)\frac{t^n}{n!}\times\\
&\quad\times\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!}\\
&= e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0}\frac{\mathrm{Li}_{(-k_1,-k_2,\ldots,-k_{r-1})}(1-e^{-2t})}{(1+e^t)^{r-1}}\times\\
&\quad\times\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!}\\
&= \frac{e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}}{(1+e^t)^{r-1}}\sum_{0<m_1<m_2<\ldots<m_{r-1}}(1-e^{-2t})^{m_{r-1}}\mathcal{L}(u,m_1,\ldots,m_{r-1})
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}(u,m_1,\ldots,m_{r-1}) &= \sum_{k_1+\ldots+k_{r-1}\geq 0}\frac{(um_1)^{k_1}\ldots(um_{r-1})^{k_{r-1}}}{k_1!\ldots k_{r-1}!}\\
&= \sum_{\widehat{m}\geq 0}\frac{1}{\widehat{m}!}\sum_{k_1+k_2+\ldots+k_{r-1}=\widehat{m}}\binom{\widehat{m}}{k_1,\ldots k_{r-1}}(um_1)^{k_1}\ldots(um_{r-1})^{k_{r-1}}\\
&= \sum_{\widehat{m}\geq 0}\frac{(um_1+\ldots+um_{r-1})^{\widehat{m}}}{\widehat{m}!}\\
&= e^{u(m_1+\ldots+m_{r-1})}.
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$\begin{aligned}
&= \frac{e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}}{(1+e^t)^{r-1}}\sum_{0<m_1<m_2<\ldots<m_{r-1}}(1-e^{-2t})^{m_{r-1}}e^{u(m_1+\ldots+m_{r-1})}\\
&= \frac{e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}}{(1+e^t)^{r-1}}\times
\end{aligned}$$

$$\times \frac{e^u(1-e^{-2t})}{1-e^u(1-e^{-2t})} \frac{e^{2u}(1-e^{-2t})}{1-e^{2u}(1-e^{-2t})} \cdots \frac{e^{(r-1)u}(1-e^{-2t})}{1-e^{(r-1)u}(1-e^{-2t})}$$

$$= \frac{e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t} e^{\binom{r}{2}u}(1-e^{-2t})^{r-1}}{(1+e^t)^{r-1}\prod_{i=1}^{r-1}(1-e^{iu}(1-e^{-2t}))}$$

$$= \frac{e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u} e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t} e^{\binom{r}{2}u+2(r-1)t}(1-e^{-2t})^{r-1}}{(1+e^t)^{r-1}\prod_{i=1}^{r-1}(e^{2t}+e^{iu}-e^{2t+iu})}.$$

4. Conclusion

This paper introduces certain generalization of poly-Genocchi polynomials, called multi poly-Genocchi polynomials, using the concept of multiple polylogarithm and explore some interesting properties and identities which are analogous to those of the multi-poly-Euler polynomials and multi-poly-Bernoulli polynomials. One of these is the differential identity that helps classify the multi poly-Genocchi polynomials as an Appell polynomial, which implies some interesting relations. Moreover, the multi poly-Genocchi polynomials are expressed in terms of multiple parameters poly-Bernoulli polynomials. This paper is concluded by introducing the symmetrized generalization of multi poly-Genocchi polynomials and by deriving its double generating function.

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