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Some univalence conditions of a certain general integral operator

Camelia Bărbatu[1,*], Daniel Breaz[2]

[1] *Department of Mathematics, Babe Bolyai University, Cluj-Napoca, Romania*
[2] *Department of Exact Science and Engineering, 1 Decembrie 1918 University, Alba Iulia, Romania*

Abstract. For some classes of analytic functions f, g, h and k in the open unit disk $\mathbb{U}$, we consider the general integral operator $\mathcal{T}_n$, that was introduced in a recent work [1] and we obtain new conditions of univalence for this integral operator. The key tools in the proofs of our results are the Pascu's and the Pescar's univalence criteria, as well as the Mocanu's and erb's Lemma. Some corollaries of the main results are also considered. Relevant connections of the results presented here with various other known results are briefly indicated.

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1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : | z |< 1\}$$

*Corresponding author.
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Email addresses: camipode@yahoo.com (C. Bărbatu), dbreaz@uab.ro (D. Breaz)

and satisfy the following usual normalization conditions:

$$f(0) = f^{'}(0) - 1 = 0,$$

$\mathbb{C}$ being the set of complex numbers.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ said to be in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$ in $\mathbb{U}$, if it satisfies the following inequality:

$$\mathrm{Re}\left[\frac{zf^{'}(z)}{f(z)}\right] > \alpha, \quad z \in \mathbb{U}.$$

.

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}(\lambda)$, $0 \leq \lambda < 1$, if

$$\mathrm{Re}\left[f^{'}(z)\right] > \lambda, \quad z \in \mathbb{U}.$$

Frasin and Jahangiri [14] studied the class $\mathcal{B}(\mu, \lambda)$, $\mu \geq 0$, $0 \leq \lambda < 1$, which consists of functions $f \in \mathcal{A}$ that satisfy the following conditions:

$$\left|f^{'}(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1 - \lambda, \quad z \in \mathbb{U}. \tag{2}$$

.

This class $\mathcal{B}(\mu, \lambda)$ is a comprehensive class of normalized analytic functions in $\mathbb{U}$. For instance, we have $\mathcal{B}(1, \lambda) = \mathcal{S}^*(\lambda)$, $\mathcal{B}(0, \lambda) = \mathcal{R}(\lambda)$ and $\mathcal{B}(2, \lambda) = \mathcal{B}(\lambda)$. In particular, the analytic and univalent function class $\mathcal{B}(\lambda)$ was studied by Frasin and Darus [13].

We consider the integral operator

$$\mathcal{T}_n(z) = \left\{\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \left(g_i^{\prime}(t)\right)^{\beta_i} \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \left(\frac{{h_i}'(t))}{{k_i}'(t)}\right)^{\delta_i}\right] \mathrm{dt}\right\}^{\frac{1}{\delta}}, \tag{3}$$

where f_i, g_i, h_i, k_i are analytic in $\mathbb{U}$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$ for all $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$.

Remark 1. *The integral operator $\mathcal{T}_n$ defined by (3), introduced by Bărbatu and Breaz in the paper [1] is a general integral operator of Pfaltzgraff, Kim-Merkes and Oversea types which extends also the other operators as follows:*

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [15].

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\alpha} dt,$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [27].

$$\mathcal{G}_\alpha(z) = \int_0^z \left(f'(t)\right)^\alpha dt,$$

iii) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [3].

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [23].

iv) For $\alpha_i - 1 = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [4]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left[f_i'(t)\right]^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [26] .

v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = \delta_i = 0$ we obtain the integral operator which was studied by Ularu in [28]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left({g_i}'(t)\right)^{\beta_i} dt\right]^{\frac{1}{\delta}}.$$

vi) For $\alpha_i - 1 = \beta_i = 0$, $k_i(z) = z$ and $k_i'(z) = 1$ we obtain the integral operator which was defined and studied by Pescar [25]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left({f_i}'(t)\right)^{\beta_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Frasin in [12] and by Oversea in [20].

vii) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by Pescar [25]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} \left(\frac{f_i'(t)}{g_i'(t)}\right)^{\delta_i} dt\right]^{\frac{1}{\delta}}.$$

viii) For $\delta = 1$, $\alpha_i - 1 = \gamma_i = 0$, $\beta_i = \delta_i$ and $h_i(z) = \frac{z^2}{2}$ we obtain the integral operator which was defined and studied by Bucur and Breaz in [6]

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^n \left[\frac{tg_i'(t)}{k_i'(t)}\right]^{\beta_i} dt,$$

this integral operator is a generalization of the integral operator introduced by Bucur, Andrei and Breaz in [10] and [11].

xi) For $\delta = 1$, $\alpha_i - 1 = \delta_i = 0$, $\beta_i = \gamma_i$ and $h_i(z) = f_i(z)$ we obtain the integral operator which was defined and studied by Nguyen, Oprea and Breaz in [18]

$$\mathcal{H}_{n,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{h_i(t)} g_i^{'}(t) \right)^{\alpha_i} dt.$$

Thus, the integral operator $\mathcal{T}_n$, introduced here by the formula (3), can be considered as an extension and a generalization of these operators above mentioned.

The following univalence condition was derived by Pascu.

Theorem 1. *(Pascu [22]) Let $\delta \in \mathbb{C}$ with $Re\delta > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1 - |z|^{2Re\delta}}{Re\delta} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then, for any complex γ with $Re\gamma \geq Re\delta$, the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)\, dt \right)^{\frac{1}{\gamma}},$$

is in the class $\mathcal{S}$.

Pescar, on the other hand, proved another univalent condition asserted by Theorem 2.

Theorem 2. *(Pescar [25]) Let γ be complex number, $Re\gamma > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$, $f(z) = z + a_2z^2 + \ldots$. If*

$$\left| c\, |z|^{2\gamma} + \left(1 - |z|^{2\gamma}\right) \frac{zf''(z)}{\gamma f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then the integral operator

$$F_\gamma(z) = \left(\gamma \int_0^z t^{\gamma-1} f'(t)\, dt \right)^{frac1\gamma},$$

is in the class $\mathcal{S}$.

Mocanu and erb proved the next Theorem.

Theorem 3. *(Mocanu - erb [17]) Let $M_0 = 1,5936...$ the positive solution of equation*

$$(2 - M)\, e^M = 2. \tag{4}$$

If $f \in \mathcal{A}$ and

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0,$$

for $z \in \mathbb{U}$, then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1, \qquad (z \in \mathbb{U})$$

The bound M_0 is sharp.

Finally, in our present investigation, we shall also need the familiar Schwarz Lemma.

Lemma 1. *(General Schwarz Lemma [16]) Let f be the function regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R, R > 0\}$ with $|f(z)| < M$ for a fixed number $\mathrm{M} > 0$ fixed. If $f(z)$ has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} z^m, \quad z \in \mathbb{U}_R.$$

The equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

The problem of univalence for some generalized integral operators using functions from the class $\mathcal{B}(\mu, \lambda)$ were recently obtained in papers[5], [7],[8], [11], [19].

2. Main results

Our main results give sufficient conditions for the general integral operator $\mathcal{T}_n$ defined by (3) to be univalent in the open disk $\mathbb{U}$.

Theorem 4. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ and M_i, N_i, P_i, Q_i, R_i, $S_i \geq 1$, $i = \overline{1, n}$, such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} \left\{ |\alpha_i - 1| \left[1 + (2 - \lambda_i) M_i^{\mu_i - 1} \right] + |\gamma_i| \left[2 + (2 - \eta_i) P_i^{\nu_i - 1} \right] \right\} +$$

$$+ (2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} |\gamma_i| (2 - \rho_i) Q_i^{\theta_i - 1} + 2c \sum_{i=1}^{n} \left[|\beta_i| N_i + |\delta_i| (R_i + S_i) \right] \leq c (2c+1)^{\frac{2c+1}{2c}}. \quad (5)$$

If $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i \in \mathcal{A}$, $h_i \in \mathcal{B}(\nu_i, \eta_i)$, $k_i \in \mathcal{B}(\theta_i, \rho_i)$, satisfies

$$|f_i(z)| < M_i, \ \left| \frac{g_i''(z)}{g_i'(z)} \right| \leq N_i, \ |h_i(z)| < P_i, \ |k_i(z)| < Q_i, \ \left| \frac{h_i''(z)}{h_i'(z)} \right| \leq R_i, \ \left| \frac{k_i''(z)}{k_i'(z)} \right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$, then for every δ, $Re\delta \geq Re\gamma$, the function $\mathcal{T}_n$, defined by (3) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$T_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i'(t) \right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t)}{k_i{}'(t)} \right)^{\delta_i} \right] \mathrm{dt},$$

for all $f_i, g_i, h_i, k_i \in \mathcal{A}$, $i = \overline{1, n}$.

The function T_n is regular in $\mathbb{U}$ and satisfies the following normalization condition $T_n(0) = T_n'(0) - 1 = 0$.

After we calculate the first-order and second-order derivatives, we obtain

$$\frac{zT_n''(z)}{T_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right] +$$

$$+ \sum_{i=1}^{n} \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right].$$

Therefore

$$\left| \frac{zT_n''(z)}{T_n'(z)} \right| \leq \sum_{i=1}^{n} \left(|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{z{g_i}''(z)}{{g_i}'(z)} \right| \right) +$$

$$+ \sum_{i=1}^{n} \left\{ |\gamma_i| \left[\left(\left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \right) + \left(\left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \right) \right] + |\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right) \right\}. \tag{6}$$

Thus, clearly, we find from this last inequality (6) that

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zT_n''(z)}{T_n'(z)} \right| \leq \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left(\left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) + |\beta_i| \left| \frac{zg_i^{''}(z)}{g_i(z)} \right| \right] +$$

$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left\{ |\gamma_i| \left(\left| \frac{zh_i'(z)}{h_i(z)} \right| + \left| \frac{zk_i'(z)}{k_i(z)} \right| + 2 \right) + |\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right) \right\} \leq$$

$$\leq \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} \left[|\alpha_i - 1| \left(\left| f_i^{'}(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \left| \frac{f_i(z)}{z} \right|^{\mu_i - 1} + 1 \right) + |\beta_i| \, |z| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \right] +$$

$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} |\gamma_i| \left(\left| h_i^{'}(z) \left(\frac{z}{h_i(z)} \right)^{\nu_i} \right| \left| \frac{h_i(z)}{z} \right|^{\nu_i - 1} + \left| k_i^{'}(z) \left(\frac{z}{k_i(z)} \right)^{\theta_i} \right| \left| \frac{k_i(z)}{z} \right|^{\theta_i - 1} + 2 \right) +$$

$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} |\delta_i| \left(|z| \left| \frac{h_i''(z)}{h_i'(z)} \right| + |z| \left| \frac{k_i''(z)}{k_i'(z)} \right| \right).$$

By applying the General Schwarz Lemma to the functions f_i, h_i, k_i, $i = \overline{1,n}$ we obtain

$$|f_i(z)| \leq M_i |z|, \quad |h_i(z)| \leq P_i |z|, \quad |k_i(z)| \leq Q_i |z|.$$

Next, using the hypothesis, we obtain:

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zT_n''(z)}{T_n'(z)} \right| \leq \frac{1 - |z|^{2c}}{c} \sum_{i=1}^{n} |\alpha_i - 1| \left(\left| f_i^{'}(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) M_i^{\mu_i - 1} +$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left\{|\beta_i|\,|z|\,N_i+|\gamma_i|\left[(2-\eta_i)\,P_i^{\nu_i-1}+(2-\rho_i)\,Q_i^{\theta_i-1}+2\right]\right\}+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left\{|\delta_i|\,|z|\,(R_i+S_i)\right\}. \tag{7}$$

Since

$$\max_{|z|\le 1}\frac{\left(1-|z|^{2c}\right)|z|}{c}=\frac{2}{(2c+1)^{\frac{2c+1}{2c}}},$$

we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zT_n''(z)}{T_n'(z)}\right|\le\frac{1}{c}\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[1+(2-\lambda_i)\,M_i^{\mu_i-1}\right]+|\gamma_i|\left[1+(2-\eta_i)\,P_i^{\nu_i-1}\right]\right\}+$$

$$+\frac{1}{c}\sum_{i=1}^{n}|\gamma_i|\left[1+(2-\rho_i)\,Q_i^{\theta_i-1}\right]+\frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\beta_i|\,N_i+|\delta_i|\,(R_i+S_i)\right]. \tag{8}$$

If we make use of (5), the last inequality yields

$$\frac{1-|z|^{2c}}{c}\left|\frac{zT_n''(z)}{T_n'(z)}\right|\le 1$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$.

Finally, we apply Theorem 1, we conclude that, the general integral operator $\mathcal{T}_n$ given by (3) is in the class $\mathcal{S}$.

Theorem 5. *Let $c,\delta,\alpha_i,\beta_i,\gamma_i,\delta_i\in\mathbb{C}$, $Re\delta>0$ and $M_i,N_i,P_i,Q_i,R_i,S_i\ge 1$, $i=\overline{1,n}$. Suppose that $f_i\in\mathcal{B}(\mu_i,\lambda_i)$, $g_i\in\mathcal{A}$, $h_i\in\mathcal{B}(\nu_i,\eta_i)$, $k_i\in\mathcal{B}(\theta_i,\rho_i)$, satisfies*

$$|f_i(z)|<M_i,\ \left|\frac{zg_i''(z)}{g_i'(z)}\right|<N_i,\ |h_i(z)|<P_i,\ |k_i(z)|<Q_i,\ \left|\frac{zh_i''(z)}{h_i'(z)}\right|<R_i,\ \left|\frac{zk_i''(z)}{k_i'(z)}\right|<S_i,$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$.

If

$$Re\delta\ge\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[(2-\lambda_i)\,M_i^{\mu_i-1}+1\right]+|\beta_i|\,N_i\right\}+$$

$$+\sum_{i=1}^{n}\left\{|\gamma_i|\left[(2-\eta_i)\,P_i^{\nu_i-1}+(2-\rho_i)\,Q_i^{\theta_i-1}+2\right]+|\delta_i|\,(R_i+S_i)\right\} \tag{9}$$

and

$$|c|\le 1-\frac{1}{Re\delta}\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[(2-\lambda_i)\,M_i^{\mu_i-1}+1\right]+|\beta_i|\,N_i\right\}-$$

$$-\frac{1}{Re\delta}\sum_{i=1}^{n}\left\{|\gamma_i|\left[(2-\eta_i)\,P_i^{\nu_i-1}+(2-\rho_i)\,Q_i^{\theta_i-1}+2\right]+|\delta_i|\,(R_i+S_i)\right\} \tag{10}$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$, then the function $\mathcal{T}_n$, defined by (3) is in the class $\mathcal{S}$.

Proof. Just as in the proof of Theorem 2.1, we have

$$\left|\frac{zT_n''(z)}{T_n'(z)}\right|\le\sum_{i=1}^{n}\left(|\alpha_i-1|\left|\frac{zf_i'(z)}{f_i(z)}-1\right|+|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right)+$$

$$+\sum_{i=1}^{n}\left\{|\gamma_i|\left[\left(\left|\frac{zh_i'(z)}{h_i(z)}-1\right|\right)+\left(\left|\frac{zk_i'(z)}{k_i(z)}-1\right|\right)\right]+|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right\}.$$

So, for a given constant $c\in\mathbb{C}$, we obtain

$$\left|c\,|z|^{2\mathrm{Re}\delta}+\left(1-\left|z^{2\delta}\right|\right)\frac{zT_n''(z)}{\delta T_n'(z)}\right|\le|c|+\frac{1}{|\delta|}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\beta_i|\left|\frac{zg_i''(z)}{g_i(z)}\right|\right]+$$

$$+\frac{1}{|\delta|}\sum_{i=1}^{n}\left\{|\gamma_i|\left[\left(\left|\frac{zh_i'(z)}{h_i(z)}\right|+1\right)+\left(\left|\frac{zk_i'(z)}{k_i(z)}\right|+1\right)\right]+|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right\}\le$$

$$\le|c|+\frac{1}{|\delta|}\sum_{i=1}^{n}|\alpha_i-1|\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right|\left|\frac{f_i(z)}{z}\right|^{\mu_i-1}+1\right)+$$

$$+\frac{1}{|\delta|}\sum_{i=1}^{n}\left[|\beta|\left|\frac{zg_i''(z)}{g_i'(z)}\right|+|\gamma_i|\left(\left|h_i'(z)\left(\frac{z}{h_i(z)}\right)^{\mu_i}\right|\left|\frac{h_i(z)}{z}\right|^{\mu_i-1}+1\right)\right]+$$

$$+\frac{1}{|\delta|}\sum_{i=1}^{n}\left\{|\gamma_i|\left(\left|k_i'(z)\left(\frac{z}{k_i(z)}\right)^{\nu_i}\right|\left|\frac{k_i(z)}{z}\right|^{\nu_i-1}+1\right)+|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right\}. \tag{11}$$

Now, applying the General Schwarz Lemma to the functions f_i, h_i, k_i, $i=\overline{1,n}$ we obtain

$$|f_i(z)|\le M_i\,|z|\,,\quad |h_i(z)|\le P_i\,|z|\,,\quad |k_i(z)|\le Q_i\,|z|\,, \tag{12}$$

Using the hypothesis and (12) in inequality (11), we have

$$\left|c\,|z|^{2\mathrm{Re}\delta}+\left(1-\left|z^{2\delta}\right|\right)\frac{zT_n''(z)}{\delta T_n'(z)}\right|\le$$

$$\le|c|+\frac{1}{|\delta|}\sum_{i=1}^{n}|\alpha_i-1|\left[\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}-1\right|+1\right)M_i^{\mu_i-1}+1\right]+$$

$$+\frac{1}{|\delta|}\sum_{i=1}^{n}\left\{|\gamma_i|\left[\left(\left|h_i'(z)\left(\frac{z}{h_i(z)}\right)^{\nu_i}-1\right|+1\right)P_i^{\nu_i-1}+1\right]+|\beta_i|\,N_i\right\}$$

$$+\frac{1}{|\delta|}\sum_{i=1}^{n}\left\{|\gamma_i|\left[\left(\left|k_i^{'}(z)\left(\frac{z}{k_i(z)}\right)^{\theta_i}-1\right|+1\right)Q_i^{\theta_i-1}+1\right]+|\delta_i|\left(R_i+S_i\right)\right\}\leq$$

$$\leq|c|+\frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[\left(2-\lambda_i\right)M_i^{\mu_i-1}+1\right]+|\beta_i|\,N_i\right\}+$$

$$+\frac{1}{\mathrm{Re}\delta}\sum_{i=1}^{n}\left\{|\gamma_i|\left[\left(2-\eta_i\right)P_i^{\nu_i-1}+1+\left(2-\rho_i\right)Q_i^{\theta_i-1}+1\right]+|\delta_i|\left(R_i+S_i\right)\right\}.$$

Finally, by applying Theorem 2 to the function T_n, we deduce that function $\mathcal{T}_n$ given by (3) is in the class $\mathcal{S}$.

Theorem 6. *Let $\delta,\alpha_i,\beta_i,\gamma_i,\delta_i\in\mathbb{C}$, $c=Re\delta>0$, M_0 the positive solution of the equation (4), $M_0=1,5936...$ and $f_i\in\mathcal{B}\left(\mu_i,\lambda_i\right)$, $g_i,h_i,k_i\in\mathcal{A}$ for all $z\in\mathbb{U}$, $i=\overline{1,n}$. Suppose also that*

$$|f_i\left(z\right)|<M_i,\qquad\left|\frac{g_i^{''}(z)}{g_i^{'}(z)}\right|<M_0,\qquad\left|\frac{h_i^{''}(z)}{h_i^{'}(z)}\right|<M_0,\qquad\left|\frac{k_i^{''}(z)}{k_i^{'}(z)}\right|<M_0,$$

where M_i are positive real numbers. If

$$\frac{1}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(2-\lambda_i\right)M_i^{\mu_i-1}+2\left|\gamma_i\right|\right]+\frac{2}{\left(2c+1\right)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\beta_i|\,M_0+2\left|\delta_i\right|M_0\right]\leq1,\quad(13)$$

then the function $\mathcal{T}_n$, defined by (3) is in the class $\mathcal{S}$.

Proof. It is easily seen that T_n is regular in $\mathbb{U}$.
Therefore, we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zT_n''(z)}{T_n'(z)}\right|\leq\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right|+1\right)+|\beta_i|\left|\frac{zg_i^{''}(z)}{g_i(z)}\right|\right]+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\gamma_i|\left(\left|\frac{zh_i'(z)}{h_i(z)}-1\right|+\left|\frac{zk_i'(z)}{k_i(z)}-1\right|\right)+|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right|+\left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right].$$

From hypothesis and applying Theorem 3, we have

$$\left|\frac{zh_i'(z)}{h_i(z)}-1\right|<1,\qquad\left|\frac{zk_i'(z)}{k_i(z)}-1\right|<1.$$

Also, applying the General Schwarz Lemma to the functions f_i, $i=\overline{1,n}$, we obtain

$$|f_i\left(z\right)|\leq M_i\left|z\right|.$$

Thus, we find that

$$\frac{1-|z|^{2c}}{c}\left|\frac{zT_n''(z)}{T_n'(z)}\right|\leq\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}|\alpha_i-1|\left(\left|f_i^{'}(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right|\left|\frac{f_i(z)}{z}\right|^{\mu_i-1}+1\right)+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\beta_i|\,M_0\,|z|+|\gamma_i|\,(1+1)+|\delta_i|\,(M_0\,|z|+M_0\,|z|)\right]\le$$

$$\le\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right|+1\right)M_i^{\mu_i-1}+|\beta_i|\,M_0\,|z|\right]+$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left(2\,|\gamma_i|+2\,|\delta_i|\,M_0\,|z|\right)\le$$

$$\le\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\,(2-\lambda_i)\,M_i^{\mu_i-1}+|\beta_i|\,M_0\,|z|+\left(2\,|\gamma_i|+2\,|\delta_i|\,M_0\,|z|\right)\right]. \tag{14}$$

Since

$$\max_{|z|\le 1}\frac{\left(1-|z|^{2c}\right)|z|}{c}=\frac{2}{(2c+1)^{\frac{2c+1}{2c}}}, \tag{15}$$

from (14) and (15), we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zT_n''(z)}{T_n'(z)}\right|\le\frac{1}{c}\sum_{i=1}^{n}\left[|\alpha_i-1|\,(2-\lambda_i)\,M_i^{\mu_i-1}+2\,|\gamma_i|\right]+$$

$$+\frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\beta_i|\,M_0+2\,|\delta_i|\,M_0\right]. \tag{16}$$

Using (13) from (16), we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zT_n''(z)}{T_n'(z)}\right|\le 1$$

for all $z\in\mathbb{U}$.

By Theorem Pascu it results that $\mathcal{T}_n\in\mathcal{S}$.

3. Corollaries and consequences

First of all, upon setting $\delta=1$ and $\gamma_i=0$ in Theorem 4, we immediately arrive at the following corollary:

Corollary 1. *Let $\gamma,\alpha_i,\beta_i,\delta_i\in\mathbb{C}$, $c=Re\gamma>0$ and $M_i,\ N_i,\ R_i,\ S_i\ge 1$, $i=\overline{1,n}$, such that*

$$(2c+1)^{\frac{2c+1}{2c}}\sum_{i=1}^{n}|\alpha_i-1|\left[1+(2-\lambda_i)\,M_i^{\mu_i-1}\right]+$$

$$+2c\sum_{i=1}^{n}\left[|\beta_i|\,N_i+|\delta_i|\,(R_i+S_i)\right]\le c\,(2c+1)^{\frac{2c+1}{2c}}. \tag{17}$$

If $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $g_i, h_i, k_i \in \mathcal{A}$, *satisfies*

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le N_i, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \le R_i, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \le S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the integral operator* $\mathcal{Y}_n$, *defined by*

$$\mathcal{Y}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} (g_i(t)')^{\beta_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{18}$$

is in the class $\mathcal{S}$.

Remark 2. *Taking in (18)* $\delta_i = 0$, *we obtain Theorem that was obtained in [28].*

If we consider $\delta = 1$ and $\beta_i = 0$ in Theorem 4, obtain the next corollary:

Corollary 2. *Let* $\gamma, \alpha_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ *and* M_i, P_i, Q_i, R_i, $S_i \ge 1$, $i = \overline{1,n}$, *such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^n \left\{|\alpha_i - 1| \left[1 + (2-\lambda_i) M_i^{\mu_i - 1}\right] + |\gamma_i| \left[2 + (2-\eta_i) P_i^{\nu_i - 1}\right]\right\} +$$

$$+ (2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^n |\gamma_i| (2-\rho_i) Q_i^{\theta_i - 1} + 2c \sum_{i=1}^n |\delta_i| (R_i + S_i) \le c(2c+1)^{\frac{2c+1}{2c}}. \tag{19}$$

If $f_i \in \mathcal{B}(\mu_i, \lambda_i)$, $h_i \in \mathcal{B}(\nu_i, \eta_i)$, $k_i \in \mathcal{B}(\theta_i, \rho_i)$, *satisfies*

$$|f_i(z)| < M_i, \quad |h_i(z)| < P_i, \quad |k_i(z)| < Q_i, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \le R_i, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \le S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, *then the integral operator* $\mathcal{X}_n$, *defined by*

$$\mathcal{X}_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{20}$$

is in the class $\mathcal{S}$.

Remark 3. *To the integral operator given by (20) if we take* $\alpha_i - 1 = 0$, *we obtain another known result proven in [25].*

If we consider $\delta = 1$ and $\alpha_i - 1 = 0$ in Theorem 4, obtain the next corollary:

Corollary 3. *Let* $\gamma, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ *and* N_i, P_i, Q_i, R_i, $S_i \ge 1$, $i = \overline{1,n}$, *such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^n |\gamma_i| \left[2 + (2-\eta_i) P_i^{\nu_i - 1} + (2-\rho_i) Q_i^{\theta_i - 1}\right] +$$

$$+2c\sum_{i=1}^{n}\left[|\beta_i|\,N_i+|\delta_i|\,(R_i+S_i)\right]\leq c\,(2c+1)^{\frac{2c+1}{2c}}\,. \tag{21}$$

If $g_i\in\mathcal{A}$, $h_i\in\mathcal{B}(\nu_i,\eta_i)$, $k_i\in\mathcal{B}(\theta_i,\rho_i)$, *satisfies*

$$\left|\frac{g_i''(z)}{g_i'(z)}\right|\leq N_i,\quad |h_i(z)|<P_i,\quad |k_i(z)|<Q_i,\quad \left|\frac{h_i''(z)}{h_i'(z)}\right|\leq R_i,\quad \left|\frac{k_i''(z)}{k_i'(z)}\right|\leq S_i,$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$, *then the integral operator* $\mathcal{D}_n$, *defined by*

$$\mathcal{D}_n(z)=\int_0^z\prod_{i=1}^{n}\left[\left(g_i(t)'\right)^{\beta_i}\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right]dt, \tag{22}$$

is in the class $\mathcal{S}$.

Remark 4. *If in (22) we put* $\beta_i=0$, *than we obtain Theorem that was obtained in [25].*

If we consider $\delta=1$ and $\delta_i=0$ in Theorem 4, obtain the next corollary:

Corollary 4. *Let* $\gamma,\alpha_i,\beta_i,\gamma_i\in\mathbb{C}$, $c=Re\gamma>0$ *and* M_i, N_i, P_i, $Q_i\geq 1$, $i=\overline{1,n}$, *such that*

$$(2c+1)^{\frac{2c+1}{2c}}\sum_{i=1}^{n}\left\{|\alpha_i-1|\left[1+(2-\lambda_i)\,M_i^{\mu_i-1}\right]+|\gamma_i|\left[2+(2-\eta_i)\,P_i^{\nu_i-1}\right]\right\}+$$

$$+(2c+1)^{\frac{2c+1}{2c}}\sum_{i=1}^{n}|\gamma_i|\,(2-\rho_i)\,Q_i^{\theta_i-1}+2c\sum_{i=1}^{n}|\beta_i|\,N_i\leq c\,(2c+1)^{\frac{2c+1}{2c}}\,. \tag{23}$$

If $f_i\in\mathcal{B}(\mu_i,\lambda_i)$, $g_i\in\mathcal{A}$, $h_i\in\mathcal{B}(\nu_i,\eta_i)$, $k_i\in\mathcal{B}(\theta_i,\rho_i)$, *satisfies*

$$|f_i(z)|<M_i,\quad \left|\frac{g_i''(z)}{g_i'(z)}\right|\leq N_i,\quad |h_i(z)|<P_i,\quad |k_i(z)|<Q_i,$$

for all $z\in\mathbb{U}$, $i=\overline{1,n}$, *then the integral operator* $\mathcal{S}_n$, *defined by*

$$\mathcal{S}_n(z)=\int_0^z\prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\left(g_i(t)'\right)^{\beta_i}\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\right]dt, \tag{24}$$

is in the class $\mathcal{S}$.

Remark 5. *Taking in (24)* $\gamma_i=0$, *we obtain a known result proven in [28].*

Letting $\mu_i=\nu_i=\theta_i=M_i=N_i=P_i=Q_i=R_i=S_i=1$ and $\rho_i=\eta_i=\lambda_i$ for all $i=\overline{1,n}$ in Theorem 4, we have:

Corollary 5. *Let $\delta, \gamma, \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$, $c = Re\gamma > 0$ and $0 \leq \lambda_i < 1$, $i = \overline{1,n}$, such that*

$$(2c+1)^{\frac{2c+1}{2c}} \sum_{i=1}^{n} (3-\lambda_i)\left(|\alpha_i - 1| + 2|\gamma_i|\right) + 2c \sum_{i=1}^{n} \left(|\beta_i| + 2|\delta_i|\right) \leq c\,(2c+1)^{\frac{2c+1}{2c}}. \quad (25)$$

If $g_i \in \mathcal{A}$, f_i, h_i, $k_i \in \mathcal{S}^*(\lambda_i)$ and

$$|f_i(z)| < 1, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq 1, \quad |h_i(z)| < 1, \quad |k_i(z)| < 1, \quad \left|\frac{h_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{k_i''(z)}{k_i'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$, then for every δ, $Re\delta \geq Re\gamma$, the function $\mathcal{T}_n$, defined by (3) is in the class $\mathcal{S}$.

Letting $n = 1$, $\delta = \gamma$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 5, we obtain:

Corollary 6. *Let $c, \delta \in \mathbb{C}$ with $Re\delta > 0$ and $M, N, P, Q, R, S \geq 1$. Suppose that $f \in \mathcal{B}(\mu, \lambda)$, $g \in \mathcal{A}$, $h \in \mathcal{B}(\nu, \eta)$, $k \in \mathcal{B}(\theta, \rho)$, such that*

$$|f(z)| < M, \ \left|\frac{zg''(z)}{g'(z)}\right| < N, \ |h(z)| < P, \ |k(z)| < Q, \ \left|\frac{zh''(z)}{h'(z)}\right| < R, \ \left|\frac{zk''(z)}{k'(z)}\right| < S,$$

for all $z \in \mathbb{U}$. If

$$Re\delta \geq |\delta| \left[(2-\lambda) M^{\mu-1} + (2-\eta) P^{\nu-1} + (2-\rho) Q^{\theta-1} + N + R + S + 3\right] \quad (26)$$

and

$$|c| \leq 1 - \frac{|\delta|}{Re\delta} \left[(2-\lambda) M^{\mu-1} + (2-\eta) P^{\nu-1} + (2-\rho) Q^{\theta-1} + N + R + S + 3\right], \quad (27)$$

then the integral operator $\mathcal{T}$, defined by

$$\mathcal{T}(z) = \left[\alpha \int_0^z t^{\alpha-1} \left(f(t) g'(t) \frac{h(t)}{k(t)} \frac{h'(t))}{k'(t)}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}}, \quad (28)$$

is analytic and univalent in $\mathbb{U}$.

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