EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 3, 2020, 390-402
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

E-J Summability of Orthogonal Series

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Abstract. In this paper we obtain a sufficient condition for the E-J summability of certain orthogonal series. Our results generalize the corresponding theorems for ordinary Hausdorff summability obtained by Kalaivana and Youvaraj.

2020 Mathematics Subject Classifications: 40G05

Key Words and Phrases: E-J matrices, orthogonal series

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1. Preliminaries

The set of all real or complex sequences $\{x_n\}$ for which $A_n(x) := \sum_k a_{nk}x_k$ converges is called the convergence domain of A, written c_A, where A is an infinite matrix. A matrix A is said to be conservative if it maps each convergent sequence into a convergent sequence, not necessarily with the same limit. If the limit is also preserved, then the matrix is called regular. Silverman and Toeplitz established necessary and sufficient conditions for a matrix to be conservative[4]. They are

(i) $\|A\|_\infty := \sup_n \sum_k |a_{nk}| < \infty$,
(ii) $t := \lim_n \sum_k a_{nk}$ exists,
(iii) $a_k := \lim_n a_{nk}$ exists for each k.

A Hausdorff matrix $H = (h_{nk})$ is a lower triangular matrix with nonzero entries

$$h_{nk} = \binom{n}{k}\Delta^{n-k}\mu_k,$$

where $\{\mu_n\}$ is any real sequence and Δ is the forward difference operator defied by $\Delta\mu_k = \mu_k - \mu_{k+1}$ and $\Delta^{n+1}\mu_k = \Delta(\Delta^n\mu_k)$. For every Hausdorff matrix each row sum is equal to μ_0[3].

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DOI: https://doi.org/10.29020/nybg.ejpam.v13i3.3687

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F. Hausdorff [2] proved that a Hausdorff matrix is conservative if and only if

$$\mu_n = \int_0^1 x^n d_\chi(x), \tag{1}$$

where the mass function $\chi \in BV[0,1]$.

The E-J generalized Hausdorff matrices, denoted by $H_\mu^\alpha = (h_{nk}^{(\alpha)})$, were defined independently by Endl [1] and Jakimovski [5], with nonzero entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k^{(\alpha)}, 0 \le k \le n,$$

for any $\alpha \ge 0$. For $\alpha = 0$, the E-J matrices reduce to the ordinary Hausdorff matrices.

If the $\mu_n^{(\alpha)}$ satisfy the condition

$$\mu_n^{(\alpha)} = \int_0^1 x^{n+\alpha} d\chi(x),$$

where $\chi \in BV[0,1]$, then the corresponding E-J matrix is conservative.

Definition 1. *Let $\gamma : [1,\infty) \to [0,\infty)$ be a nondecreasing function, $A = (a_{nk})$ an infinite matrix. Then a series $\sum_n b_n$ is said to be $|A,\gamma|_k$ summable, if*

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\sigma_n - \sigma_{n-1}|^k$$

converges, where $\sigma_n : \sum_n a_{nk} b_k$.

Definition 2. *Let $\gamma := \{\gamma_n\}$ be a positive sequence, β a real positive number. Then γ is called quasi- β-power monotone decreasing if there exists a number $M = M(\beta,\gamma) \ge 1$ such that*

$$n^\beta \gamma(n) \le M m^\beta \gamma(m)$$

for each $m \le n$.

For any real number β, Γ_β denotes the set of all increasing functions $\Gamma_\beta : [1,\infty) \to [0,\infty)$ such that each $\{\gamma_n\}$ is a quasi β-power monotone decreasing sequence.

2. Main Results

Theorem 1. *Let $\{\varphi_n\}_{n=0}^\infty \subset Ł_2[0,1]$ be an orthonormal system, H_μ^α an E-J Hausdorff matrix with χ monotone decreasing, $\gamma \in \Gamma_\beta$ for $\beta > 1 - 1/k$, $1 \le k \le 2$. Then every orthogonal series $\sum_{n=0}^\infty b_n \varphi_n$ is $|H^\alpha,\gamma|$ summable.*

The following lemmas will be needed in the proof of Theorem 1.

Lemma 1. *Let H^α be an E-J matrix with entries (h_{nk}), where χ is a monotonically increasing mass function on $[0,1]$ associated with the μ_n. Then*

(i) $a_{mn} = K\binom{n-1+\alpha}{m-1+\alpha}\xi^{m+\alpha}(1-\xi)^{n-m}$ for some $\xi \in (0,1)$,

(ii) $\sum_{m=0}^{n} |a_{mn}|^2|b_m|^2 \leq K^2 \sum_{m=0}^{n} |b_m|^2$ for all $b_n \in \mathbb{C}$ and $n \in \mathbb{N}$, where $K = \chi(1) - \chi(0)$

and $a_{mn} = \sum_{k=m}^{n} |h^{(\alpha)}_{nk} - h^{(\alpha)}_{n-1,k}|$.

Here $\mathbb{C}$ = complex numbers and $\mathbb{N}$ = natural numbers.

Proof. (i) We consider

$$h^{(\alpha)}_{nk} - h^{(\alpha)}_{n-1,k} \tag{2}$$

$$= \Big[\int_0^1 \mu^{k+\alpha}(1-\mu)^{n-k}\binom{n+\alpha}{k+\alpha}d\chi(\mu) - \int_0^1 \mu^{k+\alpha}(1-\mu)^{n-1-k}\binom{n-1+\alpha}{k+\alpha}d\chi(\mu)\Big]$$
$$= \int_0^1 \mu^{k+\alpha}(1-\mu)^{n-k}\Big[\binom{n+\alpha}{k+\alpha} - \binom{n-1+\alpha}{k+\alpha}\frac{1}{1-\mu}\Big]d\chi(\mu),$$

where $0 \leq k \leq n$. Since

$$\binom{n+\alpha}{k+\alpha} - \binom{n-1+\alpha}{k+\alpha} = \binom{n-1+\alpha}{k-1+\alpha},$$

from (2),

$$\int_0^1 \mu^{k+\alpha}(1-\mu)^{n-1-k}\left[\binom{n+\alpha}{k+\alpha}(1-\mu) - \binom{n-1+\alpha}{k+\alpha}\right]d\chi(\mu)$$
$$= \int_0^1 \mu^{k+\alpha}(1-\mu)^{n-1-k}\left[\binom{n+\alpha}{k+\alpha} - \binom{n-1+\alpha}{k+\alpha} - \mu\binom{n+\alpha}{k+\alpha}\right]d\chi(\mu)$$
$$= \int_0^1 \mu^{k+\alpha}(1-\mu)^{n-1-k}\left[\binom{n-1+\alpha}{k-1+\alpha} - \mu\binom{n+\alpha}{k+\alpha}\right]d\chi(\mu).$$

Thus

$$a_{mn} = \sum_{k=m}^{n}(h^{(\alpha)}_{nk} - h^{(\alpha)}_{n-1,k}) = \sum_{k=m}^{n}\int_0^1 \mu^{k+\alpha}(1-\mu)^{n-1-k}\left[\binom{n-1+\alpha}{k-1+\alpha} - \mu\binom{n+\alpha}{k+\alpha}\right]d\chi(\mu)$$
$$= \int_0^1 \sum_{k=m}^{n}\mu^{k+\alpha}(1-\mu)^{n-1-k}\left[\binom{n-1+\alpha}{k-1+\alpha} - \mu\binom{n+\alpha}{k+\alpha}\right]d\chi(\mu).$$

From the above inequality,

$$\sum_{k=m}^{n}\mu^{k+\alpha}(1-\mu)^{n-1-k}\left[\binom{n-1+\alpha}{k-1+\alpha} - \mu\binom{n+\alpha}{k+\alpha}\right]$$

$$= \sum_{k=m}^{n} \mu^{k+\alpha}(1-\mu)^{n-1-k}\binom{n-1+\alpha}{k-1+\alpha} - \sum_{k=m}^{n} \mu^{k+\alpha+1}(1-\mu)^{n-1-k}\binom{n+\alpha}{k+\alpha}$$
$$= \sum_{k=m}^{n} \mu^{k+\alpha}(1-\mu)^{n-1-k}\binom{n-1+\alpha}{k-1+\alpha} - \sum_{k=m}^{n} \mu^{k+\alpha+1}(1-\mu)^{n-1-k}\left[\binom{n-1+\alpha}{k-1+\alpha} + \binom{n-1+\alpha}{k+\alpha}\right]$$
$$= \sum_{k=m}^{n} \mu^{k+\alpha}(1-\mu)^{n-k}\binom{n-1+\alpha}{k-1+\alpha} - \sum_{k=m}^{n} \mu^{k+\alpha+1}(1-\mu)^{n-1-k}\binom{n-1+\alpha}{k+\alpha}$$
$$= \sum_{k=m}^{n} \mu^{k+\alpha}(1-\mu)^{n-k}\binom{n-1+\alpha}{k-1+\alpha} - \sum_{g=m+1}^{n+1} \mu^{g+\alpha}(1-\mu)^{n-g}\binom{n-1+\alpha}{g-1+\alpha}$$
$$= \mu^{m+\alpha}(1-\mu)^{n-m}\binom{n-1+\alpha}{m-1+\alpha},$$

and

$$0 \le a_{mn} \le \int_0^1 \mu^{m+\alpha}(1-\mu)^{n-m}\binom{n-1+\alpha}{m-1+\alpha} d\chi(\mu). \tag{3}$$

Using the first mean value theorem for integrals, for some $0 < \xi < 1$,

$$\int_0^1 \mu^{m+\alpha}(1-\mu)^{n-m}\binom{n-1+\alpha}{m-1+\alpha} d\chi(\mu) = \xi^{m+\alpha}(1-\xi)^{n-m}\binom{n-1+\alpha}{m-1+\alpha}\int_0^1 d\chi(\mu)$$
$$= K\xi^{m+\alpha}(1-\xi)^{n-m}\binom{n-1+\alpha}{m-1+\alpha},$$

where $0 < K \le 1$, and (i) is satisfied.

To prove (ii) we need the following lemma.

Lemma 2. *For $0 < K < 1, a_{mn} \le 1$.*

Proof. From (3)

$$\begin{aligned} a_{mn} &= K\int_0^1 \mu^{m+\alpha}(1-\mu)^{n-m}\binom{n-1+\alpha}{m-1+\alpha} d\mu \\ &= K\binom{n-1+\alpha}{m-1+\alpha}\int_0^1 \mu^{m+\alpha}(1-\mu)^{n-m} d\mu \\ &= K\binom{n-1+\alpha}{m-1+\alpha}\frac{\Gamma(m+\alpha+1)\Gamma(n-m+1)}{\Gamma(n+\alpha+2)} \\ &= K\frac{\Gamma(n+\alpha)}{\Gamma(m+\alpha)\Gamma(n-m+1)}\,\frac{\Gamma(m+\alpha+1)\Gamma(n-m+1)}{\Gamma(n+\alpha+2)} \\ &= K\frac{(m+\alpha)}{(n+\alpha+1)(n+\alpha)}. \end{aligned}$$

Since $n \ge m$,

$$a_{mn} \le K < 1. \tag{4}$$

Using equation (4) we can write

$$\sum_{m=0}^{n} |a_{mn}|^2 |b_m|^2 \leq K^2 \sum_{m=0}^{n} |b_m|^2 \leq \sum_{m=0}^{n} |b_m|^2, \tag{5}$$

which is a proof of (ii).

Lemma 3. *Let $\{\varphi_n\}_{n=0}^{\infty} \subset L_2[0,1]$ be an orthonormal system, H_{μ}^{α} be an E-J Hausdorff matrix with monotonically increasing function χ on $[0,1]$. Then, for $n \in \mathbb{N}$ and*

$$K = \int_0^1 d\chi(\mu),$$

(i) there exists $\xi \in (0,1)$ such that

$$\int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^2 = K^2 \sum_{m=0}^{n} \xi^{2m+2\alpha}(1-\xi)^{2n-2m} \binom{n-1+\alpha}{m-1+\alpha}^2 |b_m|^2$$

and

(ii)
$$\int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^2 dx = K^2 \sum_{m=0}^{n} |b_m|^2,$$

for all $b_m \in \mathbb{C}$ where, for $n \in \mathbb{N}$, $\sigma_n(x) = \sum_{k=0}^{n} h_{nk} S_k(x)$, where S_k denotes the k^{th} partial sum of the orthogonal series $\sum_{m=0}^{\infty} b_m \varphi_m$.

Proof.

$$\begin{aligned}
\sigma_n(x) - \sigma_{n-1}(x) &= \sum_{k=0}^{n} (h_{nk}^{(\alpha)} - h_{n-1,k}^{(\alpha)}) S_k(x) \\
&= \sum_{k=0}^{n} (h_{nk}^{(\alpha)} - h_{n-1,k}^{(\alpha)}) \sum_{m=0}^{k} b_m \varphi_m \\
&= \sum_{m=0}^{n} \sum_{k=m}^{n} (h_{nk}^{(\alpha)} - h_{n-1,k}^{(\alpha)}) b_m \varphi_m \\
&= \sum_{m=0}^{n} a_{mn} b_m \varphi_m.
\end{aligned}$$

Since $\{\varphi_n\}_{n=0}^{\infty}$ is an orthonormal system, using Parseval's identity,

$$\begin{aligned}
\int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^2 dx &= \sum_{m=0}^{n} |a_{mn}|^2 |b_m|^2 \\
&= K^2 \sum_{m=0}^{n} \xi^{2m+2\alpha}(1-\xi)^{2n-2m} \binom{n-1+\alpha}{m-1+\alpha}^2 |b_m|^2.
\end{aligned}$$

From Lemma 2,

$$\int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|^2 dx = \sum_{m=0}^{n} |a_{mn}|^2 |b_m|^2 \leq K^2 \sum_{m=0}^{n} |b_m|^2.$$

Proof. To prove Theorem 1, from Definition 1 we need to show that

$$\sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} |\sigma_n - \sigma_{n-1}|^k$$

converges for $1 \leq k < 2$, where, for $n \in \mathbb{N}, \sigma_n(x) \sum_{k=0}^{n} h_{nk}^{(\alpha)} S_k(x)$.

Using Lemma 3, and Hölder's inequality with $p = 2/k$, for any $1 \leq k \leq 2$, and for all $b \in \ell_2(\mathbb{Z}^+)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \gamma(n)^k n^k \int_0^n |\sigma_n(x) - \sigma_{n-1}(x)|^k dx &\leq \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1} \{K^2 \|b\|_2^2\}^{k/2} \\ &\leq \{K\|b\|_2\}^k \sum_{n=1}^{\infty} \gamma(n)^k n^{k-1}.\end{aligned}$$

Here $\{\gamma(n)\}$ is a quasi β-power monotone decreasing sequence with $\beta > 1 - 1/k$, and, since for $\epsilon = \beta - 1 + 1/k$, the sequence $\{n^{k-1}\gamma(n)^k\}$ is quasi $k\epsilon$-power monotone decreasing. Using Lemma 1 of [6], we have

$$\begin{aligned}&\leq \{K\|b\|_2\}^k \sum_{n=1}^{\infty} \gamma(2^n)^k (2^n)^{k-1} \\ &\leq \{K\|b\|_2\}^k B\gamma(2)^k (2)^{k-1},\end{aligned}$$

where $B \geq 1$.

Theorem 2. *Let $\{\varphi\}_{n=0}^{\infty} \subset L_2[0,1]$ be an orthogonal system and H_μ^α the corresponding E-J Hausdorff matrix. For $1 \leq k \leq 2$ and $\gamma \in \Gamma(\beta)$ with $\beta > 1 - 1/k$, every orthogonal series $\sum_{n=0}^{\infty} b_n \varphi_n$ is $|H^\alpha, \gamma|_k$ summable.*

Proof. Let $\chi \in BV[0,1]$ be the mass function corresponding to the E-J matrix H^α. By the Jordan decomposition theorem, $\chi = \chi_1 - \chi_2$, where χ_1 and χ_2 are monotone increasing functions. To prove the theorem we apply Theorem 1 to χ_1 and χ_2. Theorems 1 and 2 are generalizations of Theorems 1 and 2, respectively, in [6].

Theorem 3. *Let $\{\varphi\}_{n=0}^{\infty} \subset L_2[0,1]$ be an orthogonal system and H_μ^α an E-J Hausdorff matrix with $\chi \in [0,1]$ and monotone increasing. For $1 \leq k \leq 2$ and $\gamma \in \Gamma(\beta)$ with $\beta > 1 - 1/k$, a sufficient condition for the orthogonal series $\sum_{n=0}^{\infty} b_n \varphi_n$ to be $|H^\alpha, \gamma|_k$ summable is*

$$\sum_{s=0}^{\infty} \gamma(2^s)^k \left\{ \sum_{m=2^s+1}^{2^{s+1}} \sqrt{m+\alpha} |b_m|^2 \right\}^{k/2} < \infty. \tag{6}$$

Proof. Let $\chi \in BV[0,1]$ and monotonically increasing on $[0,1]$. By Lemma 1(i) there exists a $\xi \in (0,1)$ such that

$$\int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|dx \leq K^2 \sum_{m=0}^{n} \binom{n+\alpha-1}{m+\alpha-1}^2 \xi^{2m+2\alpha}(1-\xi)^{2n-2m}|b_m|^2, \qquad (7)$$

where

$$\sigma_n(x) = \sum_{k=0}^{n} h_{nk}^{(\alpha)} S_k.$$

For $1 \leq k \leq 2$, by using Hölder's inequality and equation (1),

$$\sum_{n=2}^{\infty} \gamma(n)^k n^{k-1} \left\{ \int_0^1 |\sigma_n(x) - \sigma_{n-1}(x)|dx \right\}^k$$

$$\leq \sum_{n=2}^{\infty} \gamma(n)^k n^{k-1} \left\{ K^2 \sum_{m=0}^{n} \binom{n+\alpha-1}{m+\alpha-1}^2 \xi^{2m+2\alpha}(1-\xi)^{2n-2m}|b_m|^2 \right\}^{k/2}. \qquad (8)$$

Replacing ξ by $1/(1+q)$ in (8), we obtain

$$= K^k \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^k n^{k-1} \left\{ \sum_{n=0}^{n} \binom{n+\alpha}{m+\alpha}^2 \left(\frac{m+\alpha}{n+\alpha}\right)^2 q^{2n-2m}(1+q)^{-2n-2\alpha}|b_m|^2 \right\}^{k/2}. \qquad (9)$$

O.A. Ziza [7] proved that, for $q > 0$, there exists a constant $C_q > 0$ such that

$$\max_{0\leq k\leq n} \binom{n}{k} q^k \leq C_q \frac{(1+q)^n}{\sqrt{n}}, \quad n = 1, 2, \ldots .$$

We shall generalize this Lemma for E-J matrices.

Lemma 4. *For $q > 0$ there exists a $C_q > 0$ such that*

$$\max_{0\leq k\leq n} \binom{n+\alpha}{k+\alpha} q^{k+\alpha} \leq C_q \frac{(1+q)^{n+\alpha}}{\sqrt{n+\alpha}}, \quad n = 1, 2, ...$$

Proof.

$$\frac{\binom{n+\alpha+1}{k+\alpha}}{\binom{n+\alpha}{k+\alpha-1}} = \frac{n+\alpha+1}{k+\alpha}.$$

Let

$$d_k = \Big(\frac{n+\alpha+1}{k+\alpha} - 1\Big)q.$$

The d_n are decreasing in k. Let k_n denote the largest value of $k+\alpha$ for which $d_{k_n} \geq 1$. Then $d_{k_{n+1}} < 1$, and

$$\max_{0\leq k\leq n} \binom{n+\alpha}{k+\alpha} q^{k+\alpha} = \binom{n+\alpha}{k_n} q^{k_n}.$$

It then follows that one can write

$$k_n = \frac{q}{1+q}(n+\alpha) + \nu_n,$$

where $0 < \nu_n < 1$.

Then

$$\binom{n+\alpha}{k_n} \le C_1 \frac{(n+\alpha)!}{k_n!(n+\alpha-k_n)!} = \frac{(n+\alpha)^{n+\alpha} e^{-(n+\alpha)}\sqrt{n+\alpha}}{(k_n)^{k_n} e^{-(k_n)}\sqrt{k_n}(n+\alpha-k_n)^{(n+\alpha-k_n)} e^{-(n+\alpha-k_n)}\sqrt{n+\alpha-k_n}}. \tag{10}$$

With $p = q/(1+q)$, the right hand side of (10) equals

$$\frac{(n+\alpha)^{(n+\alpha)} e^{-(n+\alpha)}\sqrt{n+\alpha}}{(p(n+\alpha)+\nu_n)^{(p(n+\alpha)+\nu_n)} e^{-(p(n+\alpha)+\nu_n)}\sqrt{(p(n+\alpha)+\nu_n)}}$$
$$\times \frac{1}{(n+\alpha-p(n+\alpha)-\nu_n)^{(n+\alpha-p(n+\alpha)-\nu_n)} e^{-(n+\alpha-p(n+\alpha)-\nu_n)}\sqrt{(n+\alpha-p(n+\alpha)-\nu_n)}}$$

$$= \frac{(n+\alpha)^{p(n+\alpha)+\nu_n}}{(p(n+\alpha)+\nu_n)^{(p(n+\alpha)+\nu_n)}} \times \frac{(n+\alpha)^{n+\alpha-p(n+\alpha)-\nu_n}}{(n+\alpha-p(n+\alpha)-\nu_n)^{n+\alpha-p(n+\alpha)-\nu_n}} \tag{11}$$
$$\times \frac{\sqrt{n+\alpha}}{\sqrt{(p(n+\alpha)+\nu_n)(n+\alpha-p(n+\alpha)-\nu_n)}}.$$

Note that

$$\frac{(n+\alpha)}{(p(n+\alpha)+\nu_n)(n+\alpha-p(n+\alpha)-\nu_n)} = \frac{(n+\alpha)}{(n+\alpha)^2(p+\frac{\nu_n}{n+\alpha})(1-(p+\frac{\nu_n}{n+\alpha}))}.$$

Set $a = p + \nu_n/(n+\alpha)$ and define a function f by $f(a) = a(1-a)$. Then $f(a)$ has a minimum value of $1/4$ at $a = 1/2$. Therefore $1/\sqrt{f(a)} \le 2$. From (11)

$$\binom{n+\alpha}{k_n} \le 2\frac{1}{\left(p+\frac{\nu_n}{n+\alpha}\right)^{p(n+\alpha)+\nu_n}} \times \frac{1}{\left(1-p-\frac{\nu_n}{n+\alpha}\right)^{(1-p)(n+\alpha)-\nu_n}} \times \frac{1}{\sqrt{n+\alpha}}$$
$$= 2\frac{1}{p^{p(n+\alpha)+\nu_n}\left(1+\frac{\nu_n}{p(n+\alpha)}\right)^{p(n+\alpha)+\nu_n}} \times \frac{1}{(1-p)^{(1-p)(n+\alpha)-\nu_n}\left(1-\frac{\nu_n}{(1-p)(n+\alpha)}\right)^{(1-p)(n+\alpha)-\nu_n}}$$
$$\times \frac{1}{\sqrt{n+\alpha}}.$$

Since $(1-p)/p = (1/p)-1$ and p is a fixed positive constant between 0 and 1, $(p/(1-p))^{-\nu_n}$ is clearly bounded. So also is $(1+\nu_n/(p(n+\alpha)))^{-p(n+\alpha)-\nu_n}$.

Let $g(p) = 1 - \nu_n/(1-p)(n+\alpha)$. Then

$$g'(p) = \frac{\nu_n}{(1-p)^2(n+\alpha)},$$

and g is decreasing in p. Since $0 < p < 1$ and fixed, $g(p)$ is bounded, Using the above facts,

$$\left(\frac{1-p}{p}\right)^{-\nu_n} \text{and} \left(\frac{p}{1-p}\right)^{-\nu_n} \left(1+\frac{\nu_n}{p(n+\alpha)}\right)^{-(p(n+\alpha))+\nu_n} \left(1-\frac{\nu_n}{(1-p)n+\alpha}\right)^{-((1-p)(n+\alpha)+\nu_n)}$$

are bounded. Also,

$$\binom{n+\alpha}{k_n} \leq C_3 \frac{1}{p^{p(n+\alpha)}} \times \frac{1}{(1-p)^{(1-p)(n+\alpha)}} \times \frac{1}{\sqrt{n+\alpha}}. \tag{12}$$

We can write (12) as

$$\binom{n+\alpha}{k_n} \leq C_3 \frac{1}{\frac{q}{1+q}^{(\frac{q}{1+q})(n+\alpha)}} \times \frac{1}{(\frac{1}{1+q})^{(\frac{1}{1+q})(n+\alpha)}} \times \frac{1}{\sqrt{n+\alpha}}. \tag{13}$$

From (13)

$$\begin{aligned} \binom{n+\alpha}{k_n}^{q^{k_n}} &\leq C_3 \frac{1}{(\frac{q}{1+q})^{(\frac{q}{1+q})(n+\alpha)}} \times \frac{1}{(\frac{1}{1+q})^{(\frac{1}{1+q})(n+\alpha)}} \times \frac{1}{\sqrt{n+\alpha}} q^{k_n} \\ &= C_3 \frac{1}{(\frac{q}{1+q})^{(\frac{q}{1+q})(n+\alpha)}} \times \frac{1}{(\frac{1}{1+q})^{(\frac{1}{1+q})(n+\alpha)}} \times \frac{1}{\sqrt{n+\alpha}} q^{(\frac{1}{1+q})(n+\alpha)+\nu_n} \\ &= C_3 \frac{(1+q)^{n+\alpha}}{\sqrt{n+\alpha}} q^{\nu_n} \leq C_q \frac{(1+q)^{n+\alpha}}{\sqrt{n+\alpha}}. \end{aligned}$$

Thus

$$\max_{0\leq k\leq n} \binom{n+\alpha}{k+\alpha} q^{k+\alpha} \leq C_q \frac{(1+q)^{n+\alpha}}{\sqrt{n+\alpha}}.$$

From (9)

$$K^k \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^k n^{k-1} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha}^2 \left(\frac{m+\alpha}{n+\alpha}\right)^2 q^{2n-2m} (1+q)^{-2n-2\alpha} |b_m|^2 \right\}^{k/2}$$

$$= K^k \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^k n^{k-1} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} \left(\frac{m+\alpha}{n+\alpha}\right)^2 \binom{n+\alpha}{n-m} q^{n-m} q^{n-m} (1+q)^{-2n-2\alpha} |b_m|^2 \right\}^{k/2}. \tag{14}$$

Using Lemma 4, equation (14) can be written as

$$\leq K^k C_q^{k/2} \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^k n^{k-1} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} \left(\frac{m+\alpha}{n+\alpha}\right)^2 \frac{1}{\sqrt{n+\alpha}} q^{n-m} (1+q)^{-2n-2\alpha} |b_m|^2 \right\}^{k/2}$$

$$= K^k C_q^{k/2} \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^k n^{k-1} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} (m+\alpha)^2 \frac{1}{(n+\alpha)^{5/2}} q^{n-m} (1+q)^{-2n-2\alpha} |b_m|^2 \right\}^{k/2}. \quad (15)$$

For $k = 2$, the above inequality becomes

$$\Omega \leq K^2 C_q \sum_{r=0}^{\infty} \sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^2 n (n+\alpha)^{-5/2} \left\{ \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} (m+\alpha)^2 q^{n-m} (1+q)^{-2n-2\alpha} |b_m|^2 \right\}^{5/2}. \quad (16)$$

Lemma 5. *There exists a $D_q > 0$ such that*

$$\sum_{n=m}^{\infty} \binom{n+\alpha}{m+\alpha} q^{n-m} (1+q)^{-n-\alpha} \leq D_q$$

for all $m \in \mathbb{Z}^+$ and $1 \leq k \leq 2$.

Proof. The proof of the lemma is easy to verify and it is a generalization of Theorem B of [6] to E-J matrices. Using Theorem B of [6], we can write (16) as

$$\Omega \leq K^2 C_q D_q \sum_{r=0}^{\infty} \gamma(2^{r+1})^2 2^r (2^r+\alpha)^{-5/2} \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2$$
$$\leq K^2 C_q D_q \sum_{r=0}^{\infty} \gamma(2^{r+1})^2 (2^r+\alpha)^{-3/2} \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2, \quad \alpha > 0.$$

Let $p = 2/k$. By Hölder's inequality, for $\alpha > 0$ and $1 \leq k \leq 2$,

$$\Omega = K^k C_q^{k/2} \sum_{r=0}^{\infty} \Big(\sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^{kq} n^{q(\frac{-k}{4}-1)} \Big)^{1/q} \Big\{ \sum_{n=2^r+1}^{2^{r+1}} \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} (m+\alpha)^2 q^{n-m} (1+q)^{-n-\alpha} |b_m|^2 \Big\}. \quad (17)$$

Since $\gamma \in \Gamma(\beta)$ and, for $\beta \in \mathbb{R}$, by Theorem A in [6], we can write the expression in the first bracket in (17) as

$$\left(\sum_{n=2^r+1}^{2^{r+1}} \gamma(n)^{kq} n^{q(\frac{-k}{4}-1)} \right)^{1/q} \leq K^{1/q} \gamma(2^r+1)^k (2^r+1)^{(-\frac{k}{4}-1)}$$
$$\leq K^{1/q} \gamma(2^r+1)^k (2^r)^{(-\frac{k}{4}-1)}.$$

Thus, from (17),

$$\Omega \leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(\frac{-k}{4}-1)} \left\{ \sum_{n=2^r+1}^{2^{r+1}} \sum_{m=0}^{n} \binom{n+\alpha}{m+\alpha} (m+\alpha)^2 q^{n-m} (1+q)^{-n-\alpha} |b_m|^2 \right\}^{k/2}. \quad (18)$$

Changing the order of summation inside the brackets in the above inequality, (18) is equal to

$$= K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \left\{ \gamma(2^{r+1})^{kq} (2^r)^{q(-\frac{k}{4}-1)} \right\}^{\frac{1}{q}} \left\{ \sum_{m=0}^{2^{r+1}} \sum_{n=2^r+1}^{2^{r+1}} \binom{n+\alpha}{m+\alpha} (m+\alpha)^2 q^{n-m} (1+q)^{-n-\alpha} |b_m|^2 \right\}^{k/2}$$

$$\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{k}{4}-1)} \left\{ \sum_{m=0}^{2^r+1} \sum_{n=2^r+1}^{2^{r+1}} \binom{n+\alpha}{m+\alpha} (m+\alpha)^2 q^{n-m} (1+q)^{-n-\alpha} |b_m|^2 \right\}^{k/2}$$

$$+ K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{k}{4}-1)} \left\{ \sum_{m=2^r+1}^{2^{r+1}} \sum_{n=2^r+1}^{2^{r+1}} \binom{n+\alpha}{m+\alpha} (m+\alpha)^2 q^{n-m} (1+q)^{-n-\alpha} |b_m|^2 \right\}^{k/2}.$$

Using Lemma 5,

$$\begin{aligned}
\Omega &\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{k}{4}-1)} \left\{ \sum_{m=0}^{2^r+1} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&+ K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{k}{4}-1)} \left\{ \sum_{m=2^r+1}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{k}{4}-1)} \left\{ \sum_{m=0}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{3k}{4})} \left\{ \sum_{m=0}^{2^{r+1}} D_q (m+\alpha)^2 |b_m|^2 \right\}^{k/2} \\
&\leq K^{k+\frac{1}{q}} C_q^{k/2} D_q^{k/2} \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2r)^{(-\frac{3k}{4})} \left\{ \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2}.
\end{aligned}$$

For $1 \leq k \leq 2$,

$$\Omega \leq L \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{3k}{4})} \Big\{ \sum_{m=0}^{2^{r+1}} (m+\alpha)^2 |b_m|^2 \Big\}^{k/2}, \tag{19}$$

where $L = K^{k+\frac{1}{q}} C_q^{k/2} D_q^{k/2}$. From (19),

$$\Omega \leq L \sum_{r=0}^{\infty} \gamma(2^{r+1})^k (2^r)^{(-\frac{3k}{4})} \left\{ \alpha^2 |b_0|^2 + \sum_{s=0}^{r} \sum_{m=2^s+1}^{2^{s+1}} (m+\alpha)^2 |b_m|^2 \right\}^{k/2}$$

$$\leq L\sum_{s=0}^{\infty}\sum_{r=s}^{\infty}\gamma(2^{r+1})^k(2^r)^{(-\frac{3k}{4})}\left\{\sum_{m=2^s+1}^{2^{s+1}}(m+\alpha)^2|b_m|^2\right\}^{k/2}+L\sum_{r=0}^{\infty}\gamma(2^{r+1})^k(2^r)^{(-\frac{3k}{4})}\alpha^k|b_0|^k.$$

Here $\{\gamma(n)\}$ is quasi β-power monotone decreasing and $\{n^{-3/4}\gamma(n)\}$ is quasi ϵ-power monotone decreasing, where $\beta > 3/4$ and $\epsilon = \beta + 3/4$. Thus, by using Lemma 1 of [6],

$$\sum_{n=m}^{\infty}\gamma(2^n)^k(2^n)^{-3k/4}\leq M\gamma(2^m)^k(2^m)^{-3k/4}, M\in\mathbb{Z}^+.$$

Therefore

$$\begin{aligned}\Omega &\leq L\sum_{s=0}^{\infty}\gamma(2^{s+1})^k(2^s)^{(-3k/4)}\left\{\sum_{m=2^s+1}^{2^{s+1}}(m+\alpha)^2|b_m|^2\right\}^{k/2}+L\gamma(2)^k\alpha^k|b_0|^k\\ &\leq 2^kL\sum_{s=0}^{\infty}\gamma(2^s)^k(2^s)^{(-3k/4)}\left\{\sum_{m=2^s+1}^{2^{s+1}}(m+\alpha)^2|b_m|^2\right\}^{k/2}+L\gamma(2)^k\alpha^k|b_0|^k\\ &\leq 2^kL\sum_{s=0}^{\infty}\gamma(2^s)^k\left\{\sum_{m=2^s+1}^{2^{s+1}}(m+\alpha)^2|b_m|^2\right\}^{k/2}+L\gamma(2)^k\alpha^k|b_0|^k\end{aligned}$$

and thus

$$\begin{aligned}&\sum_{n=2}^{\infty}\gamma(n)^kn^{k-1}\int_0^1|\sigma_n(x)-\sigma_{n-1}(x)|^kdx\\ &\leq 2^{(-\frac{3k}{4})}L\sum_{s=0}^{\infty}\gamma(2^s)^k\left\{\sum_{m=2^s+1}^{2^{s+1}}(m+\alpha)^2|b_m|^2\right\}^{k/2}+L\gamma(2)^k\alpha^k|b_0|^k.\end{aligned}$$

The following Corollaries can be verified by taking $\alpha = 0$ in the above theorems.

Corollary 1. *Every orthogonal series* $\sum_{n=0}^{\infty}c_n\psi_n$, $c_n\in\ell_2(\mathbb{Z}^+)$ *is* $|H,\psi|_k$ *summable for* $1\leq k\leq 2$ *and* $\gamma\in\Gamma_\beta$ *with* $\beta > 1-l/k$, *where* $\{\psi_n\}_{n=0}^{\infty}\subset L_2[0,1]$ *and H is a Hausdorff matrix with entries* $(h_{nk})_{n,k}\in\mathbb{Z}^+$.

This is Theorem 2 of [6].

Corollary 2. *Let* $1\leq k\leq 2$ *and* $\gamma\in\Gamma_\beta$ *with* $\beta > -3/4$, *where* $\{\phi_n\}_{n=0}^{\infty}\subset L_2[0,1]$ *and H is a Hausdorff matrix. Then, for any* $c_n\in\ell^2(\mathbb{Z}^+)$, *a sufficient condition for the orthogonal series* $\sum_{n=0}^{\infty}c_n\psi_n$ *to be* $|H,\gamma|_k$ *summable is*

$$\sum_{m=0}^{\infty}\gamma(2^m)^k\Big\{\sum_{n=2^m+1}^{2^{m+1}}\sqrt{n}|c_n|^2\Big\}^{k/2}<\infty. \tag{20}$$

This includes the results of Theorem 3 of [6].

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